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Modelling biplanes on surfaces

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Abstract

A biplane is a geometry corresponding to a symmetric $\left(\binom{k}{2}+1, k, 2\right)$ block design. Nontrivial biplanes are known to exist only for $k = 3, 4, 5, 6, 9, 11$ and 13. Group difference set constructions exist for the unique biplanes having $k = 3, 4,$ and 5; for all three biplanes having $k = 6$; and for one of the four biplanes having $k = 9$. We find models for these seven biplanes, using Cayley graph imbeddings on closed 2-manifolds.

1. Introduction

A biplane is a symmetric $(v, k, 2)$ design, that is a (v, b, r, k, λ) design with $b = v$, $r = k$, and $\lambda = 2$. Since $\lambda(v-1) = r(k-1)$ in general, here we find $v = \binom{k}{2}+1$. As a geometry, therefore, a biplane has, for some positive integer k:

(1) $\binom{k}{2} + 1$ points;
(2) $\binom{k}{2} + 1$ lines;
(3) each point on exactly k lines;
(4) each line containing exactly k points;
(5) each pair of distinct points in exactly two common lines.

It can also be shown that:

(6) each pair of lines meet in exactly two points.

As trivial biplanes exist for $k = 1$ and 2, we restrict our attention to $k \geq 3$. Hall Jr. [7] gives a useful specialization of the Bruck–Ryser–Chowla theorem to biplanes:

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Theorem 1. *For a* $(v, k, 2)$ *biplane:*

(i) *If* $k \equiv 2$ *or* $3 \pmod 4$, *then* v *is even and* $(k-2)$ *is a square.*

(ii) *If* $k \equiv 0$ *or* $1 \pmod 4$, *then* $z^2 = (k-2)x^2 + (-1)^{(v-1)/2}2y^2$ *has a solution in integers not all zero.*

Thus by (i) there are no biplanes for $k = 7, 10, 14, 15, 19, \ldots$; by (ii) there are no biplanes for $k = 8, 12, 17, \ldots$. Here is what else is known:

(a) There are unique biplanes for $k = 3, 4$ and 5. Since the dual of a biplane is a biplane, all three of these are self-dual. Following Salwach [11], we denote them by $B(3)$, $B(4)$, and $B(5)$ respectively.

(b) There are exactly three biplanes for $k = 6$.

(c) There are exactly four biplanes for $k = 9$.

(d) There are at least five biplanes for $k = 11$, and at least two for $k = 13$.

(e) No biplanes are known to exist, for $k > 13$. The open cases start with $k = 16, 18, 20, \ldots$.

As for any geometry (block design, hypergraph), a model can be provided by listing the lines (which thereby determine the points). The nicest situation occurs when one line generates the others, as will happen when one line is a λ-fold difference set (see [7]) for a group having the same order, v, as the point set. Biplanes are natural candidates for this construction, as the $k(k-1)$ ordered differences from a k-set might produce each nonidentity element of a group of order $\binom{k}{2} + 1$ exactly twice. Such difference sets exist, for the six biplanes having $k = 3, 4, 5,$ and 6 and for one of the four biplanes having $k = 9$. For these seven biplanes, techniques from topological graph theory allow concrete models arising from surface imbeddings of Cayley graphs for the groups giving the difference sets, and in the following sections we describe such models. For surface models of other geometries, see [4, 5, 12, 14, 15]. For schematic and other models of biplanes and other geometries, see [10].

2. $k = 3$, 4 and 5

2.1. $k = 3$

The tetrahedron serves as a model for $B(3)$. Note that the usual representation, in Fig. 1, readily depicts not only property (5)—corresponding to $\lambda = 2$—but also property (6). (The vertices of $K_4 = G_\Delta(Z_4)$, $\Delta = \{1, 2\}$, represent points and the region boundaries give the lines.) The line $\{0, 1, 2\}$, for example, is a 2-fold difference set for Z_4. This line generates the other three lines, using Z_4, as displayed in Table 1. This is the simplest nontrivial case of the observation of Heffter [8] and of Alpert [1] (see also Theorem 12-3 of [13]) that 2-fold triple systems correspond to triangular imbeddings of complete graphs. But as $\binom{3}{2} + 1 = 4$, the other 2-fold triple systems are not biplanes.

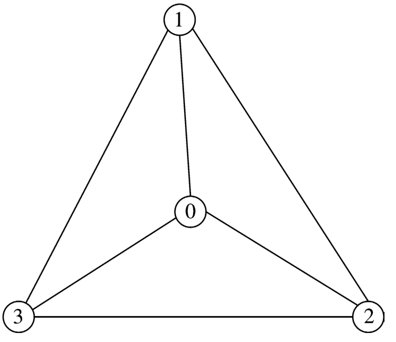

Fig. 1. The biplane $B(3)$.

Table 1
The lines of $B(3)$

0	1	2
1	2	3
2	3	0
3	0	1

2.2.

The unique biplane $B(4)$ is the complement of the Fano plane, $PG(2, 2)$, the symmetric $(7, 3, 1)$ design. As the latter has difference set $\{0, 1, 3\}$ in Z_7, the former will have difference set $\{2, 4, 5, 6\}$, reduced to $\{0, 2, 3, 4\}$ in Z_7. The toroidal voltage graph imbedding of Fig. 2 lifts to the toroidal Cayley multigraph imbedding of Fig. 3 (for background on voltage graph imbeddings, see Gross [6] or Chapter 10 of [13]). The multigraph is $K_7 \cup C_7$ (on seven vertices) as $G_\Delta(Z_7)$, $\Delta = \{1, 1', 2, 3\}$, ultimately imbedded in $S(1; 7(2))$: the pseudosurface resulting from the torus S_1 by identifying seven pairs of vertices (the two vertices labelled i, $0 \leq i \leq 6$). The seven lifts of the unshaded region are the seven lines of the biplane, generated by difference set $\{0, 2, 3, 4\}$. The seven lifts of the shaded region are the seven lines of the same (isomorphically) biplane, now generated by difference set $\{0, 4, 2, 1\}$.

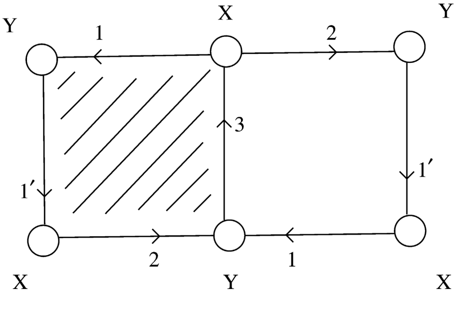

Fig. 2. A toroidal voltage graph imbedding for $B(4)$.

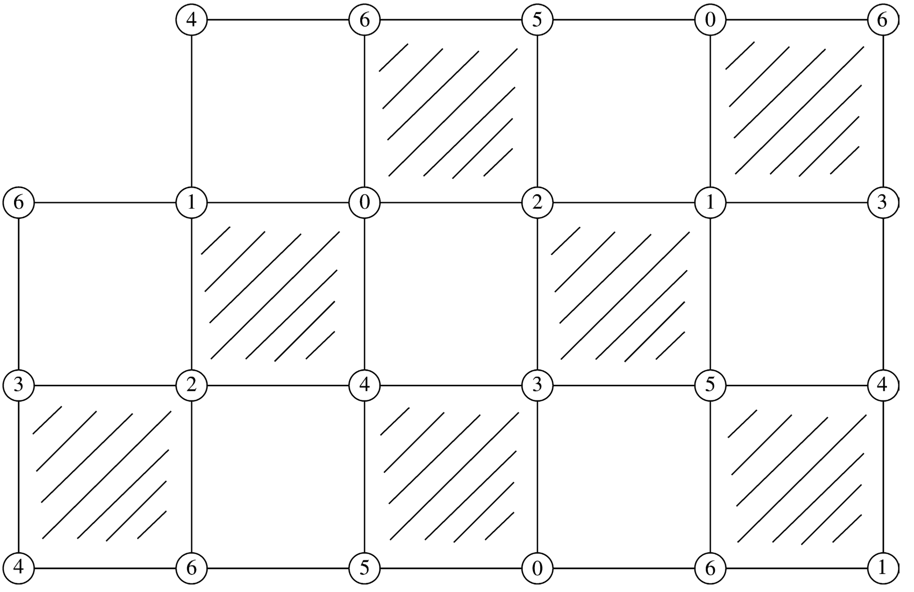

Fig. 3. The covering space.

As the pseudosurface $S(1; 7(2))$ fails to be a 2-manifold, at the seven points of identification, we seek an alternative model, as in Fig. 4. Now the voltage graph imbedding is into the Klein bottle, N_2, and the lift (again using Z_7) gives $K_7 \cup C_7 = G_{\{1,1,2,3\}}(Z_7)$ in the nonorientable surface N_8. (We apply Theorem 11-16 of [13] to distinguish between S_4 and N_8 as possible covering surfaces.) Here only the unshaded region lifts to (the seven) lines of the biplane, again generated by the difference set $\{0, 2, 3, 4\}$. The shaded regions lift to seven triangular hyperregions (covering the triangle), and one heptagonal hyperregion (covering the loop). These hyperregions are not part of the model, but are what is left over in the surface when the model is removed.

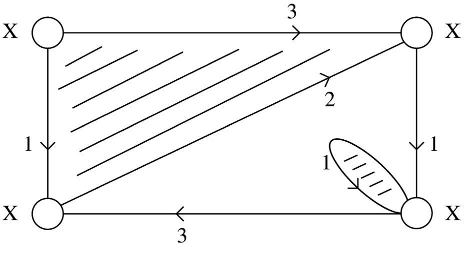

Fig. 4. A klein bottle voltage graph imbedding for $B(4)$.

Thus we can avoid the pseudosurface model. Moreover, we have improved the Euler characteristic by one (from -7 to -6); this is desirable, as increased characteristic indicates a simpler topological space. But now, as our surface is nonorientable, it inhabits R^4 rather than R^3. However, if this is not objectionable, we can do even better. The projective planar voltage graph imbedding of Fig. 5 lifts to an imbedding of $K_7 = G_{\{1,2,3\}}(Z_7)$ in N_7 (of characteristic -5), with the seven lines of the biplane generated as before by $\{0, 2, 3, 4\}$, covering the unshaded region.

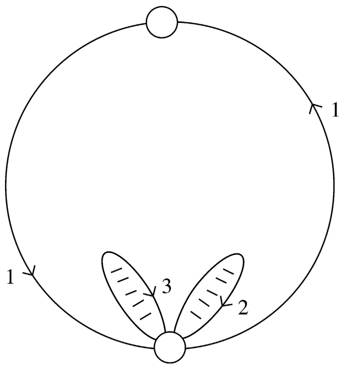

Fig. 5. A projective planar voltage graph imbedding for $B(4)$.

2.3.

The unique biplane $B(5)$ is modelled as $K_{11} = G_\Delta(Z_{11}), \Delta = \{1, 2, 3, 4, 5\}$, in S_{12}, covering Fig. 6—which also appears as Fig. 12-3 of [13]. The eleven lines are generated either by difference set $\{0, 3, 4, 8, 2\}$ and cover the unshaded region of the voltage graph imbedding, or by difference set $\{0, 2, 10, 9, 5\}$ and cover the shaded region. In combination, the two isomorphic biplanes form one (11, 22, 10, 5, 4)-BIBD.

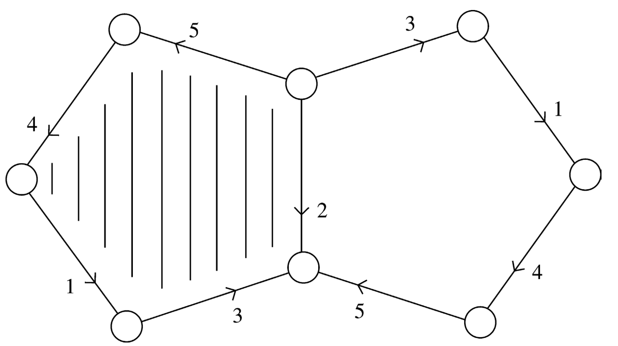

Fig. 6. A voltage graph imbedding for $B(5)$.

3. $k = 6$

There are exactly three biplanes for $k = 6$. All three of these are self-dual [7]. Each corresponds to a difference set. There are 14 groups of order 16 $\left(=\binom{6}{2}+1\right)$, five of which are Abelian. Kibler [9] states that Z_{16} and D_8 have no difference sets. For the other 12 groups of order 16, he gives all the corresponding difference sets (there are 27 in all). We note that there are **no** other noncyclic difference sets for biplanes having $k < 20$ [9]. Below we choose three of the 27 noncyclic difference sets which generate the three nonisomorphic biplanes for $k = 6$. One of these is necessarily nonAbelian, and this will cause complications.

For a noncyclic group G of order $\binom{k}{2}+1$ (now written multiplicatively), the condition for a k-subset D of G to be a biplane difference set is that each nonidentity element of G occurs exactly twice in the set $DD^{-1} = \{d_i d_j^{-1} \mid d_i \neq d_j \text{ in } D\}$. Then for D such a difference set and $g \in G$, Dg is also a difference set (since $d_i g(d_j g)^{-1} = d_i g g^{-1} d_j^{-1} = d_i d_j^{-1}$). Moreover (see Salwach [11], for example) the sets Dg, as g ranges over G, form the blocks of a projective (v, k, λ) design, where $v = |G|$. (Of course, in this paper $\lambda = 2$ and $v = \binom{k}{2} + 1$.)

3.1.

The group we select is $Z_2 \times Z_2 \times Z_2 \times Z_2$, generated by the standard basis. (There are 23 other options, for this biplane, which we denote by $B(6, 1)$.) Writing (a, b, c, d) as $abcd$, we give the difference set as $\{0000, 1000, 0100, 0010, 0001, 1111\}$. Then the spherical voltage graph of Fig. 7 lifts to an imbedding of $G_{\Delta}((Z_2)^4)$ in S_9, where $\Delta = \{1000, 1100, 0110, 0011, 1110, 1111\}$. (Note that we could represent each involution as a loop, lifting to eight digons, each of which we could then collapse to a single edge. Instead each involution is represented by a "half-edge", lifting to an eight-edge 1-factor. The result, in the covering surface, is the same.) There are 16 regions, all 6-gons, exactly giving the lines of this biplane, This is the most efficient surface possible for this geometry, which we therefore say has **genus** nine.

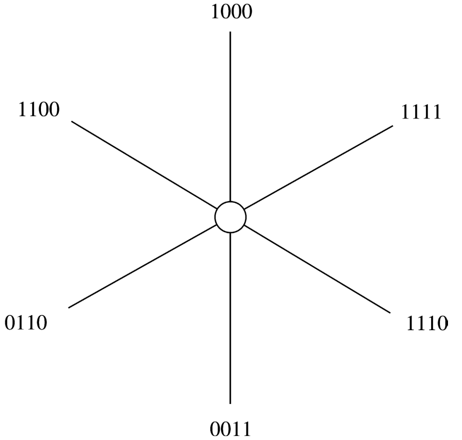

Fig. 7. A voltage graph imbedding for $B(6, 1)$.

This model has additional interest. As $B(6, 1)$ is self-dual, then so is this imbedding. The imbedding also appears as R9.18 in [3] and thus is regular (both rotary and reflexible); its automorphism group has maximal order 192. The underlying graph is the Cartesian product $K_4 \times K_4$, which can be seen by noting that Δ splits into two triples, $\{1000, 0110, 1110\}$ and $\{1100, 0011, 1111\}$, each inducing $4K_4$. Since our group is Abelian, these 48 edges complete the Cartesian product. The graph $K_4 \times K_4$ is strongly regular, and in Example 4 of Section 12-7 of [13] an imbedding is constructed in the generalized pseudosurface $(8S_0, 16(2))$, yielding a (16, 32, 6, 3; 0, 2) partially balanced incomplete block design. For $B(6, 1)$, we have constructed a (16, 16, 6, 6, 2) balanced incomplete block design.

3.2.

The only option for our second biplane for $k = 6$, $B(6, 2)$, is the group $Z_8 \times Z_2$, generated by 10 and 01. The difference set is $\{00, 10, 20, 40, 11, 61\}$. The toroidal voltage graph of Fig. 8 lifts to an imbedding of $G_\Delta(Z_8 \times Z_2)$ in S_{13}, where $\Delta = \{01, 11, 21, 30, 40, 41\}$. We check that one lift of the unshaded region, starting at the indicated vertex and proceeding clockwise, is (00, 61, 20, 11, 10, 40), as desired.

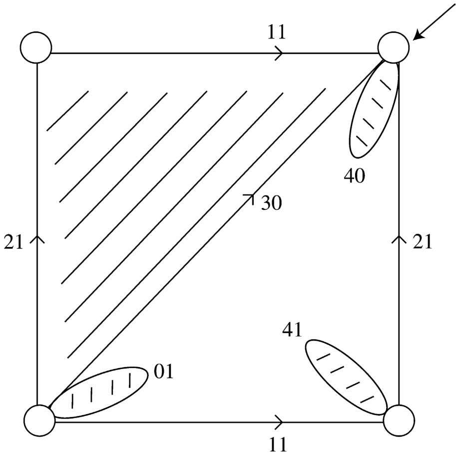

Fig. 8. A voltage graph imbedding for $B(6, 2)$.

3.3.

For the final biplane for $k = 6$, $B(6, 3)$, we have two options for the group G, both nonAbelian. Since this will be our first nonAbelian group difference set, we need a short digression. The usual approach (as in Salwach [10]) is to consider the differences $g_i g_j^{-1}$ from a subset $D = \{g_1, g_2, \ldots, g_k\}$ of a group G. Then in general the sets $Dg = \{g_1 g, g_2 g, \ldots, g_k g\}$, as g ranges over G, form the blocks of a projective (v, k, λ) design, where $v = |G|$. But the voltage graph lifting process produces regions gD. In the Abelian case these give the required lines Dg. But in the nonAbelian case we need some adjustment. We pattern the development here after that in [11].

Definition. Let $D = \{g_1, g_2, \ldots, g_k\}$ be a k-subset of a group G of order v. If for each nonidentity element d of G there are exactly λ ordered pairs $(g_i,\ g_j)$ from D such that $g_j^{-1} g_i = d$, then we say that D is a (v, k, λ) *group difference set*.

Theorem 2. *If $D = \{g_1, g_2, \ldots, g_k\}$ is a (v, k, λ) group difference set for group G, then so is gD, for each g in G.*

Proof. We need only check that $(gg_j)^{-1}(gg_i) = g_j^{-1} g_i = d$. $\square$

Theorem 3. *If D is a (v, k, λ) group difference set for group G, then the sets $gD = \{gg_1, gg_2, \ldots, gg_k\}$, as g ranges over G, give the blocks of a projective (v, k, λ) design.*

Proof. Let $a \neq b$ in G. Let $d = b^{-1}a$. Since there are exactly λ ordered pairs (g_i, g_j) having $g_j^{-1} g_i = d = b^{-1}a$, then $ag_i^{-1} = bg_j^{-1}$ are precisely the λ elements g so that $a, b \in gD$. $\square$

Returning to the case at hand, we choose the direct product $Z_2 \times Q$ (where Q is the quaternion group), generated by $\{1e, 0a, 0b\}$—where e is the identity of Q and $a^2 = b^2 = (ab)^2$ in Q. The set $D = \{0e,\ 0a^2,\ 1e,\ 1ab,\ 0b,\ 0a\}$ is a group difference set under either definition ($d = g_i g_j^{-1}$ or $d = g_j^{-1} g_i$). Here we use the latter, and the 16 lines gD, as g ranges over $Z_2 \times Q$. We choose $\Delta = \{0a^2,\ 1a^2,\ 0ab,\ 1a,\ 0a^3\}$ for $G_\Delta(Z_2 \times Q)$, so that one region covering the unshaded region of the voltage graph imbedding (in the Klein bottle) of Fig. 9 will be $R = (0e,\ 0a^2,\ 1e,\ 1ab,\ 0b,\ 0a)$, corresponding to D. The collection of left translates gD then gives the lines for a biplane with $k = 6$. But which one of the three? Is the geometry generated by gD the same as that generated by Dg? The covering surface here is S_{13}, by another application of Theorem 11-16 of [13].

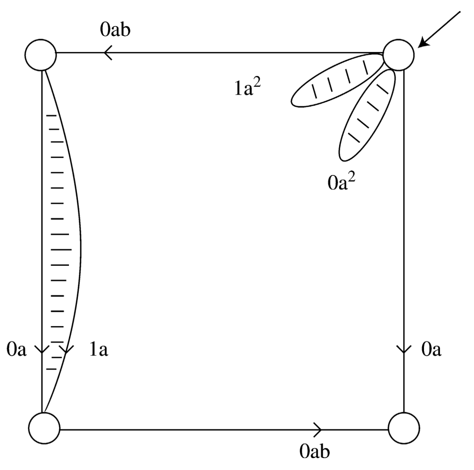

Fig. 9. A voltage graph imbedding for $B(6, 3)$.

Fortunately, the voltage graph imbedding of Fig. 9 also serves as a heuristic for a surgical construction using the 16 lines Dg, which definitely give the third biplane we seek to model with $k = 6$. For g in (respectively not in) the $Z_2 \times Z_4$ subgroup of $Z_2 \times Q$ generated by $\{1e,\ 0a\}$, orient the region corresponding to Dg clockwise (respectively counterclockwise). Sew these 16 hexagons, together with eight "shaded" squares, to form an imbedding of $G_\Delta(Z_2 \times Q)$ in S_{13} that does model the third biplane for $k = 6$.

We note that the underlying multigraphs for these three surface models, all of order 16, have degrees of regularity 6, 12 and 10 respectively (or 6, 9 and 8—after collapsing all digons corresponding to generators of order two).

4. $k = 9$

There are exactly four biplanes having $k = 9$. Two of these are dual to each other. The other two are self-dual. One of the latter, which we denote by $B(9, 1)$, is constructed by using the difference set consisting of the nine quadratic residues modulo 37 $\left(=\binom{9}{2} + 1\right)$: $D = \{1, 7, 9, 10, 12, 16, 26, 33, 34\}$ (see Hall [7]). Let $\Delta = \{6, 5, 2, 14, 13, 17, 8, 18, 15\}$. Then the voltage graph imbedding (on S_3) of Fig. 10 lifts to an imbedding of $G_\Delta(Z_{37})$ on S_{75} modelling this biplane having $k = 9$. (No difference set is known for the other three; see Baumert [2] and Kibler [9].) The imbedding has the feature that Z_{37} acts regularly, as a group of map automorphisms, not only on the points and lines of the biplane, but also on each orbit of the set of hyperregions (all of which are triangular). The latter property occurs because the Kirchoff voltage law (KVL) holds in each shaded region. This part of the construction was facilitated by noting that 7 is a multiplier for the difference set D and that the orbits under multiplication by 7^3 (=10 in Z_{37}) are the triples $\{1, 10, 26\} \cup \{7, 33, 34\} \cup \{12, 9, 16\}$. Taking successive differences $7 - 1,\ 12 - 7,\ 10 - 12,\ 33 - 10,\ 9 - 33,\ 26 - 9,\ 34 - 26,\ 16 - 34,\ 1 - 16$ produces the generating set Δ and the ordered voltage labelling of the 9-gon. Taking every third difference then gives the KVL triples $6 - 14 + 8 = 0,\ 5 + 13 - 18 = 0$, and $-2 + 17 - 15 = 0$. Two of these triples are re-ordered, so that the voltage graph imbedding will be into a surface rather than a pseudosurface.

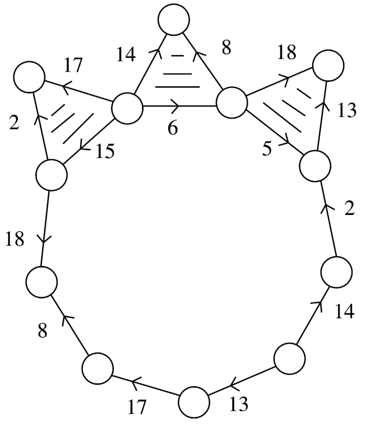

Fig. 10. A voltage graph imbedding for $B(9, 1)$.

This construction is reminiscent of that of [5, 12] for $PG(2, 8)$: the voltage graph imbeddings and methods of obtaining the KVL triples agree, but the lifting groups, voltages, covering graphs and covering surfaces, geometries and block designs all differ. The seemingly magical construction of the KVL triples in both cases relies on a multiplier q partitioning the relevant difference set into triples, such as $\{1, q, q^2\}$, whose sum is 0 (mod 37 for the biplane with $k = 9$, mod 73 for $PG(2, 8)$). In both cases, from $q^3 = 1$ it is easy to see why $q^2 + q + 1 = 0$.

We observe that the graph $G = G_{\Delta}(Z_{37})$ for our model of $B(9, 1)$ is self-complementary, as multiplication by 2 gives an isomorphism of G with $\overline{G}$. This is a unique situation for biplanes, in the sense that $\binom{k}{2}/4 = k$ is solvable for $k = 9$ uniquely.

5. Conclusion

We have used topological graph theory to model every biplane known both to exist and to correspond to a group difference set. Our constructions have involved primarily voltage graph lifting, but also surgery in one instance. The ambient spaces have included one orientable pseudosurface $S(1;\ 7(2))$, the two nonorientable surfaces N_7 and N_8, and the orientable surfaces $S_0,\ S_9,\ S_{12},\ S_{13}$ (three times), and S_{75}. The models on S_0 and S_9 are of maximum efficiency, and those on $S_0,\ S(1;\ 7(2)),\ S_9,\ S_{12}$, and S_{75} have interesting symmetries.

Questions for further study include:

(1) There are 17 known biplanes. We have modelled seven of them. The other ten are not known to have associated group difference sets. What suitable topological models might be found for these?
(2) Are there more than five biplanes for $k = 11$? More than two for $k = 13$?
(3) Do biplanes exist for $k = 16$? 18? 20? $\cdots$.
(4) For biplanes constructed using nonAbelian group difference sets (or for such (v, k, λ) designs in general):
   (a) Is the geometry having lines gD (and $d = g_j^{-1} g_i$) isomorphic to that having lines Dg (and $d = g_i g_j^{-1}$)?
   (b) For $d = g_i g_j^{-1}$, do the lines gD yield a (v, k, λ) design in general? That is, if one form for d establishes a difference set, does the other do so as well? (That this occurred in Section 3.3 follows from the happy circumstance that $a^2 = b^2$ in that context.)
(5) What can be said about the graphs underlying $B(6, 2)$ and $B(6, 3)$?

Acknowledgements

The author thanks the referees, for several valuable comments and suggestions.

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