Testing permutation properties through subpermutations

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There has been great interest in deciding whether a combinatorial structure satisfies some property, or in estimating the value of some numerical function associated with this combinatorial structure, by considering only a randomly chosen substructure of sufficiently large, but constant size. These problems are called property testing and parameter testing, where a property or parameter is said to be testable if it can be estimated accurately in this way. The algorithmic appeal is evident, as, conditional on sampling, this leads to reliable constant-time randomized estimators. Our paper addresses property testing and parameter testing for permutations in a subpermutation perspective; more precisely, we investigate permutation properties and parameters that can be well approximated based on a randomly chosen subpermutation of much smaller size. In this context, we use a theory of convergence of permutation sequences developed by the present authors [C. Hoppen, Y. Kohayakawa, C.G. Moreira, R.M. Sampaio, Limits of permutation sequences through permutation regularity, Manuscript, 2010, 34pp.] to characterize testable permutation parameters along the lines of the work of Borgs et al. [C. Borgs, J. Chayes, L. Lovász, V.T. Sós, B. Szegedy, K. Vesztergombi, Graph limits and parameter testing, in: STOC’06: Proceedings of the 38th Annual ACM Symposium on Theory of Computing, ACM, New York, 2006, pp. 261–270.] in the case of graphs. Moreover, we obtain a permutation result in the direction of a famous result of Alon and Shapira [N. Alon, A. Shapira, A characterization of the (natural) graph properties testable with one-sided error, SIAM J. Comput. 37 (6) (2008) 1703–1727.] stating that every hereditary graph property is testable.

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1. Introduction and main results

The study of very large combinatorial structures has been a thriving research topic, not least due to the multitude of applications, ranging from the analysis of large networks such as the Internet to the study of disease evolution and control, in addition to the description of molecular interactions. One of the main algorithmic challenges in the study of these objects is to describe them, or capture their essential properties, in a succinct yet accurate way.

A great deal of research has been devoted to the study of probabilistic models whose characteristics mimic the behavior of real-world networks, as illustrated in a book by Chung and Lu [15] and in a survey by Bollobás and Riordan [9], and
the references therein. For a particular application, a model is very often chosen according to the qualitative properties aimed at, and its parameters are then calibrated through real data to fit this application. In a different direction, researchers have looked for properties that are inherently robust, in the sense that one may accurately predict if a large object satisfies one of these properties simply by looking at a randomly chosen substructure of a much smaller size. Now, in a particular application, one such property may be estimated for a large structure by collecting data in a much smaller substructure.

Regarding the second approach, a rich theory has been developed for dense graphs, that is, for \( n \)-vertex graphs with \( \Omega(n^2) \) edges. For instance, one may ask about qualitative properties of a large graph \( G \). Is it bipartite? Is it connected? Can it be properly colored with a specified number of colors? In general, it is hopeless to obtain a precise answer to such a question by looking only at a randomly chosen induced subgraph on a negligible proportion of the original vertex set. Indeed, the graph could be the union of a large bipartite graph and a triangle, and, in order to establish that it is not bipartite, it would be necessary to identify this triangle. However, can one at least decide whether the graph is close to satisfying the property, or far from satisfying it? Such questions have been treated in the study of property testing, first presented in the seminal paper of Goldreich et al. [21]. A graph property \( \mathcal{P} \) is said to be testable if, for every \( \varepsilon > 0 \), there exist a positive integer \( k \) and a randomized algorithm \( \mathcal{T} \), called a tester, which has the ability to query whether a desired pair of vertices in the input graph are adjacent or not. Moreover, the following properties are satisfied for any input graph \( G \) with at least \( k \) vertices:

- \( \mathcal{T} \) is bounded above by a function that depends only on the error bound \( \varepsilon \), and not on the size of the input;
- \( G \) satisfies \( \mathcal{P} \), then the tester identifies this with probability at least \( 1 - \varepsilon \);
- \( G \) is \( \varepsilon \)-far from satisfying \( \mathcal{P} \), then, with probability at least \( 1 - \varepsilon \), the tester confirms that \( G \) does not satisfy \( \mathcal{P} \), where, by being \( \varepsilon \)-far, it is meant that no graph obtained from \( G \) by the addition or removal of at most \( \varepsilon n^2 \) edges satisfies \( \mathcal{P} \).

Goldreich and Trevisan [22] have proved that, for graphs, property testing can be addressed through simpler canonical testers: the class of testable properties remains unchanged if we consider only testers of a particular form, namely those that randomly choose a \( k \)-subset \( X \) of vertices in \( G \) and then verify whether the induced subgraph \( G[X] \) satisfies some related property \( \mathcal{P}' \). For example, if the property being tested is having edge density \( 1/2 \), then the algorithm could choose a random subset \( X \) of appropriate size and check whether the edge density of \( G[X] \) is within \( \varepsilon \) of \( 1/2 \).

Several deep results have been established in this direction. For instance, using a variant of the Szemerédi Regularity Lemma, Alon and Shapira proved in [6] that every monotone graph property is testable, and, more generally, that the same holds for hereditary graph properties [5]. This generalizes the bipartiteness and colorability questions asked above, which had already been settled by Goldreich et al. [21] and by Rödl and Duke [30]. The connection between property testing and regularity has been further explored in [2] and, in a different direction, Fischer [20] has addressed testable graph properties under the optics of their logical quantification, extending the work of Alon et al. [1].

In a similar vein, one may want to determine a numerical function of a large graph \( G = (V, E) \), say its edge density or the proportion of edges in a maximum cut. It is obvious that the value of such parameters cannot be determined exactly unless the entire graph is investigated, but is there hope of finding a good approximation by looking at a random subgraph of much smaller size? This question is addressed by Borgs et al. [10,11], who introduced the concept of parameter testing. A numerical function, or parameter, associated with a combinatorial structure is said to be testable if, with high probability, it can be estimated accurately by considering only a randomly chosen substructure of sufficiently large, but constant size. More precisely, a graph parameter \( f(G) \) is testable if, for every \( \varepsilon > 0 \), there is a positive integer \( k \) such that, given a graph \( G = (V, E) \) with at least \( k \) vertices, a randomly chosen \( k \)-subset \( X \subset V \) gives an estimate \( \hat{f} \) of \( f(G) \) for which \( |f(G) - \hat{f}| \leq \varepsilon \) with probability at least \( 1 - \varepsilon \). It can be shown that, when a parameter is testable, one such accurate estimate for the random subset \( X \) is just \( f(G[X]) \), which leads to an algorithm for estimating \( f(G) \) whose time complexity conditional on sampling depends only on \( \varepsilon \), and therefore is independent of the size of the graph.

Varians of property testing have been considered in several other areas. For instance, an early piece of work of Rubinfeld and Sudan [31] considered testing in the context of polynomial functions. In the case of strings, Alon et al. [3] showed that pertinence to a regular language is testable, and Ergün et al. [19] proved that a sequence of \( n \) integers can be tested for the property of being monotone nondecreasing using \( O(\ln n) \) queries. Batu et al. [7] showed that one can test whether the edit distance between two strings is small in sublinear time, and, more recently, Lachish and Newman [25] have studied testing with respect to the string property of being periodic and having periodicity smaller than a given constant \( g \). Regarding other structures, Batu et al. [8] provided sublinear-time algorithms for testing monotone and unimodal distributions, Alon and Shapira [4] investigated the testability of the constraint satisfaction problem, while Buhrman et al. [12] studied property testing under the prism of quantum computation.

In this paper, we develop a theory of property testing and parameter testing for permutations over a finite set of integers \([n] = \{1, 2, \ldots, n\}\) based on subpermutations. A permutation parameter \( f \) is simply a function that associates a numerical value with each permutation. For instance, a parameter of a permutation \( \sigma \) may be the number of fixed points in \( \sigma \), the number of inversions in \( \sigma \), the length of a maximum increasing subpermutation of \( \sigma \), or the number of cycles in a cycle decomposition of \( \sigma \). Inspired by the graph case, we define testability for permutation parameters in terms of subpermutations: given permutations \( \tau \) on \([m]\) and \( \sigma \) on \([n]\), we say that a strictly increasing \( m \)-tuple \((x_1, \ldots, x_m) \in [n]^m\) induces the subpermutation \( \tau \) in \( \sigma \) if \( \tau(i) < \tau(j) \) if and only if \( \sigma(x_i) < \sigma(x_j) \) for every \( (i, j) \in [m]^2 \). The set \( S = (x_1, \ldots, x_m) \) is called the index set of this subpermutation, which is called the subpermutation of \( \tau \) induced by \( S = \{x_1, \ldots, x_k\} \) and is
denoted by $\sigma[S]$. As an illustration, there is a subpermutation $\tau = (3, 1, 4, 2)$ in $\sigma = (5, 6, 2, 4, 7, 1, 3)^1$, since $\sigma$ maps the index set $S = (1, 3, 5, 7)$ onto $(5, 2, 7, 3)$, which appears in the relative order given by $\tau$.

We may now declare a permutation parameter to be testable through subpermutations, or just testable for short, if, with high probability, it can be estimated accurately by looking at a randomly chosen subpermutation of sufficiently large, but constant size. We make this idea precise.

**Definition 1.1.** Let $k < n$ be integers and $\sigma : [n] \to [n]$ be a permutation. The random subpermutation $\text{sub}(k, \sigma)$ of $\sigma$ is the induced subpermutation of $\sigma$ of size $k$ whose index set $S = \{s_1 < \cdots < s_k\}$ is chosen uniformly at random in $[n]$.

As an illustration, for the permutation $\sigma = (5, 6, 2, 4, 7, 1, 3)$ defined above and for $k = 3$, the random subpermutation $\text{sub}(k, \sigma)$ is equal to $(2, 3, 1)$ when the subset $S = \{2, 5, 7\}$ is the outcome of the random selection.

**Definition 1.2.** A permutation parameter $f$ is testable if, for every $\varepsilon > 0$, there exists an integer $k$ such that, if $\sigma : [n] \to [n]$ is a permutation of size $n > k$, then we may compute an estimate $\hat{f}$ of $f(\sigma)$ based on a random subpermutation $\text{sub}(k, \sigma)$ of $\sigma$ in such a way that

$$P \left( |f(\sigma) - \hat{f}| > \varepsilon \right) \leq \varepsilon.$$

We shall prove in Section 3 that one can always use $\hat{f} = f(\text{sub}(k, \sigma))$. This definition seems to be satisfying, as it captures rather closely the essence of what a “robust permutation parameter” should be. Notwithstanding, two points need further discussion. On the one hand, we shall see that, unlike in the graph case, it is not true that any permutation parameter that can be approximated by a small number of queries may also be approximated based on a small randomly chosen subpermutation. Thus the version of testing considered here is more constrained than the general case. On the other hand, proving that some permutation parameter is testable through direct methods could be elusive. To settle this matter, we turn again to the graph case. Borgs et al. [10,11] characterized bounded graph parameters in terms of the concept of convergent graph sequences introduced by Lovász and Szegedy in [27].

Our main result in this direction, which is analogous to the graph case (see Proposition 2.12 in [10]), characterizes bounded testable permutation parameters in terms of a notion of convergence for permutation sequences (see **Definition 2.1**), where a permutation parameter $f$ is said to be bounded if there exists a constant $K$ such that $|f(\sigma)| \leq K$ for every permutation $\sigma$.

**Theorem 1.3.** A bounded permutation parameter $f$ is testable if and only if, for every convergent permutation sequence $(\sigma_n)_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} |\sigma_n| = \infty$, the real sequence $(f(\sigma_n)_{n \in \mathbb{N}}$ converges.

With this result, we can decide in a simple manner whether a series of permutation parameters are testable or not. Consider, for instance, the following parameters of a permutation $\sigma$ on $[n]$:

(i) $fp(\sigma)$, the fixed-point density, is the number of elements $x \in [n]$ such that $\sigma(x) = x$, divided by $n$;
(ii) $cyc(\sigma)$, the cycle density, is the number of cycles in $\sigma$, divided by $n$;
(iii) $inv(\sigma)$, the inversion density, is given by the number of pairs $i < j \in [n]^2$ such that $\sigma(i) > \sigma(j)$, divided by $\binom{n}{2}$;
(iv) $\text{ordmax}(\sigma)$ is given by the size of a longest increasing subpermutation of $\sigma$, divided by $n$.

One would naturally expect the parameters $fp(\sigma)$ and $cyc(\sigma)$ not to be testable, as a subpermutation can only capture the relative order of a subset of elements in the permutation. However, this is far less clear for the parameters $inv(\sigma)$ and $\text{ordmax}(\sigma)$.

**Corollary 1.4.** The parameter $inv$ is a testable permutation parameter, while the parameters $fp$, cyc and $\text{ordmax}$ are not testable.

However, it is not hard to see that the permutation parameter $fp$ can be estimated with a small number of queries. As a matter of fact, given a permutation $\sigma$ on $[n]$, where $n$ is larger than an absolute constant $n_0$, we may sample $c$ random positions $i \in \{1, \ldots, n\}$ and use the fraction $\alpha$ of points $i$ such that $\sigma(i) = i$ to estimate $fp(\sigma)$. With sharp concentration arguments, we shall prove in **Proposition 3.5** that, for any given $\varepsilon$, this estimate is within $\varepsilon$ of the correct value with high probability, provided that the absolute constants $c$ and $n_0$ are sufficiently large in terms of $\varepsilon$. This justifies our earlier claim that the notion of testing in this paper is more restrictive than a notion based solely on query complexity.

As we considered limits of permutation sequences, a notion of distance between permutations turned out to be especially useful. It is based on a normalized version of the concept of discrepancy of a permutation, introduced by Cooper [16] to measure the “randomness” of a permutation. In this definition, we use the family $l[n]$ of all intervals in $[n]$, that is, of all sets of the form $\{x \in [n] : a \leq x \leq b\}$ for some $a, b \in [n]$.

**Definition 1.5 (Rectangular Distance).** Given permutations $\sigma_1, \sigma_2 : [n] \to [n]$, the rectangular distance between $\sigma_1$ and $\sigma_2$ is given by

$$d_c(\sigma_1, \sigma_2) = \frac{1}{n} \max_{S \in l[n]} ||\sigma_1(S) \cap T| - |\sigma_2(S) \cap T||.$$

---

1 A permutation $\sigma$ on $[n]$ is represented by $\sigma = (\sigma(1), \ldots, \sigma(n))$. This integer $n$ is the length $|\sigma|$ of $\sigma$. 
Given the pivotal rôle played by the rectangular distance \( d_\ell \); in the study of convergent permutation sequences and in its connection with parameter testing, it seems natural to define a permutation property \( \mathcal{P} \) to be weakly testable through subpermutations, or just weakly testable, if there is a permutation property \( \mathcal{P}' \) satisfying the following. For any fixed \( \varepsilon > 0 \), there exists a positive integer \( k \) such that, if \( \sigma \) is a permutation on \( [n] \) with \( n \geq k \), then the two statements below hold:

(i) the random subpermutation \( \text{sub}(k, \sigma) \) satisfies \( \mathcal{P}' \) with probability at least \( 1 - \varepsilon \) whenever \( \sigma \) satisfies \( \mathcal{P} \);

(ii) the random subpermutation \( \text{sub}(k, \sigma) \) does not satisfy \( \mathcal{P}' \) with probability at least \( 1 - \varepsilon \) whenever \( \sigma \) is \( \varepsilon \)-far from satisfying \( \mathcal{P} \), that is, whenever

\[
d_\ell(\sigma, \mathcal{P}) = \min\{d_\ell(\sigma, \pi) : \pi : [n] \rightarrow [n] \text{ satisfies } \mathcal{P} \} \geq \varepsilon.
\]  

The "weakness" of this notion of testability is due to the fact that we measure the error in terms of the metric \( d_\ell \), as opposed to the usual \( \ell_1 \)-distance (see Section 5.2 for additional comments concerning this). However, with this definition, we obtain the following general result, which translates the main result of \([5]\) to the permutation setting (with the above caveat). Here, a permutation property \( \mathcal{P} \) is said to be hereditary if the fact that a permutation \( \sigma \) satisfies \( \mathcal{P} \) implies that all its subpermutations also satisfy \( \mathcal{P} \).

**Theorem 1.6.** Every hereditary permutation property is weakly testable.

One consequence of this theorem is the following. Given a possibly infinite family of permutations \( \mathcal{F} \), a natural structural question is whether an input permutation \( \sigma \) is \( \mathcal{F} \)-avoiding, that is, whether \( \sigma \) does not contain any subpermutation \( \tau \in \mathcal{F} \) as a subpermutation. As the property \( \mathcal{P}_\mathcal{F} \) of being \( \mathcal{F} \)-avoiding is hereditary, Theorem 1.6 attests that it is weakly testable. In fact, the statement that, for every family of permutations \( \mathcal{F} \), property \( \mathcal{P}_\mathcal{F} \) is weakly testable is actually equivalent to Theorem 1.6, since, given a hereditary permutation property \( \mathcal{P} \), there exists a countable family of permutations \( \mathcal{F} \) such that \( \mathcal{P} = \mathcal{P}_\mathcal{F} \), namely, the family \( \mathcal{F} \) given by all permutations that do not satisfy \( \mathcal{P} \).

The remainder of our paper is structured as follows. In Section 2, we sketch the theory of permutation sequences and their convergence to a limit object, which is a very important tool in the proofs presented here. Section 3 evinces the link between convergent permutation sequences and testable parameters with the proof of Theorem 1.3. Property testing is the subject of Section 4, while Section 5 comprises final remarks and open problems.

### 2. Limits of permutation sequences

This section is devoted to the description of a theory of convergence for permutation sequences introduced by the present authors in \([24]\), and the reader is referred to that paper for the proofs of the results stated here. The approach in \([24]\) follows quite closely the strategy used for graphs. We refer to \([23]\) for a more concise probabilistic approach.

We start with a few preliminary definitions. Let \( [n]^m \) denote the set of all strictly increasing \( m \)-tuples in \( [n] \). For positive integers \( m \leq n \), and for permutations \( \tau \) and \( \sigma \) on \( [m] \) and \( [n] \), respectively, the number of occurrences \( \Lambda(\tau, \sigma) \) of the subpermutation \( \tau \) in \( \sigma \) is the number of \( m \)-tuples \((x_1, x_2, \ldots, x_m) \in [n]^m\) such that \( \sigma(x_i) < \sigma(x_j) \) if and only if \( \tau(i) < \tau(j) \).

The **density** of the permutation \( \tau \) as a subpermutation of \( \sigma \) is given by

\[
t(\tau, \sigma) = \begin{cases} \binom{n}{m}^{-1} \Lambda(\tau, \sigma), & \text{if } m \leq n, \\ 0, & \text{otherwise}. \end{cases}
\]

This may be used to define the convergence of a permutation sequence.

**Definition 2.1.** A permutation sequence \( (\sigma_n)_{n \in \mathbb{N}} \) is convergent if, for every fixed permutation \( \tau \), the sequence of real numbers \( (t(\tau, \sigma_n))_{n \in \mathbb{N}} \) converges.

When the sequence of lengths \( (|\sigma_n|)_{n \in \mathbb{N}} \) tends to infinity, every convergent permutation sequence has a natural limit object, called a **limit permutation**. The restriction to permutation sequences with no bounded subsequence is quite natural, since, if this is not the case, it can be shown that the convergent sequence \( (\sigma_n)_{n \in \mathbb{N}} \) is eventually constant.

A limit permutation consists of an (uncountable) family of **cumulative distribution functions**, or cdf for short, where we say that a function \( F : [0, 1] \rightarrow [0, 1] \) is a cdf if \( F \) is a nondecreasing and right-continuous function with \( F(0) \geq 0 \) and \( F(1) = 1 \). Note that \( F \) is a cdf if and only if there is a \( [0, 1] \)-valued random variable \( Y \) such that \( \forall y \in [0, 1] : F(y) = \mathbb{P}(Y \leq y) \).

**Definition 2.2.** A **limit permutation** is a Lebesgue measurable function \( Z : [0, 1]^2 \rightarrow [0, 1] \) satisfying the following conditions:

(a) for every \( x \in [0, 1] \), the restriction \( Z(x, \cdot) \) is a cdf;

(b) for every \( y \in [0, 1] \), the restriction \( Z(\cdot, y) \) satisfies

\[
\int_0^1 Z(x, y) \, dx = y.
\]

We observe that condition (a) follows from condition (b) for almost all \( x \in [0, 1] \).
This transition from discrete to continuous objects appears as a natural consequence of the following limiting procedure. A permutation \( \sigma \) is first encoded as a bipartite graph \( G_\sigma \), called the graph of \( \sigma \), whose color classes \( A \) and \( B \) are disjoint copies of \([n]\), and where \([a, b] \) is an edge, with \( a \in A \) and \( b \in B \), if and only if \( \sigma(a) < b \). Now, the \([n] \times [n] \) bipartite adjacency matrix \( Q_\sigma \) of \( G_\sigma \) is such that rows are nondecreasing and each column \( i \) sums to \( i - 1 \). Note that this matrix can be naturally encoded as a step function \( Z_\sigma : \{0, 1\}^2 \rightarrow \{0, 1\} \) given by

\[
Z_\sigma(x, y) = \begin{cases} 
0, & \text{if } x = 0 \text{ or } y = 0; \\
Q_\sigma([x\pi], [y\pi]), & \text{otherwise.} 
\end{cases}
\]  

Clearly, the function \( Z_\sigma \) is Lebesgue measurable in \([0, 1]^2\) and it is such that, on the one hand, the functions \( Z_\sigma(x, \cdot) \) are cdfs and, on the other hand, we have \( \int_{[0,1]^2} Z_\sigma(x, y) \, dx = (i - 1)/n \) for every \( y \) in the interval \((i - 1)/n, i/n]\). A limit permutation \( Z \) is derived from a limiting procedure involving a sequence of step functions.

**Theorem 2.3.** Given a convergent permutation sequence \((\sigma_n)_{n \in \mathbb{N}}\) for which \( \lim_{n \rightarrow \infty} |\sigma_n| = \infty \), there exists a limit permutation \( Z : [0, 1]^2 \rightarrow [0, 1] \) such that

\[
\lim_{n \rightarrow \infty} t(\tau, \sigma_n) = t(\tau, Z) \quad \text{for every } \tau.
\]

The statement of Theorem 2.3 depends on \( t(\tau, Z) \), the subpermutation density of \( \tau \) in the function \( Z \). Roughly speaking, if \( \tau \) has length \( m \), an occurrence of \( \tau \) in \( Z \) is a point \((x_1, \ldots, x_m, y_1, \ldots, y_m) \in [0, 1]^m \) such that \( x_1 < x_2 < \cdots < x_m \) and \( y_{\tau^{-1}(1)} < y_{\tau^{-1}(2)} < \cdots < y_{\tau^{-1}(m)} \). The density of \( \tau \) in \( Z \) is given by the size of this set with respect to a specific measure associated with \( Z \). The precise definition of this concept requires a few basic measure-theoretical pre-requisites. With every cdf \( F : [0, 1] \rightarrow [0, 1] \), we associate a Lebesgue-Stieltjes probability measure \( \mu_F \) over the Borel sets of \([0, 1] \), namely the measure satisfying \( \mu_F([0,1]) = 1 \) and \( \mu_F((a,b)) = F(b) - F(a) \), for \( 0 \leq a < b \leq 1 \). Moreover, given \( k \) cdfs \( F_1, \ldots, F_k \) and their respective measures \( \mu_1, \ldots, \mu_k \), the product measure \( \mu = \mu_1 \times \cdots \times \mu_k \) is a probability measure in \([0, 1]^k \) over the Borel sets of \([0, 1]^k \).

The usual notation for the Lebesgue-Stieltjes integral of a Borel-measurable function \( g : [0, 1] \rightarrow \mathbb{R} \) over a measurable set \( S \subseteq [0, 1] \) with respect to \( \mu_1 \) is \( \int_S g \, d\mu_1 = \int_S g \, dF_1 \). When \( S = [a, b] \), we use \( \int_{[a,b]} g \, dF_1 = \int_a^b g \, dF_1 \). For a Borel-measurable function \( g : [0, 1]^k \rightarrow \mathbb{R} \), the integral over the measurable set \( S \subseteq [0, 1]^k \) with respect to \( \mu = \mu_1 \times \cdots \times \mu_k \) is denoted by \( \int_S g \, d\mu = \int_{[a,b]} g \, dF_1 \cdots dF_k \).

**Definition 2.4.** For a permutation \( \tau : [m] \rightarrow [m] \) and a limit permutation \( Z : [0, 1]^2 \rightarrow [0, 1] \), the subpermutation density of \( \tau \) in \( Z \) is given by

\[
t(\tau, Z) = \# \int_{[0,1]^m} \left( \int_{[0,1]^m} dZ(x_{\tau^{-1}(1)}, \cdot) \cdots dZ(x_{\tau^{-1}(m)}, \cdot) \right) \, dx_1 \cdots dx_m.
\]

Note that integration is taken over the \( m \)-simplex \([0, 1]^m \subseteq [0, 1]^m\), which is the set of \( m \)-tuples \((y_1, \ldots, y_m) \) such that \( 0 \leq y_1 < y_2 < \cdots < y_m \leq 1 \). Each term \( dZ(x_{\tau^{-1}(1)}, \cdot) \cdots dZ(x_{\tau^{-1}(m)}, \cdot) \) of the inner integral comes from the measure associated with \( Z(x_{\tau^{-1}(1)}, \cdot) \), and the order of the integrating factors \( dZ(x_{\tau^{-1}(1)}, \cdot) \cdots dZ(x_{\tau^{-1}(m)}, \cdot) \) in the product measure reflects the connection between the integration variable corresponding to \( y_i \) and the measure associated with the cdf \( Z(x_{\tau^{-1}(i)}, \cdot) \).

An interesting fact about this limiting procedure is the following converse of Theorem 2.3.

**Theorem 2.5.** For every limit permutation \( Z : [0, 1]^2 \rightarrow [0, 1] \), there is a convergent permutation sequence \((\sigma_n)_{n \in \mathbb{Z}}\) converging to \( Z \).

To demonstrate this result, limit permutations are used to define a new model of random permutations, which we call \( Z \)-random permutations. This resembles the concept of \( W \)-random graphs introduced by Lovász and Szegedy in [27], defined for each fixed limit object, or grapthon, \( W \) of a convergent graph sequence.

**Definition 2.6 (\( Z \)-Random Permutation).** Given a limit permutation \( Z : [0, 1]^2 \rightarrow [0, 1] \) and a positive integer \( n \), a \( Z \)-random permutation \( \sigma(n, Z) \) is a permutation on \([n] \) generated as follows. A sequence of \( n \) real numbers \( X_1 < \cdots < X_n \) is generated uniformly in the simplex \([0, 1]^n \). A sequence \( a_1, \ldots, a_n \) in \([0, 1] \) is then generated, where the \( a_i \) are drawn independently with probabilities induced by the cdf \( Z(X_i, \cdot) \). The permutation \( \sigma(n, Z) \) is given by the indices of the real numbers \( a_i \) as these are listed in increasing order.

We mention that \( Z \)-random permutations are well defined with probability 1, for, as proved in [24], a property of limit permutations is that the probability that the sequence \( a_1, \ldots, a_n \) has repeated elements is zero. We observe that, for a limit permutation \( Z \), the order of the definitions of the density \( t(\tau, Z) \) and of the \( Z \)-random permutation \( \sigma(n, Z) \) could be interchanged. In this case, having fixed a permutation \( \tau \) of length \( m \), one could give an equivalent and perhaps more intuitive definition of the density \( t(\tau, Z) \) in terms of random permutations, namely

\[
t(\tau, Z) = \mathbb{P}(\sigma(m, Z) = \tau).
\]

Using random permutations, Theorem 2.5 may be established in the following form.

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2 For example, if \( n = 3 \) and the generation of the \( a_i \) yields \( a_2 < a_1 < a_3 \), then \( \sigma(3, Z) = (2, 1, 3) \).
Theorem 2.7. Given a limit permutation $Z : [0, 1]^2 \to [0, 1]$, the sequence of random permutations $(\sigma(n, Z))_{n \in \mathbb{Z}}$ is convergent with probability one, and its limit is $Z$.

In the proof of the above result, the following is a crucial step, which will also be used later in this paper.

Lemma 2.8. Let $Z$ be a limit permutation and let $n \geq m$ be positive integers. For any permutation $\tau : [m] \to [m]$ and any constant $\varepsilon > 0$, we have
\[
\mathbb{E}(t(\tau, \sigma(n, Z))) = t(\tau, Z) = \mathbb{P}(\sigma(m, Z) = \tau),
\]
\[
\mathbb{P}\left(\left| t(\tau, \sigma(n, Z)) - t(\tau, Z) \right| > \varepsilon \right) \leq 2 \exp(-\varepsilon^2 n/2m^2).
\]

Another important question regarding limits of permutation sequences is to characterize the set of limit permutations that are limits to a given convergent permutation sequence $(\sigma_n)$. It is clear that this limit is not unique in a strict sense, because, if $Z$ is a limit of a given permutation sequence $(\sigma_n)$ and $A$ is a measurable subset of $[0, 1]$ with measure zero, then any limit permutation obtained through the replacement of the cdf $Z(x, \cdot)$ by a cdf $Z'(x, \cdot)$ for every $x \in A$ is also a limit of $(\sigma_n)$, as the value of the integrals will not be affected. A similar situation occurred in the case of graph limits in [10], and uniqueness was captured by a natural pseudometric induced by a distance $d_Z$ between the graphs generated by them. For permutations, the question of relabeling does not occur due to the fact that there is a natural order constraint. However, it is possible to use a similar proof technique, and even more can be achieved: two limit permutations are limits of the same sequence if and only if they only differ in a set with Lebesgue measure zero.

To extend this graph-theoretical approach to permutations, one may naturally define a notion of rectangular distance for limit permutations that generalizes the distance $d_Z$ for permutations, as we shall do in Definition 2.10. Note that the study of convergent permutation sequences and the rectangular distance are deeply interconnected, as evinced by the following results.

Theorem 2.9. Let $(\sigma_n)_{n \in \mathbb{N}}$ and $(\pi_n)_{n \in \mathbb{N}}$ be convergent permutation sequences such that $|\sigma_n| = |\pi_n|$ for every $n$ and $\lim_{n \to \infty} |\sigma_n| = \infty$. These two sequences converge to the same limit $Z$ if and only if the real sequence $(d_Z(\sigma_n, \pi_n))_{n \in \mathbb{N}}$ converges to zero.

More generally, as we consider the extension of the rectangular distance to limit permutations, the restriction to measuring distances between permutations of the same size may be dropped, and, in particular, Theorem 2.9 holds without this hypothesis.

Definition 2.10. Given functions $Z_i : [0, 1]^2 \to [0, 1], i \in \{1, 2\}$, such that $Z_i(x, \cdot)$ is a cdf for every $x \in [0, 1]$, the rectangular distance between $Z_1$ and $Z_2$ is defined by
\[
d_{Z}(Z_1, Z_2) = \sup_{x_1 < x_2, y_1 < y_2 \in [0, 1]} \left| \int_{x_1}^{x_2} \int_{y_1}^{y_2} Z_1(x, \cdot) dx \cdot \int_{x_1}^{x_2} \int_{y_1}^{y_2} Z_2(x, \cdot) dx \right|.
\]

With this definition, the following continuity result may also be established.

Theorem 2.11. A permutation sequence $(\sigma_n)_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} |\sigma_n| = \infty$ converges to a limit permutation $Z$ if and only if
\[
\lim_{n \to \infty} d_{\sigma_n}(Z_0, Z) = 0.
\]

To end this section, a technical result involving convergence of permutation sequences, random permutations and the rectangular distance is stated for future reference. Note that, in the statement of this result, reference is made to the rectangular distance between permutations of distinct lengths. This can be formally defined through the extension of rectangular distance to step functions and limit permutations by defining $d_{Z}(\sigma, \pi) = d_{\pi}(Z_0, Z_2)$.

Lemma 2.12. Let $k$ be a sufficiently large positive integer and let $\pi : [n] \to [n]$ be a permutation with $n > k$. Then we have
\[
d_{\pi}(\pi, \pi(k, \pi)) \leq 7k^{-1/4} \text{ with probability at least } 1 - 2e^{-\sqrt{k} / 3}.
\]

3. Parameter testing for permutations

In this section, we address the problem of determining whether a permutation parameter is testable or not through subpermutations. Using the results stated in the previous section, we describe a family of testable parameters. Moreover, we prove Theorem 1.3 and use it to determine whether a series of permutation parameters are testable.

For the first part, consider a fixed limit permutation $Z$. We may use the concept of rectangular distance introduced in Definition 2.10 to define a bounded permutation parameter $f_Z$ given by
\[
f_Z(\sigma) = d_{\sigma}(\sigma, Z) = d_{\sigma}(Z_0, Z).
\]

The following is an easy consequence of the results in the previous section.
Proposition 3.1. For every limit permutation $Z$, the permutation parameter $f_2$ is testable.

Proof. Let $\epsilon > 0$ and fix $k_0 > 0$ such that $\max\{7k_0^{-1/4}, e^{-\sqrt{k_0}/3}\} < \epsilon$. Fix $k > e^{k_0}$. For a permutation $\sigma$ of length $n > k$, we estimate the value of $f_2(\sigma) = d_{\epsilon}(\sigma, Z)$ as follows. A subpermutation $\pi$ of $\sigma$ of size $k$ is chosen uniformly at random, and the estimate is given by $\tilde{f}(\pi) = f_2(sub(k_0, \pi))$.

It is clear that $|f_2(\sigma) − \tilde{f}| = |f_2(\sigma) − f_2(sub(k_0, \sigma))|$, as the direct choice of a subpermutation of size $k_0$ is equivalent to the two-step random selection of the previous paragraph. Moreover, by the definition of $f_2$ and by the fact that $d_{\epsilon}$ is a pseudometric, and therefore satisfies the triangle inequality, we have

$$|f_2(\sigma) − f_2(sub(k_0, \sigma))| = |d_{\epsilon}(\pi, Z) − d_{\epsilon}(sub(k_0, \sigma), Z)| < d_{\epsilon}(sub(k_0, \sigma), \sigma).$$

By Lemma 2.12 and our choice of $k_0$, this error is smaller than $\epsilon$ with probability at least $1 − \epsilon$, concluding the proof. □

There is a special instance of $f_2$ that deserves additional attention. As mentioned in the introduction, Cooper [16] defined the discrepancy $D(\sigma)$ of a permutation $\sigma$ on $[n]$ in the context of quasirandom permutations. As a matter of fact, we have

$$D(\sigma) = \max_{S,T \subseteq [n]} |\sigma(S) \cap T| − \frac{|S||T|}{n}.$$  

This parameter can be turned into a bounded parameter $D_1$, the normalized discrepancy, by considering $D_1(\sigma) = D(\sigma)/n$. Note that

$$D_1(\sigma) = \frac{1}{n} \max_{S,T \subseteq [n]} |\sigma(S) \cap T| − \frac{|S||T|}{n} = \frac{1}{n} \max_{S = \{x_1, x_2, \ldots\}, T = \{a_1, a_2\} \subseteq [n]} \left| \left( \sum_{i=x_1}^{x_2} Z_{i/n} − Z_{\sigma(i/n, a_2)} − \frac{(x_2 − x_1)(a_2 − a_1)n}{a_1} \right) \right|$$

$$= \sup_{x_1 < x_2 \in [0,1]} \left| \int_{x_1}^{x_2} \int_{a_1}^{a_2} dZ_{\sigma}(x, \cdot) dx − \int_{x_1}^{x_2} \int_{a_1}^{a_2} dy dx \right| = d_{\epsilon}(\sigma, U),$$

where $U$ is the limit permutation given by $U(x, y) = y$ for every $(x, y) \in [0,1]^2$. This implies the following result.

Corollary 3.2. The normalized discrepancy $D_1$ is a testable permutation parameter.

We now concentrate on the proof of Theorem 1.3. The steps of this proof are very similar to those of (a) $\implies$ (b), (b) $\implies$ (c) and (c) $\implies$ (e) in [10, Theorem 6.1]

Proof of Theorem 1.3. We first prove that the testability of a bounded parameter $f$ implies the convergence of the sequence $(f(\sigma_n))$ for every convergent permutation sequence $(\sigma_n)$ such that $\lim_{n \to \infty} |\sigma_n| = \infty$.

Let $f$ be a testable parameter bounded by the positive constant $M$ and fix $\epsilon > 0$. Definition 1.2 tells us that there exist a positive integer $k$ and an estimation function $\tilde{f}$ for which, if $\sigma : [n] \to [n]$ is a permutation with length $n > k$, then the inequality $|f(\sigma) − \tilde{f}(sub(k, \sigma))| < \frac{\epsilon}{2M}$ holds with probability at least $1 − \frac{\epsilon}{2M}$, where, as before, $sub(k, \sigma)$ is the random subpermutation of $\sigma$ given by the relative order of the sequence $(\sigma(s_1), \ldots, \sigma(s_k))$, where the subset $S = \{s_1 < \cdots < s_k\}$ of size $k$ is chosen uniformly at random in $[n]$. As a consequence,

$$E(f(sub(k, \sigma))) \leq \left(1 − \frac{\epsilon}{2M}\right) E(f(\sigma)) + \frac{\epsilon}{2M} \cdot M \leq f(\sigma) + \frac{\epsilon}{2M}$$

for $\epsilon$ chosen sufficiently small. A similar argument leads to a lower bound on $E(f(sub(k, \sigma)))$, and we have

$$|f(\sigma) − E(f(sub(k, \sigma)))| \leq \epsilon.$$

Let $(\sigma_n)_{n \in \mathbb{N}}$ be a convergent permutation sequence. Thus, for every permutation $\tau : [k] \to [k]$, there exists a limit $t(\tau)$ for the real sequence $(t(\tau, \sigma_n))_{n \in \mathbb{N}}$. In particular, as $n$ tends to infinity, we have

$$P(sub(k, \sigma_n) = \tau) \to t(\tau, \sigma_n) \to t(\tau).$$

Now,

$$E(\tilde{f}(sub(k, \sigma))) = \sum_{\tau : [k] \to [k]} P(sub(k, \sigma_n) = \tau) \cdot \tilde{f}(\tau) \to \sum_{\tau : [k] \to [k]} t(\tau) \cdot \tilde{f}(\tau) = a_k.$$
Hence, for every $n$ sufficiently large, we have
\[ |f(\sigma_n) - a_k| \leq |f(\sigma_n) - \mathbb{E}(f(sub(k, \sigma_n)))| + \varepsilon \leq 2\varepsilon, \]
so that $f(\sigma_n)$ oscillates at most $4\varepsilon$ for $n$ sufficiently large. As $\varepsilon > 0$ is arbitrary, the sequence $(f(\sigma_n))_{n \in \mathbb{N}}$ is a Cauchy sequence, and therefore is convergent.

We now prove the converse. Suppose for a contradiction that the sequence $(f(\sigma_n))_{n \in \mathbb{N}}$ converges whenever $(\sigma_n)_{n \in \mathbb{N}}$ is a convergent permutation sequence, but the parameter $f$ is not testable. In particular, the function $\tilde{f}$ defined by $\tilde{f}(sub(k, \sigma_n)) = f(sub(k, \sigma_n))$ is not an estimation function for $f$. As a consequence, there exists $\varepsilon > 0$ such that, for all positive integers $k$, there is a permutation $\sigma$ of length $n > k$ such that $|f(\sigma_n) - f(sub(k, \sigma_n))| > \varepsilon$ with probability greater than $\varepsilon$. In particular,

there is a permutation sequence $(\sigma_n)_{n \in \mathbb{N}}$, $|\sigma| \geq e^n$, for which
\[ |f(\sigma_n) - f(sub(n, \sigma_n))| > \varepsilon \text{ with probability greater than } \varepsilon \text{ for every } n. \tag{4} \]

To reach a contradiction, we first prove an auxiliary claim.

**Claim 3.3.** Suppose that $(f(\sigma_n))_{n \in \mathbb{N}}$ converges whenever $(\sigma_n)_{n \in \mathbb{N}}$ is a convergent sequence. Then, for every $\varepsilon > 0$, there exist $\delta > 0$ and $n_0$ such that any given permutations $\sigma$ and $\sigma'$ on $[n]$, satisfying $d_\infty(\sigma, \sigma') < \delta$ must also satisfy $|f(\sigma) - f(\sigma')| < \varepsilon$.

Claim 3.3 is in fact an instance of the Heine–Cantor Theorem, which states that a continuous function on a compact metric space is uniformly continuous. Before proving this claim, we show that it leads to the desired result. Let $\varepsilon$ and $n_0$ be given in (4) and fix $\delta > 0$ such that $d_\infty(\sigma, \sigma') < \delta$ implies $|f(\sigma) - f(\sigma')| < \varepsilon$. By Lemma 2.12, we have $d_\infty(\sigma_n, sub(n, \sigma_n)) < \delta$ with probability at least $1 - \varepsilon/2$ if $n$ is sufficiently large. This contradicts (4). \qed

**Proof of Claim 3.3.** Suppose that this is not the case and fix $\varepsilon > 0$ and sequences $(\sigma_n)$ and $(\sigma'_n)$ for which $d_\infty(\sigma_n, \sigma'_n) \to 0$, but $|f(\sigma_n) - f(\sigma'_n)| > \varepsilon$ for every $n$. Observe that these two sequences may be chosen to be convergent, as the set of all permutations on all positive numbers is countable, hence a diagonalization argument may be applied. Indeed, let $(\tau_m)_{m \in \mathbb{N}}$ be an ordering of these permutations. If $(\sigma_n)$ does not converge, starting with $\tau_1$, let $(\sigma'_1)$ be a subsequence of $(\sigma_n)$ for which the bounded real sequence $(t(\tau_1, \sigma'_1))$ converges. Inductively, we let $(\sigma'_m)$ be a subsequence of $(\sigma'_n)$ such that $(t(\tau_m, \sigma'_m))$ converges. It is now easy to see that the diagonal sequence $(\sigma'_m)_{m \in \mathbb{N}}$ is a convergent subsequence of $(\sigma_n)$.

Now, we know that $|\sigma_n|$ and $|\sigma'_n|$ are convergent sequences and that $d_\infty(\sigma_n, \sigma'_n) \to 0$. Theorem 2.9 implies that the sequence $(\sigma'_n) = (\sigma'_1, \sigma'_2, \sigma'_3, \ldots)$ must also converge. This is a contradiction, as the sequence $(f(\sigma'_n))$ is convergent by assumption, but cannot converge due to our choice of $(\sigma_n)$ and $(\sigma'_n)$. \qed

Before looking more closely at classes of testable permutation parameters, we observe that the above proof of Theorem 1.3 tells us more about estimates $\tilde{f}$ for bounded testable permutation parameters. Indeed, when we showed that there is an estimator for every bounded testable parameter $f$ for which $(f(\sigma_n))$ is convergent whenever $(\sigma_n)$ is convergent, we established that one such estimator applies $\tilde{f}$ to a sufficiently large random subpermutation of the input permutation. In particular, we may only consider estimates of this type.

With the characterization of Theorem 1.3, one immediately obtains a class of testable permutation parameters: for every fixed permutation $\tau$, the parameter $f(\tau)$, that associates with every permutation $\sigma$ its subpermutation density $t(\tau, \sigma)$ is clearly testable, as the convergence of the sequence $f(\sigma_n)$ is part of the assumption that $(\sigma_n)$ is convergent. This easy observation can be used to derive the following “converse” of Theorem 1.3.

**Proposition 3.4.** A permutation sequence $(\sigma_n)_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} |\sigma_n| = \infty$ is convergent if and only if the sequence $(f(\sigma_n))_{n \in \mathbb{N}}$ converges for every bounded testable permutation parameter $f$.

**Proof.** The fact that the convergence of a permutation sequence $(\sigma_n)$ implies the convergence of the sequences $(f(\sigma_n))$ is part of the statement of Theorem 1.3. For the converse, we use the fact that the bounded parameters given by the subpermutation densities of each fixed permutation $\tau$ are testable. As a consequence, our hypothesis tells us that $(t(\tau, \sigma_n))$ must converge for every $\tau$, and $(\sigma_n)$ is convergent by definition. \qed

Theorem 1.3 also yields an easy proof of Corollary 1.4.

**Proof of Corollary 1.4.** It is very easy to establish the testability of the parameter inv, since $\text{inv}(\sigma) = t(\tau, \sigma)$ for the permutation $\tau = (2, 1)$.

Now, given a positive integer $n$, consider two sequences $\sigma_n = (1, 2, 3, \ldots, n)$ and $\sigma'_n = (n, 1, 2, 3, \ldots, n - 1)$. Note that, for $k \leq n$, $t(\tau, \sigma_n)$ is equal to 1 whenever $\tau$ is of the form $(1, 2, \ldots, k)$ is and is equal to zero for every other $\tau$. On the other hand, we have $t(\tau, \sigma'_n) = (n - 1)^{-1}(n - 1 - k) = \frac{n-k}{n}$ if $\tau = (1, 2, \ldots, k)$, $t(\tau, \sigma'_n) = (n - 1)^{-1}(n - k - 1) = \frac{k}{n}$ if $\tau = (k, 1, 2, \ldots, k - 1)$, and $t(\tau, \sigma'_n) = 0$ for every other choice of $\tau$. This implies that the permutation sequence $(\sigma'_n) = (\sigma'_1, \sigma'_2, \sigma'_3, \ldots)$ converges, since we have $t(\tau, \sigma'_1) \to 1$, if $\tau$ is monotone increasing, and $t(\tau, \sigma'_n) \to 0$ for any other choice of $\tau$. However, for every positive integer $n$, we have $\text{fp}(\sigma_n) = 1$ and $\text{cyc}(\sigma_n) = 1$, while $\text{fp}(\sigma'_n) = 0$ and $\text{cyc}(\sigma'_n) = 1/n$, so that the sequences $(\text{fp}(\sigma'_n))$ and $(\text{cyc}(\sigma'_n))$ do not converge. This establishes the non-testability of the parameters fp and cyc.
Regarding the parameter \( \text{ordmax}(\sigma) \), for each positive integer \( n \), consider the permutation \( \pi_n : [n] \rightarrow [n] \) given by \( \pi_n(1) = 1, \pi_n(i) = i + 1 \) for \( i < n \) even, \( \pi_n(i) = i - 1 \) for \( i > 1 \) odd, and \( \pi_n(n) = n \) if \( n \) is even. For instance, we have \( \pi_8 = (1, 3, 2, 5, 4, 7, 6, 8) \) and \( \pi_9 = (1, 3, 2, 5, 4, 7, 6, 9, 8) \). With an easy counting argument, we may show that, for any fixed permutation \( \tau \), we have \( \lim_{n \to \infty} \ell(\tau, \pi_n) = \lim_{n \to \infty} \ell(\tau, \sigma_n) \), where \( \sigma_n \) is again the identity permutation on \([n] \). This clearly implies that the permutation parameter \( \text{ordmax}(\sigma) \) is not testable, as \( \text{ordmax}(\sigma_n) = 1 \) and \( \text{ordmax}(\pi_n) = \lceil (n + 1)/2 \rceil/n \), which converges to 1/2 as \( n \) tends to infinity. \( \square \)

As mentioned in the introduction, our notion of testability through subpermutations is more restrictive than a notion of testability based on query complexity. As a matter of fact, the fixed-point density of a permutation may be accurately approximated with a small number of queries, even if it is not testable through subpermutations.

**Proposition 3.5.** Let \( \varepsilon > 0 \) and let \( \sigma \) be a permutation on \([n] \) where \( n > \varepsilon^{-3} \). Then there is a randomized tester \( \mathcal{T} = \mathcal{T}(\varepsilon) \) that, based on \( \varepsilon^{-3} \) random variables, provides an estimate \( \tilde{f} \) of \( \text{fp}(\sigma) \) such that \( |\text{fp}(\sigma) - \tilde{f}| < \varepsilon \) with probability at least \( 1 - \varepsilon \).

To prove this result, it suffices to apply the weak law of large numbers for binomial random variables to the tester \( \mathcal{T} = \mathcal{T}(\varepsilon) \) that chooses \( k(\varepsilon) = \lceil \varepsilon^{-3} \rceil \) elements \( t_1, \ldots, t_k \) of \([n] \) uniformly at random, with repetition, and outputs the value

\[
\tilde{f} = \frac{1}{k} \sum_{i \in \{1, \ldots, k\}} \sigma(t_i).
\]

The details are omitted.

There are other permutation parameters whose testability or non-testability through subpermutations may be established. Let \( S_n \) be the set of all permutations on \([n] \). Given a metric \( d : S_n^2 \rightarrow \mathbb{R} \) in \( S_n \), the weight \( w_d(\pi) \) of a permutation \( \pi \in S_n \) is defined in [13] as \( w_d(\pi) = d(\pi, \varepsilon) \), the distance between \( \pi \) and the identity permutation \( \varepsilon = (1, 2, \ldots, n) \). This notion of weight is especially useful for a metric that is right-invariant, that is, for a metric \( d \) such that \( d(\pi, \sigma) = d(\pi \cdot \tau, \sigma \cdot \tau) \) for all \( \pi, \sigma, \tau \in S_n \). Hence, if \( d \) is right-invariant, we have \( d(\pi, \sigma) = d(\pi \cdot \sigma^{-1}, \varepsilon) = w_d(\pi \cdot \sigma^{-1}) \).

There are several classical right-invariant metrics in \( S_n \). Quite often, for \( \pi, \sigma \in S_n \), such a metric \( d(\pi, \sigma) \) is defined as the minimum number of instances of some pre-defined basic operation to turn \( \pi \) into \( \sigma \). The following examples are among the best-known distances between two permutations \( \pi \) and \( \sigma \) in \( S_n \); see Diaconis [18]:

(a) normalized Hamming distance \( d_H(\pi, \sigma) = \frac{1}{n} \sum_{i \in [n]} \chi(\pi(i) \neq \sigma(i)) \);
(b) normalized Ulam distance \( d_U(\pi, \sigma) \), where the operation allowed is to choose an element and insert it in a new position, normalized through division by \( n \);
(c) normalized transposition distance \( d_T(\pi, \sigma) \), where the operation allowed is to switch any pair of elements, normalized through division by \( n \);
(d) normalized Kendall’s tau distance \( d_K(\pi, \sigma) \), where the operation allowed is to switch any pair of consecutive elements, normalized through division by \( \binom{n}{2} \);
(e) \( \ell_\infty \)-distance, \( \ell_\infty(\pi, \sigma) = \frac{1}{n} \max_{i \leq n} |\pi(i) - \sigma(i)| \).

The next corollary classifies the weights associated with these distances in terms of testability. Its proof is straightforward, along the lines of the proof of Corollary 1.4.

**Corollary 3.6.** The weight \( w_K(\cdot) \) associated with the normalized Kendall’s tau distance \( d_K \) is a testable permutation parameter. The weights \( w_U(\cdot), w_T(\cdot) \) and \( \ell_\infty(\cdot) \), associated with the normalized Hamming, Ulam and transposition distances, and with the \( \ell_\infty \)-distance, respectively, are non-testable permutation parameters.

**Proof.** Fix a permutation \( \sigma \) on \([n] \) and let \( \varepsilon \) denote the identity permutation on \([n] \).

The fact that the normalized Kendall’s tau distance \( d_K \) yields a testable permutation parameter is trivial, as \( w_K(\sigma) = d_K(\sigma, \varepsilon) \) is equal to the number of inversions in \( \sigma \), which we have seen to be a testable parameter in Corollary 1.4.

We now look at the other parameters. First, it is easy to see \( w_U(\sigma) = d_U(\sigma, \varepsilon) = (n - \text{fp}(\sigma))/n \), which leads to the non-testability of \( w_U \), as the fixed-point density \( \text{fp}(\sigma) \) is not testable by Corollary 1.4. Similarly, the Ulam distance between \( \sigma \) and \( \varepsilon \) is the minimum number of times one has to choose an element of \( \sigma \) and insert it in a new position to obtain the identity permutation \( \varepsilon \). It is not hard to see that it suffices to keep a longest increasing subpermutation of \( \sigma \) intact, while all the other elements need to be inserted in their correct position, so that \( w_U(\sigma) = (n - \text{ordmax}(\sigma))/n \). Again, Corollary 1.4 asserts the non-testability of \( w_U \), as \( \text{ordmax}(\sigma) \) is not testable. One may also argue that \( w_T(\sigma) = 1 - \text{cyc}(\sigma) \), hence the weight associated with the normalized transposition distance is not testable.

For \( w_\infty \), it suffices to see that, with \( \sigma_0, \sigma_0', \sigma_0'' \) defined as in the proof of Corollary 1.4, the sequence \( (w_\infty(\sigma_0')) \) does not converge, even though \( (\sigma_0') \) is convergent. This is because \( w_\infty(\sigma_0) = 0 \) for every positive integer \( n \), while \( w_\infty(\sigma_0') = (n - 1)/n \to 1 \) as \( n \) tends to infinity. \( \square \)

4. Property testing for permutations

The aim of this section is to establish Theorem 1.6, which asserts the weak testability of hereditary permutation properties. The proof of this result is inspired by similar work on graphs due to Lovász and Szegedy [26].
Theorem 1.6. Let \( P \) be a hereditary permutation property and, for every integer \( k \), let \( T(k) \) be the tester defined as follows: given a permutation \( \sigma \) with length at least \( k \), a permutation \( \tilde{\sigma} \) is generated according to the distribution of \( \text{sub}(k, \sigma) \). If \( \tilde{\sigma} \) satisfies \( P \), then the tester returns that \( \sigma \) satisfies \( P \), otherwise it returns that \( \sigma \) does not satisfy \( P \).

The heredity of \( P \) implies that the tester always provides the correct answer if \( \sigma \) satisfies \( P \). Therefore, in order to establish the testability of this property, it suffices to show that, given \( \varepsilon > 0 \), there exists a positive integer \( k = k(\varepsilon) \) such that, whenever a permutation \( \sigma \) with length \( n \geq k \) satisfies \( d_\varepsilon(\sigma, P) \geq \varepsilon \), the tester returns that \( \tilde{\sigma} \) does not satisfy \( P \) with probability at least \( 1 - \varepsilon \). This fact follows from the following lemma.

Lemma 4.1. Let \( P \) be a hereditary property. Then, given \( \varepsilon > 0 \), there exist positive integers \( n_0 \) and \( M \), and a positive constant \( c \) such that the following holds. If \( \sigma \) is a permutation with length \( n \geq n_0 \) satisfying \( d_\varepsilon(\sigma, P) \geq \varepsilon \), then, for some permutation \( \tau \) with length \( m \leq M \) that does not satisfy \( P \), we have \( t(\tau, \sigma) \geq c \).

Before proving Lemma 4.1, we see why it implies Theorem 1.6. Let \( n_0, M \) and \( c \) be given by Lemma 4.1 with \( \eta = \varepsilon \). With foresight, consider the tester \( T(k) \) with \( k = 2KM^2/c \), where \( K = K(\varepsilon) \) is a sufficiently large constant in terms of \( \varepsilon \).

Let \( \sigma \) be a permutation on \([n]\), \( n > k \), such that \( d_\varepsilon(\sigma, P) \geq \varepsilon \). We have to show that a random subpermutation \( \text{sub}(k, \sigma) \) does not satisfy \( P \) with probability at least \( 1 - \varepsilon \). To this end, let \( \tau \) be the permutation with length \( m \leq M \) given by Lemma 4.1 for our permutation \( \sigma \), which does not satisfy \( P \). Since \( P \) is hereditary, it suffices to prove that \( \text{sub}(k, \sigma) \) contains a copy of \( \tau \) with probability at least \( 1 - \varepsilon \).

Fix an arbitrary \( m\lfloor K/c \rfloor \)-element subset of the domain of \( \text{sub}(k, \sigma) \) and consider it as the union of \( \lfloor K/c \rfloor \) disjoint blocks of size \( m \). Since \( \sigma \) contains at least \( c \left( \frac{n}{m} \right) \) copies of \( \tau \), the first block induces a copy of \( \tau \) with probability at least \( c \). As every element in the permutation can lie in at most \( \left( \frac{n-1}{m-1} \right) \) copies of \( \tau \), there are at most \( m \left( \frac{n-1}{m-1} \right) \) copies of \( \tau \) containing elements of the first block of \( \text{sub}(k, \sigma) \). Thus the probability that the second block induces a copy of \( \tau \) is at least

\[
c - m \left( \frac{n}{m} \right)^{-1} \left( \frac{n-1}{m-1} \right) = c - \frac{m^2}{n},
\]

regardless of the outcome of the first block. In general, for \( 0 \leq j \leq \lfloor K/c \rfloor \), the \( j \)th block induces a copy of \( \tau \) with probability at least \( c - jm^2/n \geq c/2 \) regardless of the outcome of the previous blocks, as we chose \( n \) larger than \( 2m^2K/c \). Hence \( \text{sub}(k, \sigma) \) does not contain \( \tau \) with probability at most \( (1 - c/2)^\lfloor K/c \rfloor \), which is smaller than \( \varepsilon \) for \( K \) sufficiently large in terms of \( \varepsilon \), as required. \( \square \)

Proof of Lemma 4.1. Suppose for a contradiction that the statement does not hold. In other words, there is \( \eta > 0 \) such that, for every \( n_0 = M = \ell \) and \( c_\ell = 1/\ell \), there exist a positive integer \( n_\ell > n_0 \) and a permutation \( \sigma_\ell \) on \([n_\ell] \) such that \( d_\varepsilon(\sigma_\ell, P) \geq \eta \), but \( t(\tau, \sigma_\ell) < 1/\ell \) for every permutation \( \tau \) with length at most \( \ell \) that does not satisfy \( P \).

Using a straightforward diagonalization argument, we may show that \( (\sigma_\ell) \) has a convergent subsequence, which, for convenience, is called the same. Let \( Z \) be its limit.

Our construction clearly implies that \( t(\tau, Z) = 0 \), for every fixed permutation \( \tau \) that does not satisfy \( P \). Now, consider the sequence \( (\pi_\ell) \) where \( \pi_\ell = \text{sub}(n_\ell, Z) \) is a \( Z \)-random permutation with length \( n_\ell \). As seen in Lemma 2.8, given \( \tau \) not satisfying \( P \), the expected number of copies of \( \tau \) in \( \pi_\ell \) is \( t(\tau, Z) = 0 \). In particular, all subpermutations of \( \pi_\ell \) satisfy \( P \), and therefore we have that \( d_\varepsilon(\pi_\ell, \sigma_\ell) \geq \eta \) for every \( \ell \). However, the sequence \( (\sigma_\ell) \) converges to \( Z \), while \( (\pi_\ell) \) converges to \( Z \) with probability one by Theorem 2.7, so that \( d_\varepsilon(\sigma_\ell, \pi_\ell) \) must converge to zero by Theorem 2.9, a contradiction. \( \square \)

An immediate corollary of Theorem 1.6 is the following result, which asserts that it is testable to determine whether a permutation \( \sigma \) avoids a fixed family of permutations. More precisely, let \( F \) be a family of permutations, possibly infinite, and say that a permutation \( \sigma \) is \( F \)-avoiding if \( \sigma \) does not contain any permutation in \( F \) as a subpermutation.

Corollary 4.2. Let \( F \) be a family of permutations. The property \( P_F \) of being \( F \)-avoiding is testable.

5. Further remarks and open problems

In this final section, we discuss a few additional questions related with our theory.

5.1. Cayley distances and testing

As a first topic, we again consider testing distances between permutations. In Section 3, we established that the normalized Kendall tau distance between two permutations can be effectively estimated through subpermutations, as it is right-invariant and the weight associated with it is a testable permutation parameter. We have also seen that this is not the case for the normalized transposition distance, whose corresponding weight is not testable. One common feature of these two distances is that they are normalized Cayley distances, where a Cayley distance between permutations of some fixed length \( n \) is defined as follows. Consider a set \( S \) of permutations on \([n]\) that is closed under taking inverses, that is, for which \( \sigma \in S \) implies that \( \sigma^{-1} \in S \). The Cayley graph \( G_S \) associated with \( S \) is the graph with vertex set given by all permutations on \([n]\) such that two vertices \( \sigma \) and \( \pi \) are adjacent if and only if \( \sigma \cdot \pi^{-1} \in S \). This naturally leads to a distance \( d_k \) between permutations on \([n]\), namely the length of a shortest path in \( G_S \) that connects \( \sigma \) to \( \pi \). Hence a Cayley distance between
permutations is just a distance defined through a Cayley graph. Its normalized counterpart is divided by the diameter of the corresponding Cayley graph, i.e., by the biggest weight. For instance, Kendall's tau distance is derived from the Cayley graph generated by $S = \{\sigma_i\}_{i=1}^{n-1}$, where the $\sigma_i$ are the adjacent transpositions, i.e., each $\sigma_i$ is the identity permutation but for the elements $i$ and $i + 1$, which are mapped to each other. The transposition distance may be defined through the Cayley graph generated by the set of ordinary transpositions $S = \{\sigma_{ij}\}_{1 \leq i < j \leq n}$, that is, $\sigma_{ij}$ coincides with the identity everywhere but for $i$ and $j$, which are mapped to each other.

**Question 5.1.** Characterize the set of testable weights associated with normalized Cayley distances.

5.2. Rectangular distance vs. edit distance

A second topic concerns the use of our theory to prove a permutation counterpart of the removal lemma for graphs. In its simplest form, the removal lemma states that, if a graph $G$ contains at most $o(n^m)$ copies of some $m$-vertex graph $H$, then one may remove $o(n^s)$ edges of $G$ to destroy all copies of $H$. A translation of this to permutations has been given by Cooper [17]. If $\binom{[n]}{2}$ denotes the set of all pairs $\{i, j\}$ for which $i < j$, a set $S \subseteq \binom{[n]}{2}$ is said to remove all copies of the permutation $\tau$ in $\sigma$ if the index set of every superpermutation $\tau$ in $\sigma$ contains both points of some element of $S$. Now, consider a permutation $\sigma$ on $[n]$ such that $\Lambda(\tau, \sigma) = o(n^m)$ for some fixed permutation $\tau$ on $[m]$. Cooper [17] proves that we may choose a set $S \subseteq \binom{[n]}{2}$ of size $|S| = o(n^m)$ that removes all copies of $\tau$ in $\sigma$. As pointed out by Mathias Schacht [32], the following generalization of this result may be obtained through a slight modification of Cooper's proof.

**Theorem 5.2.** Let $F$ be an arbitrary family of permutations. Given $\eta > 0$, there exist positive constants $n_0$, $M$, and $c_0$ with the following property. Let $\sigma$ be a permutation on $[n]$, $n \geq n_0$, such that every set $S \subseteq \binom{[n]}{2}$ that removes all copies of every permutation in $F$ has cardinality at least $n^2$. Then there exists $\varepsilon \in F$ for which $|\varepsilon| = m \leq M$ and $\Lambda(\varepsilon, \sigma) \geq c_0 n^m$.

Alternatively, this question could be addressed with our theory in the following way. Let $\sigma$ and $\pi$ be permutations on $[n]$. We say that a set $S \subseteq \binom{[n]}{2}$ removes all distinct superpermutations of $\sigma$ and $\pi$ if, whenever a set of indices $a_1 < \cdots < a_k \in [n]$ induces different superpermutations of $\sigma$ and $\pi$, there are indices $i < j \in [k]$ with $(a_i, a_j) \in S$. It is easy to see that, for a set $S$ to remove all distinct superpermutations of $\sigma$ and $\pi$, it suffices that it removes all their relative inversions, that is, all of their distinct superpermutations of length two. Based on this, we may define the edit distance $d_1 : S_n^2 \to [0, 1]$ by

$$d_1(\sigma, \pi) = \frac{1}{\binom{n}{2}} \min \{|S| : S \text{ removes all relative inversions of } \sigma \text{ and } \pi\}.$$

It is a fact that this distance coincides with Kendall’s tau distance introduced in Section 3. Moreover, in the sense of Cooper [17], this distance is the permutation analogue of the edit distance for graphs, and therefore inherits the name.

As with the rectangular distance, we may define the edit distance between a permutation $\sigma$ on $[n]$ and a property $P$ by

$$d_1(\sigma, P) = \min \{d_1(\sigma, \pi) : \pi : [n] \to [n] \text{ satisfies } P\}.$$  

We conjecture that, if $P$ is a hereditary property, then the function $f_P$ mapping each permutation $\sigma$ to the real number $d_1(\sigma, P)$ is continuous with respect to the distance $d_\infty$. More precisely, we have the following.

**Conjecture 5.3.** Let $P$ be a hereditary permutation property. Then, for every $\eta > 0$, there exists $\delta > 0$ such that any permutation $\sigma$ satisfying $d_\infty(\sigma, P) < \delta$ also satisfies $d_1(\sigma, P) < \eta$.

It is easy to see that this conjecture is true when the roles of $d_\infty$ and $d_1$ are interchanged. Furthermore, it is known that the fact that $d_\infty(\sigma_n, \pi_n)$ is small does not imply $d_1(\sigma_n, \pi_n)$ to be small. Indeed, if for every $n$ the permutations $\sigma_n$ and $\pi_n$ are independent and uniformly distributed on $S_n$, then $\lim_{n \to \infty} d_\infty(\sigma_n, \pi_n) = 0$, but

$$\lim_{n \to \infty} d_1(\sigma_n, \pi_n) = \lim_{n \to \infty} d_k(\sigma_n, \pi_n) = \lim_{n \to \infty} d_k(\sigma_n \pi_n^{-1}, e) = \lim_{n \to \infty} t((2, 1), \sigma_n \pi_n^{-1}) = \frac{1}{2}.$$

The validity of Corollary 5.3 would easily lead to an alternative proof of Theorem 5.2. To see why this is the case, let $F$ be a family of permutations, fix $\eta > 0$ and suppose for a contradiction that the statement of Theorem 5.2 is false. Thus there is a sequence $(\sigma_\ell)$ with $|\sigma_\ell| = n_\ell \to \infty$ for which $d_1(\sigma_\ell, P) \geq \eta$, but $\Lambda(\tau, \sigma_\ell) < \frac{1}{n_\ell^m}$ for every permutation $\tau \in F$ such that $m = |\tau| < \ell$. However, Cooper [17] implies that $d_\infty(\sigma_\ell, P) \geq \delta$ for every $\sigma_\ell$, which contradicts Lemma 4.1 with $\eta_{4.1} = \delta$.

It should be pointed out that the analogue of Corollary 5.3 is true for graphs when $d_\infty$ and $d_1$ are replaced by the rectangular and the edit distances for graphs, respectively, as established by Lovász and Szegedy [29]. Furthermore, the correctness of Corollary 5.3 would also imply that Theorem 1.6 holds for the notion of testability where $d_\infty$ is replaced by $d_1$ in (1), establishing the $\ell_1$-permutation counterpart of Alon and Shapira's heredity result [5]. This would further unify the theory developed for permutations with the graph case, as “$\varepsilon$-farness” in the definition of testable graph properties is given in terms of the edit distance.
5.3. Finite forcibility

Another topic of interest in the graph context are the so-called finitely forcible properties (see the work of Lovász and Szegedy [28]). In the study of finite forcibility, one wishes to characterize families of graph properties that are determined if we prescribe the homomorphism density of a finite number of subgraphs. This can be easily generalized to permutations: a family of permutations \( A \) forces the permutation property \( P \) if there exists a function \( c : A \to [0, 1] \) for which any large permutation whose subpermutation densities in \( A \) are close to the values prescribed by \( c \) must also be close to satisfying \( P \). More precisely, for every \( \varepsilon > 0 \), there exist positive constants \( n_0 \) and \( \delta \) with the following property. If \( \sigma \) is a permutation on \( [n] \), \( n \geq n_0 \), and \( \| t(\tau, \sigma) - c(\tau) \| < \delta \) for every \( \tau \in A \), then \( \sigma \) satisfies \( d_n(\sigma, P) < \varepsilon \). A permutation property \( P \) is then finitely forcible if there exists a finite family of permutations forcing \( P \).

Analogously, given a permutation parameter \( f \), one could ask whether there exists a finite family of permutations \( A \) for which the value of \( f(\sigma) \) can be arbitrarily well approximated whenever the value of \( t(\tau, \sigma) \) is known for every \( \tau \in A \). To this end, we say that a permutation parameter \( f \) is \( A \)-forcible if there is a function \( f^* : \mathbb{N} \times [0, 1]^A \to \mathbb{R} \) satisfying the following. Given \( \varepsilon > 0 \), there exist a positive integer \( n_0 \) such that, for a permutation \( \sigma \) on \( [n] \), \( n \geq n_0 \), the parameter \( f \) satisfies \( | f(\sigma) - f^*(\sigma, (t(\tau, \sigma))_{\tau \in A}) | < \varepsilon \).

If we make the additional requirement that \( f^* \) is a continuous function, then it is not hard to see that every finitely forcible permutation parameter is testable. Perhaps more surprisingly, one may show that every testable parameter is not far from being finitely forcible. To make this precise, we say that a permutation parameter \( f \) can be \((\varepsilon, \delta, n_0)\)-finitely approximated if there exists a finite family of permutations \( A \) such that, if \( \sigma \) and \( \pi \) are permutations of length at least \( n_0 \) satisfying \( \| t(\tau, \sigma) - t(\tau, \pi) \| < \delta \) for every \( \tau \in A \), then \( | f(\sigma) - f(\pi) | < \varepsilon \). In other words, the value of the parameter \( f \) is highly dependent on the subpermutation densities of the permutations in \( A \). Moreover, a permutation parameter is finitely approximable if, for every \( \varepsilon > 0 \), there exist positive constants \( n_0 \) and \( \delta \) such that \( f \) can be \((\varepsilon, \delta, n_0)\)-finitely approximated.

**Proposition 5.4.** A bounded permutation parameter \( f \) is testable if and only if it is finitely approximable.

**Proof.** We first show that every bounded permutation parameter that is testable can be \((\varepsilon, \delta, n_0)\)-finitely approximated for an arbitrary \( \varepsilon > 0 \) and conveniently chosen \( n_0 = n_0(\varepsilon) \) and \( \delta = \delta(\varepsilon) > 0 \). To this end, let \( f \) be a bounded testable permutation parameter, and suppose for a contradiction that it is not finitely approximable. Let \( \varepsilon > 0 \) be such that, for every positive integer \( \ell \), with \( n_{0, \ell} = \ell, \delta_\ell = 1/\ell \) and \( A_\ell \) being the family of all permutations with length at most \( \ell \), there exist permutations \( \sigma_{\ell} \) and \( \pi_{\ell} \) of length at least \( \ell \) such that \( \| t(\tau, \sigma_{\ell}) - t(\tau, \pi_{\ell}) \| < \delta_\ell \) for every \( \tau \in A_\ell \), but \( | f(\sigma_{\ell}) - f(\pi_{\ell}) | > \varepsilon \). As before, we may use a diagonalization argument to find a convergent subsequence \( (\sigma_{\ell_i}) \) of \( (\sigma_{\ell}) \). It is not hard to see that the sequence \( (\mu_{\ell_i}) \) with \( \mu_{2\ell_i − 1} = \sigma_{\ell_i} \) and \( \mu_{2\ell_i} = \pi_{\ell_i} \) is also convergent. However, the sequence \( (f(\mu_{\ell_i})) \) cannot converge, as we have \( | f(\mu_{\ell_i}) - f(\mu_{\ell_i+1}) | > \varepsilon \) whenever \( \ell \) is odd. This leads to a contradiction, as Theorem 1.3 implies that \( f \) is not testable.

Conversely, suppose that \( f \) is finitely approximable, fix \( \varepsilon > 0 \) and let \( n_0, \delta > 0 \) and \( A \) be such that \( f \) can be \((\varepsilon, \delta, n_0)\)-finitely approximated by \( A \). Because \( A \) is finite, given a convergent permutation sequence \( (\sigma_n) \) with \( \lim_{n \to \infty} |\sigma_n| = \infty \), we may fix \( n_{0, \ell} > 0 \) with the following two properties: (i) \( |\sigma_n| > n_{0, \ell} \) for every \( n \geq n_{0, \ell} \); (ii) \( | t(\tau, \sigma_n) - t(\tau, \sigma_m) | < \delta \) for every \( n, m \geq n_{0, \ell} \) and every \( \tau \in A \). Therefore we must have \( | f(\sigma_n) - f(\sigma_m) | < \varepsilon \) for every \( n, m \geq n_{0, \ell} \), so that \( (f(\sigma_n)) \) is a Cauchy sequence of real numbers, and hence is convergent. As \( (\sigma_n) \) is arbitrary, Theorem 1.3 implies that \( f \) is testable.

Several examples of bounded testable permutation parameters presented in this paper are known to be finitely forcible, exception made to the parameters of the form \( f_j = d_{\infty}(\cdot, Z) \). Hence a natural question seems to be whether finite forcibility holds for every bounded permutation parameter.

**Question 5.5.** Find a bounded testable permutation parameter \( f \) that is not finitely forcible, or prove that there is no such parameter.

This question is related with a question of R. L. Graham (see [16]). For every positive integer \( j \), let \( \sigma_j \) be a permutation on \( [n_j] \). We say that the sequence \( (\sigma_j) \) satisfies the property of asymptotic \( k \)-symmetry \( P(k) \) if, for every \( \varepsilon > 0 \), there exists \( j_0 \) such that, for every \( j \geq j_0 \) and every permutation \( \tau \) on \( [k] \),

\[
| t(\tau, \sigma_j) - \frac{1}{k!} | < \varepsilon.
\]

Graham conjectures that there is no positive integer \( N \) with the property that \( P(k) \) implies \( P(k+1) \) for all \( k \geq N \). The existence of such a number \( N \) would imply that one could determine whether a permutation sequence is quasirandom through the density of subpermutations of size \( N \). It is known that \( P(1) \not\Rightarrow P(2) \), \( P(2) \not\Rightarrow P(3) \) and \( P(3) \not\Rightarrow P(4) \). It is also a fact that \( P(k) \Rightarrow P(k+1) \) for all \( k > 1 \). In contrast to this conjecture, a well-known graph-theoretical result due to Chung, Graham and Wilson [14] establishes that, if we let \( G(k) \) be the property that all graphs on \( k \) vertices occur as induced subgraphs at approximately the same rate, then we have

\[
G(1) \Leftarrow G(2) \Leftarrow G(3) \Leftarrow G(4) \Leftarrow G(5) \Leftarrow \cdots
\]

In other words, property \( G(4) \) implies quasirandomness, which in turn implies \( G(k) \) for all \( k \).
Now, suppose that one could answer Question 5.5 in a way that every bounded testable permutation parameter is finitely forcible. In particular, the normalized discrepancy $D_1$ is finitely forcible, so that there exists a finite family of permutations $\mathcal{A}$ with the following property: to guarantee that a permutation $\sigma$ on $[n]$ satisfies $D_1(\sigma) < \varepsilon$ for some $\varepsilon > 0$, it suffices to verify that $|t(\tau, \sigma) - \frac{1}{n}| < \delta$ for every $\tau \in \mathcal{A}$, where $\delta$ is determined by $\varepsilon$. One of the equivalent ways to characterize a quasirandom permutation sequence $(\sigma_n)$ is through the property $\lim_{n \to \infty} D_1(\sigma_n) = 0$, as established in [16]. Hence, by setting $k = \max(|\sigma| : \sigma \in \mathcal{A})$, this would imply that a sequence satisfying property $P(k)$ is quasirandom, and therefore satisfies property $P(j)$ for every $j \geq k$, contradicting Graham’s conjecture.

On the other hand, the fact that $D_1$ is a bounded testable parameter implies that it is finitely approximable by Proposition 5.4. As a consequence, if an “error” $\varepsilon > 0$ is given in advance, it is true that there exist positive constants $N$ and $\delta$ such that $\limsup_{n \to \infty} D_1(\sigma_n) < \varepsilon$ whenever $|t(\tau, \sigma_n) - \frac{1}{n}| < \delta$ holds for every permutation $\tau$ on $[n]$ if $n$ is sufficiently large.

To end the paper, we observe that, in the language of limit permutations, proving Graham’s conjecture is equivalent to the following statement.

**Conjecture 5.6** (Graham’s Conjecture). For every $k \in \mathbb{N}$ there is a limit permutation $Z_k$ such that, for all $\tau \in S_k$, we have $t(\tau, Z_k) = 1/k!$, but $d_\infty(Z_k, U) > 0$, where $U(x, y) = y$ is the uniform limit permutation.

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