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A duality between small-face problems in arrangements of lines and Heilbronn-type problems ☆

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Abstract

Arrangements of lines in the plane and algorithms for computing extreme features of arrangements are a major topic in computational geometry. Theoretical bounds on the size of these features are also of great interest. Heilbronn's triangle problem is one of the famous problems in discrete geometry. In this paper we show a duality between extreme (small) face problems in line arrangements (bounded in the unit square) and Heilbronn-type problems. We obtain lower and upper combinatorial bounds (some are tight) for some of these problems. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Heilbronn's triangle problem; Line arrangements; Duality

1. Introduction

The investigation of arrangements of lines in the plane has attracted much attention in the literature. In particular, certain extremal features of such arrangements are of great interest. Using standard duality between lines and points, such arrangements can be mapped into sets of points in the plane, which have also been studied intensively. In this dual setting, distributions in which certain features (defined by triples of points) assume their maxima are often sought. In this paper we show a connection between these two classes of problems and summarize the known bounds for some extremal-feature problems.

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Let $\mathscr{A}(\mathscr{L})$ be an arrangement of a set $\mathscr{L}$ of n lines. We assume the lines of $\mathscr{A}$ to be in general position, in the sense that no two lines have the same slope. Thus every triple of lines define a triangle. Let $U=[0,1]^2$ be the unit square. An arrangement $\mathscr{A}$ is called *narrow* if all its lines intersect the two vertical sides of U. A narrow arrangement $\mathscr{A}$ is called *transposed* if the lines of $\mathscr{A}$ intersect the vertical sides of U in two sequences that, when sorted in increasing y order, are the reverse of each other. Clearly, every transposed arrangement is also narrow. Note that *all* the vertices of a transposed arrangement lie in U. Later in the paper we will also define the set of *convex* arrangements which is a proper subset of the set of transposed arrangements.

In this paper we investigate the 'size' of triangles defined by the lines of narrow arrangements, according to several measures of the size of a triangle. We consider the arrangements in which the minimum size of a triangle assumes its maximum, and attempt to bound this value.

Heilbronn's triangle problem is the following.

> Let $\{P_1, P_2, \ldots, P_n\}$ be a set of n points in U, such that the minimum of the areas of the triangles $P_iP_jP_k$ (for $1 \leqslant i < j < k \leqslant n$) assumes its maximum possible value $\mathscr{G}_0(n)$. Estimate $\mathscr{G}_0(n)$.

Heilbronn conjectured that $\mathscr{G}_0(n)=\mathrm{O}(1/n^2)$. The first nontrivial upper bound (better than $\mathrm{O}(1/n)$), namely, $\mathrm{O}(1/(n\sqrt{\log\log n}))$, was given by Roth [6]. Schmidt [9] improved this result 20 years later and obtained $\mathscr{G}_0(n)=\mathrm{O}(1/n\sqrt{\log n})$. Soon after that Roth improved the upper bound twice to $\mathrm{O}(1/n^{1.105\ldots})$ [7] and $\mathrm{O}(1/n^{1.117\ldots})$ [8].[1] The best known upper bound, $\mathscr{G}_0(n)=\mathrm{O}(1/n^{8/7-\varepsilon})=\mathrm{O}(1/n^{1.142\ldots})$ (for any $\varepsilon>0$), is due to Komlós et al. [3] by a further refinement of the method of [7,8]. A simple probabilistic argument by Alon et al. [1, p. 30] proves a lower bound of $\Omega(1/n^2)$. Erdős [6, appendix] showed the same lower bound by an example. However, Komlós et al. [4] show by a rather involved probabilistic construction that $\mathscr{G}_0(n)=\Omega(\log n/n^2)$. In a companion paper [2], we show a lower bound for the generalization of Heilbronn's triangle problem to higher dimensions.

In this paper we consider some variants of Heilbronn's problem, in which other measures of triangles defined by triples of points are considered, and/or some restrictions are imposed on the locations of the points in U. Specifically, in a *monotone decreasing* distribution of points, the x-coordinates and the y-coordinates of the points appear in opposite permutations, and a *convex monotone decreasing* distribution is a monotone decreasing distribution in which the points form a convex (or concave) chain according to the respective permutation.

The paper is organized as follows. In Section 2 we give a precise definition of the two classes of problems and the duality between them. In Sections 3, 4 and 5 we obtain bounds for these problems. We summarize in Section 6 the known bounds for all problems.

[1] The exponents of n in these bounds are the smaller roots $\mu=(17-\sqrt{65})/8$ and $v=2-\sqrt{0.8}$ of the equations $4\mu^2-17\mu+14=0$ and $5v^2-20v+16=0$, respectively.

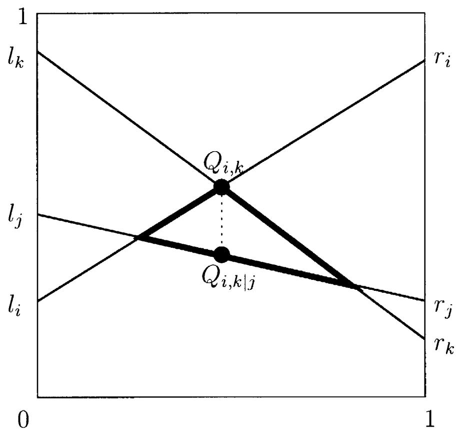

Fig. 1. Vertical distance.

2. The duality and the problems

Let $\mathscr{A}$ be a narrow arrangement of n lines. We define two measures of triangles. Let $F_1(\tau)$ be the vertical height of a triangle τ, that is, the maximum length of a vertical segment contained in τ. Let $F_2(\tau)$ be the area of τ. Our goal is to bound $\mathscr{F}_1(n) = \max_{\mathscr{A}} \min_{\tau \in \mathscr{A}} F_1(\tau)$ and $\mathscr{F}_2(n) = \max_{\mathscr{A}} \min_{\tau \in \mathscr{A}} F_2(\tau)$, where the maximum is taken over all narrow arrangements $\mathscr{A}$ of n lines, and where the minimum is taken over all triangles τ formed by triples of lines in $\mathscr{A}$. We use the superscripts '(mon)' and '(conv)' for denoting the monotone decreasing and the convex monotone decreasing variants, respectively.

Denote by l_i (resp., r_i) the y-coordinate of the intersection point of the line $\ell_i \in \mathscr{L}$ with the left (resp., right) vertical side of U. We dualize the line ℓ_i to the point $P_i = (l_i, r_i)$. This dualization maps the measures F_1 and F_2 of lines into measures of triples of points in U. The two optimization problems are mapped into generalizations of Heilbronn's problem, where the difference is in the definition of the measure of a triple of points. Note that the dual of a transposed arrangement of lines is a monotone *decreasing* set of points. The measures G_1 and G_2 (defined below) crucially rely on the *decreasing* monotonicity of the points. A 'convex' arrangement is an arrangement whose dual set of points lies in convex position.

It is easy to see that the vertical height of a triangle is the minimum length of a vertical segment that connects a vertex of the triangle to the line supporting the opposite edge. This follows from the convexity of a triangle, and indeed that segment always lies inside the triangle. We now specify (in terms of the dual representation) the vertical distance between the intersection of two lines to a third line of $\mathscr{A}$ (in the primal representation). Refer to Fig. 1. The equation of ℓ_i is $y = (r_i - l_i)x + l_i$. We compute the distance between $Q_{i,k} = (x_{i,k}, y_{i,k})$, the intersection point of ℓ_i and ℓ_k, and $Q_{i,k|j} = (x_{i,k}, y_{i,k|j})$, the vertical projection of $Q_{i,k}$ on ℓ_j. A simple calculation shows that

$$Q_{i,k} = \left(\frac{l_k - l_i}{(l_k - l_i) - (r_k - r_i)}, \frac{l_k r_i - l_i r_k}{(l_k - l_i) - (r_k - r_i)} \right).$$

By substituting $x_{i,k}$ in the equation of ℓ_j we find that $y_{i,k|j} = [r_j(l_k - l_i) - l_j (r_k - r_i)]/[(l_k - l_i) - (r_k - r_i)]$. Finally,

$$\mathrm{Dist}(Q_{i,k}, Q_{i,k|j}) = |y_{i,k} - y_{i,k|j}| = \left| \frac{r_i(l_k - l_j) - r_j(l_k - l_i) + r_k(l_j - l_i)}{(l_k - l_i) + (r_i - r_k)} \right|$$

$$= 4\,\mathrm{abs}\left(\frac{\frac{1}{2}\begin{vmatrix} l_i & r_i & 1 \\ l_j & r_j & 1 \\ l_k & r_k & 1 \end{vmatrix}}{2((l_k - l_i) + (r_i - r_k))} \right).$$

The numerator of the last term is the area of the triangle defined by the points P_i, P_j, and P_k. In case P_i and P_k are in monotone decreasing position, the denominator is the perimeter of the axis-aligned box defined by P_i and P_k. When setting $G_1(P_i, P_j, P_k) = \mathrm{Dist}(Q_{i,k}, Q_{i,k|j})$, maximizing the smallest value of F_1 over the triangles defined by triples of lines of $\mathscr{A}$ dualizes to maximizing the smallest value of G_1 over triples of points in U. We denote the asymptotic value by $\mathscr{G}_1(n) = \mathscr{F}_1(n)$.

We now compute (in the dual representation) the area of the triangle defined by three lines of $\mathscr{A}$ (in the primal representation). Recall that

$$Q_{i,j} = (x_{i,j}, y_{i,j}) = \left(\frac{l_j - l_i}{(l_j - l_i) - (r_j - r_i)}, \frac{l_j r_i - l_i r_j}{(l_j - l_i) - (r_j - r_i)} \right).$$

Tedious computation (mainly performed by Mathematica [10], see Appendix A) shows that

$$\mathrm{Area}(Q_{i,j}, Q_{j,k}, Q_{k,i})$$

$$= \frac{1}{2}\mathrm{abs} \begin{vmatrix} x_{i,j} & y_{i,j} & 1 \\ x_{j,k} & y_{j,k} & 1 \\ x_{k,i} & y_{k,i} & 1 \end{vmatrix}$$

$$= \frac{1}{2}\,\mathrm{abs}\,(x_{j,k} y_{k,i} - y_{j,k} x_{k,i} - x_{i,j} y_{k,i} + y_{i,j} x_{k,i} + x_{i,j} y_{j,k} - y_{i,j} x_{j,k})$$

$$= \frac{1}{2}\,\mathrm{abs}\left(\frac{(l_i r_j - l_j r_i + l_k r_i - l_i r_k + l_j r_k - l_k r_j)^2}{((l_j - l_i) + (r_i - r_j))((l_k - l_i) + (r_i - r_k))((l_k - l_j) + (r_j - r_k))} \right)$$

$$= 16\,\mathrm{abs}\left(\frac{\left(\frac{1}{2}\begin{vmatrix} l_i & r_i & 1 \\ l_j & r_j & 1 \\ l_k & r_k & 1 \end{vmatrix} \right)^2}{2((l_j - l_i) + (r_i - r_j)) \cdot 2((l_k - l_i) + (r_i - r_k)) \cdot 2((l_k - l_j) + (r_j - r_k))} \right).$$

The numerator of the last term is the square of the area of the triangle defined by the points P_i, P_j, and P_k. In case P_i, P_j, and P_k are in monotone decreasing position, the denominator is the product of the perimeters of the axis-aligned boxes defined by P_i, P_j, and P_k. When setting $G_2(P_i,P_j,P_k)=\text{Area}\ (Q_{i,j},Q_{j,k},Q_{k,i})$, maximizing the smallest value of F_2 over the triangles defined by triples of lines of $\mathscr{A}$ dualizes to maximizing the smallest value of G_2 over triples of points in U. We denote the asymptotic value by $\mathscr{G}_2(n)=\mathscr{F}_2(n)$.

In summary, we define a duality between narrow arrangements of lines and distributions of points in the unit square. We define measures of triangles defined by triples of lines of the arrangements, and find their dual measures of triples of points. We consider some special structures of the arrangements and their dual distributions of points. In all cases, we look for the arrangement of lines (or distribution of points) that maximizes the minimum value of the measure over all triples of lines (or points), and attempt to asymptotically bound this value.

3. Heilbronn's triangle problem

As noted in the introduction, the best known lower bound for the original Heilbronn's triangle problem, $\mathscr{G}_0(n)=\Omega(\log n/n^2)$, is obtained by Komlós et al. [4]. The same authors obtain in [3] the best known upper bound, $O(1/n^{1.142\ldots})$.

Unlike the measures G_1 and G_2, it is not essential to require the set to be *decreasing* in the monotone case. Here we have only an upper bound:

Theorem 1. $\mathscr{G}_0^{(\mathrm{mon})}(n)=O(1/n^2)$.

Proof. Refer to the sequence of segments that connect pairs of consecutive points. The total length of the sequence is at most 2. Cover the whole sequence of segments by at most $n/3$ squares with side $6/n$. Thus at least one square contains three points. The area of the triangle that they define is at most $18/n^2$. □

Note that applying this argument to the original Heilbronn's triangle problem (by covering the minimum Euclidean spanning tree of the points by squares) leads to a weak upper bound $\mathscr{G}_0(n)=O(1/n)$. This bound is obtained by covering the Euclidean minimum spanning tree of the points, whose length is $O(\sqrt{n})$,[2] by $O(n)$ squares with side $O(1/\sqrt{n})$.

For the convex case the bound is tight.

Theorem 2. $\mathscr{G}_0^{(\mathrm{conv})}(n)=\Theta(1/n^3)$.

[2] To see this, construct $\sqrt{n}$ vertical segments of height 1 and $1/\sqrt{n}$ apart, and connect them with one horizontal segment of length 1. Now, connect each point by a horizontal segment to the nearest vertical segment. The total length of this tree is at most $3\sqrt{n}/2+1$.

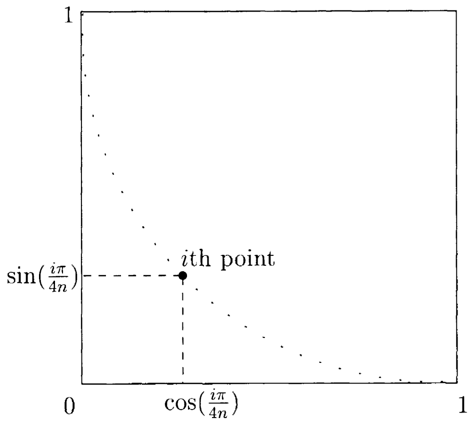

Fig. 2. Uniform distribution of points on an arc.

Proof. The lower bound is obtained by a simple example: Put n points equally spaced on an arc of radius 1 and of length $\pi/2$ (see Fig. 2). The points are ordered from bottom-right to top-left. The coordinates of the ith point are $(\cos(2i\delta), \sin(2i\delta))$, for $1 \leqslant i \leqslant n$, where $\delta = \pi/(4n)$. The area of the triangle defined by every three consecutive points is $4\cos(\delta)\sin^3(\delta) = \Theta(1/n^3)$.

It is also easy to prove the upper bound. Refer to the sequence of segments that connect pairs of consecutive points. Drop all the segments of length greater than $8/n$. Since the maximum length of such a convex and monotone chain is 2, less than $n/4$ segments are dropped. Now drop all pairs of consecutive segments whose external angle is greater than $4\pi/n$. Since the maximum turning angle of such a convex chain is $\pi/2$, less than $n/8$ pairs are dropped. That is, less than $n/4$ segments are now dropped. In total, we have dropped less than $n/2$ segments; therefore, two consecutive segments have not been dropped. The area of the triangle which these two segments define is upper bounded by $\frac{1}{2}(8/n)^2 \sin(4\pi/n) = \Theta(1/n^3)$. □

Note that the 'dropped-segments' argument also shows that $\mathscr{G}_0^{(\mathrm{mon})}(n) = O(1/n^2)$, as is proven in Theorem 1.

4. Vertical height

Theorem 3. $\mathscr{G}_1(n) = \Theta(1/n)$.

Proof. The lower bound is shown by an example.[3] Set $l_i = i/n - 1/n^i$ and $r_i = i/n$, for $1 \leqslant i \leqslant n$ (a set of almost-parallel lines). In this example

$$G_1(P_i, P_j, P_k) = \left| \frac{(j-i)n^{i+j} - (k-i)n^{i+k} + (k-j)n^{j+k}}{n^{j+1}(n^k - n^i)} \right|,$$

[3] Note that this construction is monotone and even convex, and yet it beats the $O(1/n^2)$ upper bound of $\mathscr{G}_1^{(\mathrm{conv})}(n)$. This is because it is an *increasing* construction.

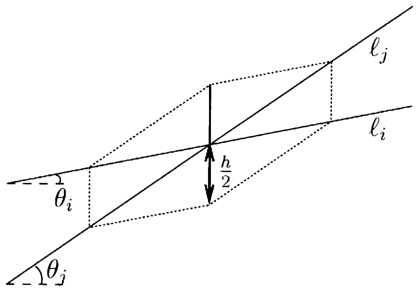

Fig. 3. Hexagonal forbidden zones induced by intersection of lines.

which is minimized when i, j, and k are consecutive integers. It is easy to verify that in this example, for every i, $G_1(P_i, P_{i+1}, P_{i+2}) = [n-1]/[n(n+1)]$. Hence $\mathscr{G}_1(n) = \Omega(1/n)$.

For the upper bound, we return to the primal representation of the problem by an arrangement $\mathscr{A}$ of n lines. Since $\mathscr{A}$ is narrow, there exists a vertical segment of length 1 which stabs all the n lines. Hence, there exists a triple of lines which are stabbed by a vertical segment of length at most $2/(n-1)$. Such a triple cannot define a triangle whose vertical height exceeds $2/(n-1)$, therefore $\mathscr{F}_1(n) = O(1/n)$. $\square$

We now refer to transposed arrangements of lines (the dual of monotone decreasing Heilbronn's sets of points). The best upper bound of which we are aware is Mitchell's [5]. Here is a simplified version of the proof of this bound (the cited reference uses more complex 'forbidden zones' and does not provide proof of Lemma 4). Let θ_i denote the slope of ℓ_i (for $1 \leqslant i \leqslant n$). Assume, without loss of generality, that at least $n/2$ lines of $\mathscr{L}$ have positive slopes. For the asymptotic analysis, we may assume that all the lines of $\mathscr{L}$ are ascending. Denote by h the minimum vertical height of a triangle in $\mathscr{A}(\mathscr{L})$. Then each pair of lines $\ell_i, \ell_j \in \mathscr{L}$ induces a hexagon of area $[3h^2\cos(\theta_i)\cos(\theta_j)]/[4\sin(\theta_j - \theta_i)]$ (see Fig. 3) through which no other line of $\mathscr{L}$ can pass. This is since such a line would form with ℓ_i and ℓ_j a triangle whose vertical height is at most $h/2$. The intersection of every pair of such hexagons is empty, for otherwise there would be a triangle whose vertical height is less than h. Denote by S the total area of the $\binom{n}{2}$ hexagons. On one hand,

$$S = \sum_{1 \leqslant i < j \leqslant n} \frac{3h^2\cos(\theta_i)\cos(\theta_j)}{4\sin(\theta_j - \theta_i)} \geqslant \frac{3h^2}{8} \sum_{1 \leqslant i < j \leqslant n} \frac{1}{\sin(\theta_j - \theta_i)}$$

$$\geqslant \frac{3h^2}{8} \sum_{1 \leqslant i < j \leqslant n} \frac{1}{\theta_j - \theta_i},$$

where we use the facts that $0 < \theta_i \leqslant \pi/4$ for $1 \leqslant i \leqslant n$ and $\sin(\theta) < \theta$.

Lemma 4. $\sum_{1\leqslant i<j\leqslant n} \frac{1}{\theta_j-\theta_i} = \Omega(n^2 \log n)$.

Proof. Let $a_i = \theta_{i+1} - \theta_i$. We have $0 < a_i < \pi/4$ (for $1 \leqslant i \leqslant n-1$). Thus,

$$\sum_{1\leqslant i<j\leqslant n} \frac{1}{\theta_j-\theta_i} = \sum_{1\leqslant i<j\leqslant n} \frac{1}{\sum_{k=i}^{j-1} a_k}.$$

We reorder the outer summation (and omit some summands) such that we have one summand with $(n-1)$ terms $(1/a_1+1/a_2+\cdots+1/a_{n-1})$, two summands with $\lfloor (n-1)/2 \rfloor$ terms (e.g., $(1/(a_1+a_2)+1/(a_3+a_4)+\cdots)$), and up to $\lfloor n/2 \rfloor$ summands with two terms $\left(\text{e.g., } \left(\frac{1}{a_1+\cdots+a_{\lfloor n/2 \rfloor}} + \frac{1}{a_{\lfloor n/2 \rfloor+1}+\cdots+a_{2\lfloor n/2 \rfloor}}\right)\right)$. Now, we minimize each summand separately by using the averages inequality and conclude that a summand with s terms is at least $4(n-1)^2/\pi s^2$. Hence

$$\sum_{1\leqslant i<j\leqslant n} \frac{1}{\sum_{k=i}^{j-1} a_k} > \frac{4}{\pi} \sum_{s=1}^{\lfloor n/2 \rfloor} \frac{(n-1)^2}{s} = \Theta(n^2 \log n). \qquad \square$$

We thus have $S = \Omega(h^2 n^2 \log n)$. On the other hand, the total area of all the hexagons along one line of $\mathscr{L}$ cannot exceed h (the area of a strip of vertical height h clipped to the unit square). By summing up for all the lines of $\mathscr{L}$ we obtain $2S \leqslant hn$. The combination of the two inequalities implies that $h = O(1/(n \log n))$. (Note that the 'dropped-segments' argument gives in this case a weaker upper bound of $O(1/n)$.) Thus we have the following:

Theorem 5. $\mathscr{G}_1^{(\mathrm{mon})}(n) = O(1/n \log n)$.

Finally, we consider convex arrangements of lines (the dual of convex monotone decreasing Heilbronn's sets). As with the convex case of the original Heilbronn's triangle problem, the bound in this case is tight.

Theorem 6. $\mathscr{G}_1^{(\mathrm{conv})}(n) = \Theta(1/n^2)$.

Proof. For the lower bound, we use the same example (points on an arc) as in the proof of Theorem 2. Clearly, the perimeter of the bounding box of every triangle defined by three consecutive points in this example is $\Theta(1/n)$. Since the area of each such triangle is $\Theta(1/n^3)$ we obtain the lower bound. For the upper bound, we follow again the argument of dropping segments. The perimeter of the bounding box of the triangle defined by the two remaining consecutive segments is linear in the length s of the longer segment. The quotient of the area of the triangle and s is upper bounded by $\frac{1}{2}(8/n)\sin(2\pi/n) = \Theta(\frac{1}{n^2})$. $\square$

Table 1 shows a few examples which show the lower bound $\mathscr{G}_1^{(\mathrm{conv})}(n) = \Omega(1/n^2)$.

Table 1
Convex line arrangements which show that $\mathscr{G}_1^{(\mathrm{conv})}(n) = \Omega(1/n^2)$

l_i	r_i	$G_1(P_i,P_j,P_k)$
$\frac{i}{n}$	$1-\left(\frac{i}{n}\right)^2$	$\left\lvert \frac{(i-j)(k-j)}{n(i+k+n)} \right\rvert$
$\frac{i}{n}$	$1-\sqrt{\frac{i}{n}}$	$\left\lvert \frac{(\sqrt{j}-\sqrt{i})(\sqrt{k}-\sqrt{j})}{(\sqrt{i}+\sqrt{k}+\sqrt{n})\sqrt{n}} \right\rvert$
$\sin\left(\frac{i\pi}{n}\right)$	$\cos\left(\frac{i\pi}{n}\right)$	$\left\lvert \frac{\sin\left(\frac{(i-j)\pi}{2n}\right)-\sin\left(\frac{(i-k)\pi}{2n}\right)+\sin\left(\frac{(i-k)\pi}{2n}\right)}{\cos\left(\frac{i\pi}{2n}\right)-\cos\left(\frac{k\pi}{2n}\right)-\sin\left(\frac{i\pi}{2n}\right)+\sin\left(\frac{k\pi}{2n}\right)} \right\rvert$

5. Area

The function $\mathscr{G}_2(n)$ is as high as we like.

Theorem 7. *$\mathscr{G}_2(n) = \Omega(f(n))$ for any $f(n)$.*

Proof. We use the construction $l_i = i/n - 1/n^{Mi}$ and $r_i = i/n$ (for $1 \leqslant i \leqslant n$ and for an arbitrarily large M). In this example,

$$G_2(P_i,P_j,P_k) = \left\lvert \frac{[(j-i)n^{M(i+j)} - (k-i)n^{M(i+k)} + (k-j)n^{M(j+k)}]^2}{2n^2(n^{Mj}-n^{Mi})(n^{Mk}-n^{Mi})(n^{Mk}-n^{Mj})} \right\rvert,$$

which is easily verified to be minimized by $i=1$, $j=2$, and $k=3$. In this example, $G_2(P_1,P_2,P_3) = [n^{2(M-1)}(n^M-1)]/(2n^M+1)$. Set $M = [\log f(n)]/(2\log n)+1$, and the claim follows. □

We now refer to transposed arrangements of lines (the dual of monotone decreasing Heilbronn's sets of points).

Theorem 8. $\mathscr{G}_2^{(\mathrm{mon})}(n) = \mathrm{O}(1/(n\sqrt{\log n}))$.

Proof. We follow an argument similar to that used in the proof of Theorem 5. Let again θ_i denote the slope of ℓ_i (for $1 \leqslant i \leqslant n$), and assume without loss of generality that all the lines of $\mathscr{L}$ are ascending. Denote by A the minimum area of a triangle in $\mathscr{A}(\mathscr{L})$. Here the 'forbidden zone' induced by each pair of lines $\ell_i, \ell_j \in \mathscr{L}$ is a rectangle whose diagonal is of length $2A/[\sin(\theta_j-\theta_i)]$ (refer to Fig. 4). Indeed, if any other line of $\mathscr{L}$ passed through this rectangle, then together with ℓ_i and ℓ_j it would form a triangle whose area is less then $A/2$. The intersection of every pair of such rectangles is empty, for otherwise there would be a triangle whose area is less than A. The area of the forbidden rectangle is $2A^2/[\sin(\theta_j-\theta_i)]$. Let again S denote the total

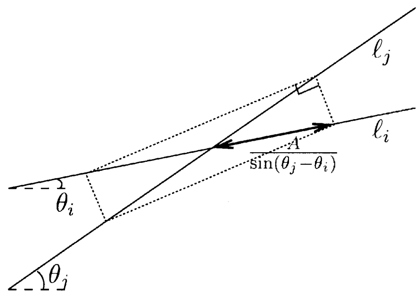

Fig. 4. Forbidden zones for $\mathscr{G}_2^{(\mathrm{mon})}$.

area of the $\binom{n}{2}$ forbidden zones. Here, we have

$$S = \sum_{1 \leqslant i < j \leqslant n} \frac{2A^2}{\sin(\theta_j - \theta_i)} = \Omega(A^2 n^2 \log n),$$

by using again Lemma 4. Since we cannot give an upper bound on S better than $S \leqslant 1$, we only obtain $A = \mathrm{O}(1/(n\sqrt{\log n}))$. $\square$

Finally, we consider convex arrangements of lines (the dual of convex monotone decreasing Heilbronn's sets). As with the convex cases of the previous problems, the bound in this case is tight.

Theorem 9. $\mathscr{G}_2^{(\mathrm{conv})}(n) = \Theta(1/n^3)$.

Proof. For the lower bound we use the construction $l_i = i/n$ and $r_i = 1 - (i/n)^2$ (for $1 \leqslant i \leqslant n$). A simple calculation shows that in this example

$$G_2^{(\mathrm{conv})}(P_i, P_j, P_k) = \left| \frac{(j-i)(k-i)(k-j)}{2(i+j+n)(i+k+n)(j+k+n)} \right|,$$

whose minimum is easily verified to be $\Theta(1/n^3)$. Here is an alternative explanation. On one hand, no three of these points are collinear, for otherwise a line would intersect the parabola $y = 1 - x^2$ in three distinct points. On the other hand, the points belong to a grid whose step is $1/n \times 1/n^2$, and every three noncollinear such grid points define a triangle whose area is at least $1/(2n^3)$.

For the upper bound, we follow again the argument of dropping segments. Then $\mathscr{G}_2(n)$ is upper bounded by $A^2/(xy^2)$, where x and y are the lengths of the two remaining consecutive segments, and A is the area of the triangle spanned by these two segments. But $A^2/(xy^2) \leqslant \frac{1}{4}(8/n)\sin^2(2\pi/n) = \Theta(1/n^3)$. $\square$

Table 2
Summary of known bounds

Measure		Construction		
Arrangement of Lines	Point Set	Narrow (Arrangement) General (Heilbronn)	Transp. (Arrangement) Mon. Dec. (Heilbronn)	"Convex" (Arng.) Conv. Dec. (Heil.)
	$\mathscr{G}_0$: A	$\Omega\left(\frac{\log n}{n^2}\right), \mathrm{O}\left(\frac{1}{n^{8/7-\epsilon}}\right)$	$\left[\Omega\left(\frac{1}{n^3}\right)\right], \mathrm{O}\left(\frac{1}{n^2}\right)$	$\Theta\left(\frac{1}{n^3}\right)$
$\mathscr{F}_1$: V.H.	$\mathscr{G}_1$: $\frac{A}{P}$	$\Theta\left(\frac{1}{n}\right)$	$\left[\Omega\left(\frac{1}{n^2}\right)\right], \mathrm{O}\left(\frac{1}{n\log n}\right)$	$\Theta\left(\frac{1}{n^2}\right)$
$\mathscr{F}_2$: Area	$\mathscr{G}_2$: $\frac{A^2}{P^3}$	Unbounded	$\left[\Omega\left(\frac{1}{n^3}\right)\right], \mathrm{O}\left(\frac{1}{n\sqrt{\log n}}\right)$	$\Theta\left(\frac{1}{n^3}\right)$

6. Summary

In this paper we show a relation between small-face problems in arrangements of lines and Heilbronn-type problems in point sets. We use a duality between the two classes for obtaining bounds for these problems. We summarize in Table 2 the best bounds that we are aware of. (There is a slight abuse of notation in the synonyms of $\mathscr{G}_1$ and $\mathscr{G}_2$.) Note that every lower (resp., upper) bound trivially applies for columns on the left (resp., right) of it. We mention such trivial bounds in square brackets. The main open problem is to obtain tight bounds for some of the problems.

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Appendix A Mathematica program

The following is a Mathematica 'program' which computes G_1 (vertical height) and G_2 (area) for the arrangement of lines defined by $l_i = i/n$, $r_i = 1-(i/n)^2$:

In[1]:=$l[i\_] = i/n$

Out[1] $= \frac{i}{n}$

In[2]:=$r[i\_] = 1-(i/n)^2$

Out[2] $= 1 - \frac{i^2}{n^2}$

In[3]:=$x[i\_, k\_] =$ Simplify$[(l[k]-l[i])/(l[k]-l[i]-r[k]+r[i])]$

Out[3] $= \frac{n}{i+k+n}$

`In[4]:=`$y[i\_, k\_] = \texttt{Simplify}[(l[k] * r[i] - l[i] * r[k])/(l[k] - l[i] - r[k] + r[i])]$

`Out[4]` $= \frac{ik+n^2}{n(i+k+n)}$

`In[5]:=`$\texttt{VertHgt}[i\_, j\_, k\_] = \texttt{Simplify}[(r[i](l[k] - l[j]) - r[j](l[k] - l[i]) + r[k](l[j] - l[i]))/(l[k] - l[i] - r[k] + r[i])]$

`Out[5]` $= \frac{(i-j)(-j+k)}{n(i+k+n)}$

`In[6]:=`$\texttt{Area}[i\_, j\_, k\_] = \texttt{Simplify}[(x[j,k] * y[k,i] - x[k,i] * y[j,k] - x[i,j] * y[k,i] + x[k,i] * y[i,j] + x[i,j] * y[j,k] - x[j,k] * y[i,j])/2]$

`Out[6]` $= \frac{(-i+j)(i-k)(j-k)}{2(i+j+n)(i+k+n)(j+k+n)}$

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