Convergent Matrix Pencils

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Abstract—This paper deals with the study of sufficient conditions to guarantee that matrices expressed as a matrix pencil of a pair of given matrices be convergent © 2001 Elsevier Science Ltd All rights reserved

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1. INTRODUCTION

The study of the stability of discrete numerical solutions of strongly coupled mixed partial differential systems constructed by means of finite difference schemes is closely related to the convergence of certain matrices [1] Dealing with systems of the type \( u_t - A u_{xx} - B u = 0 \), where \( A, B \) are complex matrices in \( \mathbb{C}^{s \times s} \) and \( u \) is a vector in \( \mathbb{C}^s \), the stability of the involved difference schemes is expressed in terms of the convergence of the matrix

\[
L(r) = I - r (\alpha^2 A - \beta^2 B),
\]

where \( \alpha, \beta \) are fixed positive numbers, and \( r \) is a positive parameter. We recall that a matrix \( P \) in \( \mathbb{C}^{s \times s} \) is said to be convergent if the sequence \( \{P^n\}_{n \geq 0} \) converges to the zero matrix in \( \mathbb{C}^{s \times s} \). If we denote by \( \sigma(P) \) the set of all the eigenvalues of \( P \), and \( \rho(P) = \max\{|z|, z \in \sigma(P)\} \), by [2, p 25], the matrix \( P \) is convergent if and only if \( \rho(P) < 1 \) This paper deals with the study of sufficient conditions on matrices \( A \) and \( B \) so that for small enough values of the parameter \( r \) the matrix \( L(r) \) defined by (1) is convergent. If \( Q \) lies in \( \mathbb{C}^{s \times s} \), we denote by \( Q^H \) its conjugate transpose. If \( Q = Q^H \), then \( Q \) is called Hermitian and in this case, we denote by \( \lambda_{\text{min}}(Q) \) the smallest of the real eigenvalues of \( Q \) and \( \lambda_{\text{max}}(Q) = \max\{z, z \in \sigma(Q)\} \).

2. ON CONVERGENT MATRIX PENCILS

Note that \( L(r) = I - (r/\beta^2)((\alpha^2 A - \tilde{B}) \) where \( \tilde{B} = \beta^4 B \) and by the spectral mapping theorem [3, p 569] it follows that

\[
\sigma(L(r)) = \left\{ 1 - \frac{rz}{\beta^2}, z \in \sigma((\alpha^2 A - \tilde{B}) \right\}, \quad \tilde{B} = \beta^4 B
\]

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If \( z \) lies in \( \sigma((\alpha\beta)^2 A - \tilde{B}) \), then
\[
1 - \frac{r^2}{\beta^2} = \left(1 - \frac{r}{\beta^2} \text{Re}(z)\right)^2 + \left(-\frac{r}{\beta^2} \text{Im}(z)\right)^2 - 1 + \frac{r^2}{\beta^2} |z|^2 - \frac{2r}{\beta^2} \text{Re}(z)
\]
(3)
By (2),(3) the matrix \( L(r) \) is convergent if
\[
\text{Re}(z) > 0 \quad \text{and} \quad \frac{r}{\beta^2} < \frac{2\text{Re}(z)}{|z|^2}, \quad \text{for all} \ z \in \sigma((\alpha\beta)^2 A - \tilde{B})
\]
(4)
If \( \tilde{B} = \tilde{B}_1 + \tilde{B}_2, \ A = A_1 + A_2 \), where \( A_i, \tilde{B}_i \), for \( i = 1,2 \), are the Hermitian matrices defined by
\[
\tilde{B}_1 = \frac{\tilde{B} + \tilde{B}^H}{2}, \quad \tilde{B}_2 = \frac{\tilde{B} - \tilde{B}^H}{2i}, \quad A_1 = \frac{A + A^H}{2}, \quad A_2 = \frac{A - A^H}{2i}
\]
(5)
then
\[
((\alpha\beta)^2 A - \tilde{B}) = \left[((\alpha\beta)^2 A_1 - \tilde{B}_1) + z \left[((\alpha\beta)^2 A_2 - \tilde{B}_2)\right]\right]
\]
(6)
By the Bromwich Theorem [4, p 389] it follows that
\[
\lambda_{\min}\left(((\alpha\beta)^2 A_1 - \tilde{B}_1)\right) \leq \text{Re}(z) \leq \lambda_{\max}\left(((\alpha\beta)^2 A_1 - \tilde{B}_1)\right),
\]
\[
\lambda_{\min}\left(((\alpha\beta)^2 A_2 - \tilde{B}_2)\right) \leq \text{Im}(z) \leq \lambda_{\max}\left(((\alpha\beta)^2 A_2 - \tilde{B}_2)\right),
\]
(7)
By Theorem 3.14 of [5, p 25] it follows that
\[
\lambda_{\min}\left(((\alpha\beta)^2 A_1 - \tilde{B}_1)\right) \geq \lambda_{\min}\left(-\tilde{B}_1\right) + (\alpha\beta)^2 \lambda_{\min}(A_1) = (\alpha\beta)^2 \lambda_{\min}(A_1) - \lambda_{\max}\left(\tilde{B}_1\right),
\]
(8)
and by (7),(8) one gets
\[
\min\left\{\text{Re}(z), \ z \in \sigma\left(((\alpha\beta)^2 A - \tilde{B})\right)\right\} \geq (\alpha\beta)^2 \lambda_{\min}(A_1) - \lambda_{\max}\left(\tilde{B}_1\right)
\]
(9)
Under the hypotheses
\[
x > 0, \quad \text{for all} \ x \in \sigma(A_1) \quad \text{and} \quad y \leq 0, \quad \text{for all} \ y \in \sigma\left(\tilde{B}_1\right),
\]
(10)
by (9) it follows that
\[
\text{Re}(z) > 0, \quad \text{for all} \ z \in \sigma\left(((\alpha\beta)^2 A - \tilde{B})\right)
\]
(11)
Under condition (10), by (11) one gets
\[
\min_{z \in \sigma((\alpha\beta)^2 A - \tilde{B})} \frac{2\text{Re}(z)}{|z|^2} = \frac{2\min\left\{\text{Re}(z), \ z \in \sigma\left(((\alpha\beta)^2 A - \tilde{B})\right)\right\}}{\max\left\{|z|^2, \ z \in \sigma\left(((\alpha\beta)^2 A - \tilde{B})\right)\right\}}
\]
(12)
By (11) and (12), condition (4) is satisfied if
\[
\frac{r}{\beta^2} < \frac{2\min\left\{\text{Re}(z), \ z \in \sigma\left(((\alpha\beta)^2 A - \tilde{B})\right)\right\}}{\max\left\{|z|^2, \ z \in \sigma\left(((\alpha\beta)^2 A - \tilde{B})\right)\right\}}
\]
(13)
By [6, p 246], if we denote \( \sigma(A_i) = \{\lambda_j(A_i), 1 \leq j \leq s\} \), \( i = 1,2 \), and
\[
G_j(t) = \left\{z \in \mathbb{C}, |z - (\alpha\beta)^2 \lambda_j(A_i)| \leq \rho\left(\tilde{B}_i\right)\right\} = D\left((\alpha\beta)^2 \lambda_j(A_i), \rho\left(\tilde{B}_i\right)\right),
\]
it follows that

\[
\sigma \left( (\alpha \beta)^2 A_1 - \bar{B}_1 \right) \subset \bigcup_{j=1}^{\gamma} G_j \left( t \right), \quad 1 \leq t \leq 2 \quad (14)
\]

Hence,

\[
\lambda_{\text{max}} \left( (\alpha \beta)^2 A_1 - \bar{B}_1 \right) \leq (\alpha \beta)^2 \lambda_{\text{max}}(A_1) + \rho \left( \bar{B}_1 \right), \quad 1 \leq t \leq 2, \quad (15)
\]

and by (7), (15) one gets

\[
\text{Re}(z) \leq (\alpha \beta)^2 \lambda_{\text{max}}(A_1) + \rho \left( \bar{B}_1 \right), \quad z \in \sigma \left( (\alpha \beta)^2 A - \bar{B} \right), \quad (16)
\]

\[
\text{Im}(z) \leq (\alpha \beta)^2 \lambda_{\text{max}}(A_2) + \rho \left( \bar{B}_2 \right), \quad z \in \sigma \left( (\alpha \beta)^2 A - \bar{B} \right), \quad (17)
\]

\[
|z|^2 \leq \left( (\alpha \beta)^2 \lambda_{\text{max}}(A_1) + \rho \left( \bar{B}_1 \right) \right)^2 + \left( (\alpha \beta)^2 \lambda_{\text{max}}(A_2) + \rho \left( \bar{B}_2 \right) \right)^2, \quad z \in \sigma \left( (\alpha \beta)^2 A - \bar{B} \right) \quad (18)
\]

By (4), (9), (13), and (18) one concludes that \( L(r) \) is convergent if

\[
r < \frac{2\beta^2 \left[ (\alpha \beta)^2 \lambda_{\text{min}}(A_1) - \lambda_{\text{max}}(\bar{B}_1) \right]}{\left[ (\alpha \beta)^2 \lambda_{\text{max}}(A_1) + \rho \left( \bar{B}_1 \right) \right]^2 + \left[ (\alpha \beta)^2 \lambda_{\text{max}}(A_2) + \rho \left( \bar{B}_2 \right) \right]^2} \quad (19)
\]

Taking into account that \( \bar{B} = \beta^4 B \), condition (19) can be expressed in the form

\[
r < \frac{2\beta^2 \left[ (\alpha \beta)^2 \lambda_{\text{min}}(A_1) - \beta^4 \lambda_{\text{max}}(B_1) \right]}{\left[ (\alpha \beta)^2 \lambda_{\text{max}}(A_1) + \beta^4 \rho(B_1) \right]^2 + \left[ (\alpha \beta)^2 \lambda_{\text{max}}(A_2) + \beta^4 \rho(B_2) \right]^2}
\]

or

\[
r < \frac{2 \left[ \alpha^2 \lambda_{\text{min}}(A_1) - \beta^2 \lambda_{\text{max}}(B_1) \right]}{\left[ \alpha^2 \lambda_{\text{max}}(A_1) + \beta^2 \rho(B_1) \right]^2 + \left[ \alpha^2 \lambda_{\text{max}}(A_2) + \beta^2 \rho(B_2) \right]^2} \quad (20)
\]

Summarizing, the following result has been established

**Theorem 2.1** Let \( \alpha, \beta \) be nonzero real numbers and \( r > 0 \), let \( A, B \) be matrices in \( \mathbb{C}^{s \times s} \) such that if \( B_1 = (B + B^H)/2 \), \( A_1 = (A + A^H)/2 \), \( B_2 = (B - B^H)/2t \), \( A_2 = (A - A^H)/2t \) one satisfies

\[
x > 0, \quad \text{for all } x \in \sigma(A_1) \quad \text{and} \quad y \leq 0, \quad \text{for all } y \in \sigma(B_1) \quad (21)
\]

Then for small enough values of \( r \) so that (20) holds true, the matrix \( L(r) = I - r(\alpha^2 A - \beta^2 B) \) is convergent

**References**