A Finite Generalized Hexagon Admitting a Group Acting Transitive on Ordered Heptagons is Classical

H. Van Maldeghem*

Department of Pure Mathematics and Computer Algebra,
University of Gent, Krijgslaan 281, B-9000 Ghent, Belgium

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Let $\Gamma$ be a thick finite generalized hexagon and let $G$ be a group of automorphisms of $\Gamma$. If $G$ acts transitively on the set of non-degenerate ordered heptagons, then $\Gamma$ is one of the Moufang hexagons $H(q)$ or $^3H(q)$ associated to the Chevalley groups $G_2(q)$ or $^3D_4(q)$ respectively, or their duals; and $G$ contains the corresponding Chevalley group. Moreover, we show that no thick generalized octagon admitting a group acting transitively on the set of ordered nonagons (enneagons) can exist. This completes the determination of all finite thick generalized $n$-gons, $n \geq 3$, with a group acting transitively on the set of ordered $(n+1)$-gons with elementary methods. Because we do not use the classification of the finite simple groups, from which these results also follow.

1. Introduction and Main Results

A finite generalized $n$-gon of order $(s, t)$, $s, t \in \mathbb{N}\setminus\{0\}$, is an incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L}, I)$ in which $\mathcal{P}$ and $\mathcal{L}$ are disjoint non-empty sets of objects called points and lines respectively, and for which $I$ is a symmetric point-line incidence relation satisfying axioms (GP1), (GP2) and (GP3).

(GP1) Each point is incident with $1+t$ lines and two distinct points are incident with at most one line.

(GP2) Each line is incident with $1+s$ points and two distinct lines are incident with at most one point.

(GP3) If the distance in the incidence graph between two elements (points and lines) $v, w$ is—strictly—smaller than $n$, then there is a unique

* Senior Research Associate of the National Fund for Scientific Research (Belgium).

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minimal (i.e., with a minimal number of elements) sequence of consecutive incident elements starting with \(v\) and ending in \(w\).

For a generalized quadrangle (4-gon), hexagon (6-gon) and octagon (8-gon), we can write (GP3) respectively as follows:

(GQ3) For every non-incident pair \((x, L) \in \mathcal{P} \times \mathcal{L}\), there exists a unique pair \((y, M) \in \mathcal{P} \times \mathcal{L}\) for which \(xIMIyIL\).

(GH3) For every non-incident pair \((x, L) \in \mathcal{P} \times \mathcal{L}\), there exists either a unique pair \((y, M) \in \mathcal{P} \times \mathcal{L}\) for which \(xIMIyIL\), or a unique quadruple \((y, M, z, N) \in \mathcal{P} \times \mathcal{L} \times \mathcal{P} \times \mathcal{L}\) for which \(xINIZIMIyIL\).

(GO3) For every non-incident pair \((x, L) \in \mathcal{P} \times \mathcal{L}\), there exists either a unique pair \((y, M) \in \mathcal{P} \times \mathcal{L}\) for which \(xIMIyIL\), or a unique quadruple \((y, M, z, N, u, X) \in \mathcal{P} \times \mathcal{L} \times \mathcal{P} \times \mathcal{L} \times \mathcal{P} \times \mathcal{L}\) for which \(xIXIUINIZIMIyIL\).

The following terminology will be used throughout. A finite generalized hexagon or octagon of order \((s, t)\) is thick if \(s, t \geq 2\) (the non-thick generalized hexagons and octagons are the flag complexes (or the doubles) of the projective planes and generalized quadrangles of order \((s, s)\) respectively) and the dual of this (order \((1, s)\) and \((s, 1)\) respectively). A heptagon in a generalized hexagon is a subconfiguration consisting of seven distinct points and seven distinct lines such that each line (respectively point) is incident with exactly two points (respectively lines). An ordered heptagon is a heptagon in which the elements are ordered in such a way that two consecutive elements are incident. Similar definitions for nonagons and ordered nonagons in generalized octagons. A sub-\(n\)-gon of order \((1, 1)\) (a “usual” \(n\)-gon) in a generalized \(n\)-gon, \(n = 6, 8\) is an apartment. A skeleton is a sub-configuration \(\Omega = (\Sigma; L, p)\) where \(\Sigma\) is an apartment and \(L\) (respectively \(p\)) is a line (respectively point) not in \(\Sigma\) but incident with a point \(p_1\) (respectively line \(L_1\)) of \(\Sigma\), where \(p_1IL_1\). We will always use upper case letters, such as \(L, M, N\) for lines and lower case ones, such as \(p, x, z, b\) for points.

If in a generalized hexagon, a point \(x\) is collinear with two non-collinear points \(p_1\) and \(p_2\), then by axiom (GH3), \(x\) is unique with that property and we denote \(x = p_1 \vee p_2\).

There are presently, up to duality, only two classes of thick finite generalized hexagons known and they are related to the Chevalley groups \(G_2(q)\) and \(^3D_4(q)\). We denote the first one by \(H(q)\) (distinguishing it from its dual \(H(q)\)) by saying that \(H(q)\) is naturally embedded in the quadric \(Q_1(q)\), see Tits [16] and the second one by \(^3H(q)\) (distinguishing it by its dual \(^3H(q)\)) by telling that it has order \((q, q^3)\). From this, it follows that the dual of \(H(q)\) is a subhexagon of \(^3D_4(q)\) (see Tits [16] or Kantor [7]).
We call the members of these 4 classes of finite generalized hexagons the
finite classical hexagons. They were all discovered by Tits [16] (but the
name “Tits hexagon” would cause confusion with the hexagons satisfying
the Tits property, which was introduced by Buekenhout & Van
Maldeghem [1]).

As for finite thick generalized octagons, the situation is even simpler.
Only one such class is presently known (up to duality). It is also due to Tits
[19] and it is related to the class of Ree groups of characteristic 2. These
octagons have order \((q, q^2)\) and we call them the Ree-Tits octagons.

It follows easily from the main result of Van Maldeghem [20] that the
finite classical hexagons admit an automorphism group \(G\) acting tran-
sitively on the set of skeletons. This in fact is equivalent with \(G\) acting
transitively on the set of ordered heptagons, see below. The converse is also
true. Suppose the finite generalized hexagon \(\Gamma\) admits a group \(G\) acting
transitively on the set of ordered heptagons. Then \(G\) is a group with a
\((B, N)\)-pair of type \(G_2\) and using the classification of the finite simple
groups one can show that \(\Gamma\) must be classical (for an explicit proof, see
Buekenhout & Van Maldeghem [1]). The aim of this paper is to give a
proof of this result without using the classification of the finite simple
groups. Once we have shown that the generalized hexagon must be classi-
cal, then a result of Seitz [12] immediately implies that \(G\) must contain the
simple group \(G_2(q)\) \((q \geq 3)\), \(3D_4(q)\) or \(G_2(2) \cong U_3(3)\). In the latter case, the
order of \(H(2)\) is \((2, 2)\), the full automorphism group is \(G_2(2)\) and the num-
ber of ordered heptagons is exactly equal to the order of \(G_2(2)\) (which is
12, 096). So \(G\) must contain the corresponding Chevalley group.

Our first main result is:

**Theorem 1.** Let \(\Gamma\) be a finite thick generalized hexagon and let \(G\) be a
group of automorphisms of \(\Gamma\). Then \(G\) acts transitively on the set of ordered
heptagons if and only if \(\Gamma\) is one of the classical generalized hexagons \(H(q),
H(q)^o\) or \(H(q)^o\) and \(G\) contains the corresponding Chevalley group.

A similar result is proved for finite generalized quadrangles by Thas &
Van Maldeghem [15]. Of course, for finite projective planes, an analogous
result (transitivity on ordered quadrangles) follows immediately from the
well-known theorem of Ostrom & Wagner [8]. By a well known result of
Feit & Higman [3], finite thick generalized \(n\)-gons exist only for
\(n = 2, 3, 4, 6, 8\). So we finally turn our attention to octagons. We will show
as second main result:

**Theorem 2.** There does not exist a finite thick generalized octagon
admitting a group of collineations acting transitively on the set of ordered
nonagons.
As a result, we have the following corollary:

**Corollary.** A finite thick generalized \( n \)-gon \( \Gamma \) admitting a group \( G \) acting transitively on the set of all ordered \((n+1)\)-gons is Moufang and a complete list of such pairs \((\Gamma, G)\) is determined by elementary methods (i.e., without using the classification of the finite simple groups).

We remark that the finiteness assumption cannot be dispensed with in the preceding results; indeed, this follows from a well-known construction method of Kegel and Schleiermacher as adapted by Tits [18].

We also remark that Theorem 1 improves on the main result of Van Maldeghem [20] in which all finite generalized hexagons with transitive apartments are classified. These hexagons satisfy automatically the hypothesis of Theorem 1.

The motivation for studying generalized polygons admitting an automorphism group acting transitively on ordered circuits of a certain length stems from the need of a classification-free proof of the fact that all finite rank 2 Tits systems are known. In the case of quadrangles, Payne & Thas [9] have developed a geometric machinery which can be used to try to do so. A large part of that machinery must be used to show that all finite generalized quadrangles with a group acting transitively on ordered pentagons are known, see Thas & Van Maldeghem [15]. No such machinery is available for hexagons and octagons, but this paper wants to show that in spite of that, geometric reasonings can prove a lot. Also, by the geometric nature of our proof, certain substructures turn up (mainly affine planes), and a more systematic investigation of those must lead to a better understanding of the fact that so few finite hexagons are known. For octagons, the new idea of distance-\( t \) regularity (see Van Maldeghem [21]) is here successively applied.

In fact, it is the author's belief that a classification-free proof of the above mentioned fact is within reach, at least for the case of equal parameters (i.e., \( s = t \)). The geometry needed in the case of hexagons would not be much different from that turning up in the proof of our main result.

2. **Proof of Theorem 1**

In this section, we denote by \( \Gamma = (\mathcal{P}, \mathcal{L}, I) \) a finite thick generalized hexagon of order \((s, t)\) and by \( G \) a group of automorphisms of \( \Gamma \) acting transitively on the set of ordered heptagons. Note that we may assume \( s, t \geq 3 \) by Cohen & Tits [2].

Recall that the *distance in \( \Gamma \)* is the one inherited from the incidence graph. Elements at distance 6 (the maximal distance in \( \Gamma \)) are called *opposite*. 

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We notice that the proof is not longer than the one of the equivalent result for generalized quadrangles, despite the lack of a comparable machinery for finite generalized hexagons. With such a machinery (mainly on anti-regular points (see below) and on the half Moufang condition), our proof would be significantly shorter.

2.1. Some General Facts

2.1.1. Skeletons. Let $\gamma = (p_1, L_1, \ldots, p_7, L_7)$, with $p_i \in L_i$, be an ordered heptagon in $G$. Let $p'_i$ be the point of $L_{i+1}$ at distance 4 from $p_i$ (where indices are taken modulo 7). Consider the skeleton $\Omega = (\Sigma; L_2, p'_2)$, where $\Sigma$ is the apartment $(p_1, L_1, \ldots, p_6, L_6, p'_1)$. Then $\Omega$ completely determines $\gamma$ and vice versa. Hence $G$ acts transitively on the set of skeletons, and conversely, every group acting transitively on the set of skeletons also acts transitively on the set of ordered heptagons.

2.1.2. The Moufang Condition. Consider an apartment $\Sigma = (p_1, L_1, \ldots, p_6, L_6, p_1)$ (where consecutive elements are incident). If the group of collineations of $\Gamma$ fixing every point incident with either $L_1$ or $L_2$, and also every line incident with either $p_1$, $p_2$, or $p_3$ acts transitively on the set of apartments containing $p_1$, $p_2$, $p_3$, $L_6$, $L_1$, $L_2$, and $L_3$, then we say that $\Gamma$ is $(p_1, L_1, p_2, L_2, p_3)$-transitive (and these collineations are called $(p_1, L_1, p_2, L_2, p_3)$-elations). Dually, one defines $(L_1, p_2, L_2, p_3, L_3)$-transitivity. If $\Gamma$ is $(p_1, L_1, p_2, L_2, p_3)$-transitive for all possible choices of the points $p_1$, $p_2$, $p_3$ and the lines $L_1$ and $L_2$ in $\Gamma$, and if moreover, also the dual transitivity property holds for every possible choice, then one calls $\Gamma$ Moufang. From a theorem of Fong & Seitz [4, 5] follows that finite Moufang generalized hexagons are classical. The converse is also true, see Tits [17]. We will use that characterization in the proof of Theorem 1.

2.1.3. Half Regular and Anti-regular Points. Let $x$ be a point of $\Gamma$ and consider the set of points $\Gamma(x)$ collinear with $x$. We define blocks in this set as follows: for every point $y$ opposite $x$, the block $x'y'$ is the set of points collinear with $x$ and at distance 4 from $y$.

1. If this geometry is a semi-linear space (i.e., two points in $\Gamma(x)$ determine at most one such block), then we say that $x$ is half regular. This notion is introduced by Van Maldeghem & Bloemen [22], where the authors remark that from a theorem of Ronan [10] follows that, if every point of a generalized hexagon is half regular, then it is Moufang. An alternative (and actually better) name for half regular is distance-2 regular, see Van Maldeghem [21] (it is better since one can generalize this to arbitrary distance from $x$). Anyway, we will also use this characterization of the classical hexagons in the finite case (Ronan's result is indeed also valid in the infinite case).
2. If in this geometry two blocks meet in at most 2 points, and if every 3 pairwise non-collinear points in \( I(x) \) are contained in a least (and hence exactly) one block, then we call \( x \) anti-regular. This definition is modelled on the same situation in generalized quadrangles (see Payne & Thas [9]). We will meet this property in our proof thus giving a motivation to study anti-regularity in generalized hexagons separately without assuming a group.

The geometry obtained in this paragraph will be denoted by \( \Gamma_x \).

2.2. Reduction to Two Main Cases

Our proof is inspired by Ronan [11], although we cannot use his results (because, as we will see, we will have the intersection set property, but not the regulus property).

For two points \( x, y \) at distance 4 from each other, we denote by \( x * y \) the unique point at distance 2 from both \( x \) and \( y \).

Consider two points \( x, y \) at distance 4 from each other via the chain \( x \text{IL} x \text{I} (x * y) \text{IL} y \). Let \( p \) be a point collinear with \( x \) but not incident with \( L_x \). Let \( p_1 \) and \( p_2 \) be two points collinear with \( y \), not incident with \( L_y \), and at distance 4 from \( p \). There are two possibilities.

1. Suppose \( x^p_1 = x^p_2 \). By the transitivity property, there is a collineation fixing \( x, x * y, y \) and \( p_1 \), and mapping the line \( yp_2 \) to any desired line \( L \) through \( y \), \( yp_1 \neq L \neq L_y \). The set \( x^p_1 \) is preserved and \( p_2 \) is mapped to a point \( p_3 \) on \( L \). Obviously, \( x^p_1 = x^p_2 = x^p \). Since \( L \) was arbitrary, every point \( z \) collinear with \( y \), opposite \( x \) and at distance 4 from \( p \) has the property \( xz = x^p_1 \). By the transitivity, we can now let \( p \) vary over the set of all points collinear with \( x \) but not on \( L_x \), and hence we obtain the property that whenever \( z_1 \) and \( z_2 \) are points collinear with \( y \) and opposite \( x \), then either \( xz_1 = x^p_1 \) or \( xz_1 \cap xz_2 = [x * y] \). By transitivity, this holds for every such pair (\( x, y \)). Following Ronan [11], we say that all intersection sets (w.r.t. points) of \( \Gamma \) have order 1.

2. Suppose now \( x^p_1 \neq x^p_2 \). Then there exists a line \( L \) through \( x \) incident with two distinct points \( a_1 \) and \( a_2 \) at distance 4 from \( p_1 \) and \( p_2 \) respectively. By the transitivity property, there is a collineation \( \theta \) fixing \( x, x * y, p_1 \), and \( p_2 \), and mapping \( a_2 \) to any desired point on \( L \) distinct from \( x \) and \( a_1 \). The point \( p_2 \) will be mapped onto any point \( p_2^\theta \) incident with \( yp_2 \), except for \( y \) and \( z \), where \( z \) has distance 4 from \( a_1 \). Obviously, \( x^p_1 \cap x^p_2 \) contains \( p_\theta^2 \) and \( x^p_1 \cap x^p_2 \) contains \( a_1 \). Since \( p_\theta^2 \) was essentially arbitrary, and since by transitivity also the line \( yp_2 \) is arbitrary and for the same reason \( p_1 \) as well, we conclude that whenever \( z_1 \) and \( z_2 \) are points at distance 4, collinear with \( y \) and opposite \( x \), then \( |xz_1 \cap xz_2| \geq 2 \). Reversing the roles of \( x \) and \( y \), we also see that whenever \( u_1 \) and \( u_2 \) are two non-
collinear points in \( I(x) \) opposite \( y \), then there exists a point \( z \) collinear to \( y \) and opposite \( x \) such that \( \{u_1, u_2\} \subseteq x^z \).

Let us get back to the above situation involving \( a_1, a_2, p_1 \) and \( p_2 \). There exist also collineations fixing \( x, y, p_1, a_1 \) and \( p \) and mapping \( a_2 \) to any point \( u \) of \( L \) different from \( x \) and \( a_1 \). Of course, such mappings do not preserve the line \( yp_2 \), and in fact every other choice for \( u \) gives another image of \( yp_2 \), hence \( s - 1 \leq t - 1 \), implying \( s \leq t \).

The properties obtained in this paragraph are also valid for every choice of such a pair \((x, y)\), by transitivity. Following Ronan [11], we say that all intersection sets of \( I \) have order \( \geq 2 \).

From this, we derive two possibilities:

Case (i). In either \( I \) or its dual, all intersection sets have order 1;

Case (ii). In both \( I \) and its dual, all intersection sets have order \( \geq 2 \).

Remark that Case (ii) implies \( s = t \).

2.3. Case (i)

In fact, Ronan [11] proves that, if \( I \) has also the regular condition, then it is Moufang. We are not in a position to show directly the regular condition, but a little weaker version will do the trick here. We may assume, by duality, that all intersection sets w.r.t. points have order 1 in \( I \).

We consider an apartment \((p_1, L_1, ..., p_6, L_6, p_1)\) as in 2.1.2. Denote by \( S \) the set of points of the form \( xV y \), with \( xI L_2, yI L_5 \) and \( x \) at distance 4 from \( y \). Using the transitivity property as above, one shows completely similar to the argument in the previous paragraphs, that either \( p = p^* \), for all \( u \in S \setminus \{p_1\} \), or every point on every line \( L \) through \( p_1, L_1 \neq L \neq L_4 \), is at distance 4 from exactly 1 element of \( S \). We call these cases Subcase (a) and Subcase (b) respectively. If Subcase (a) holds for one apartment and one choice of \( p_1 \) in that apartment, then Subcase (a) holds for all apartments and choices of \( p_1 \), by the transitivity. Similarly for Subcase (b).

2.3.1. Subcase (a). We assume in this paragraph that Subcase (a) (respectively Subcase (b)) holds. Let \( p \) be any point of \( I \). We show that \( p \) is half regular—or distance-2 regular—(respectively anti-regular). Therefore, let \( a \) and \( b \) be two non-collinear points collinear with \( p \). Let \( x \) and \( y \) be two points opposite \( p \) both at distance 4 from both \( a \) and \( b \). We must show \( p = p^* \) (respectively \( p^* = p^* \) or \( p^* \cap p^* = \{a, b\} \); by the Subcase (b) assumption, we already have at least one block through every three pairwise non-collinear points in \( I(p) \)). We may suppose that either \( x \neq a \) is not collinear with \( y \neq b \), or \( x \neq b \) is not collinear with \( y \neq b \); otherwise the result follows from the Subcase (a) (respectively (b)) assumption. So suppose \( x \) and \( b \) are not collinear with \( y \). Let \( L \) be the line joining \( b \) and \( y \neq b \).
and let $M$ be the line joining $a \ast x$ and $a$. Let $u$ and $v$ be the points collinear with $y \ast b$ and $x \ast a$ respectively, and at distance 3 from $M$ and $L$ respectively.

By the Case (i) assumption, $p^{*} = p^{\ast}$ and $p^{\ast} = p^{*}$, and by the Subcase (a) assumption, $p^{*} = p^{\ast}$ (respectively $p^{\ast} = p^{*}$, or $p^{*} \cap p^{\ast} = \{a, b\}$), implying $p^{*} = p^{\ast}$ (respectively $p^{\ast} = p^{*}$, or $p^{*} \cap p^{\ast} = \{a, b\}$). This shows that $p$ is half regular (respectively anti-regular). But this means that all points are half regular and $\Gamma$ is classical.

2.3.2. Subcase (b). We show the result in four steps.

Step 1. In this step, we make the additional assumption that $G$ acts regularly on the set of ordered heptagons in $\Gamma$. Fix an apartment $\Sigma = (p_{1}, L_{1}, \ldots, p_{6}, L_{6}, p_{1})$, where consecutive elements are incident. Let $L$ be any line incident with $p_{1}, L_{1} \neq L \neq L_{6}$. The group $H_{1}$ fixing $p_{1}, p_{3}, p_{5}, L_{6}, L_{1}, L_{3}$ and $L_{5}$ has order $s(s-1)$ and acts sharply doubly transitively on the set $V$ of points incident with $L_{6}$ but different from $p_{1}$. Hence it acts on that set as a Frobenius group and so $H_{1}$ had a unique normal regular subgroup $N_{1}$ of order $s$, which is elementary abelian. So $s = p_{1}^{\#}$ with $p_{1}$ a prime and $n_{i} \in \mathbb{N}_{0}$. Similarly $t = p_{2}^{\#}$, $\pi_{2}$ prime and $n_{j} \in \mathbb{N}_{0}$.

Now let $p$ be a point incident with $L_{1}, p_{1} \neq p \neq p_{3}$. The subgroup of $H_{1}$ fixing $p$ acts regularly on the set $V$ (see above), hence this subgroup is $N_{1}$. Since $p$ was essentially arbitrary, $N_{1}$ fixes $L, L_{6}$ and $L_{3}$, fixes every point on $L_{1}$, every point on $L_{2}$ and acts regularly on $V$. Let $N_{1}'$ be the subgroup of $H_{1}$ fixing $p_{6}$. Suppose an element $\theta \in N_{1}'$ fixes some point $x$ on $L_{2}, p_{2} \neq x \neq p_{3}$. Let $x'$ be the point collinear with $x$ and at distance 3 from $L_{5}$. By the assumption of Subcase (b), the unique point at distance 4 from $x'$ on $L$ is different from the unique point at distance 4 from $p_{3}$ on $L$, but both these points are fixed by $\theta$. Hence $\theta$ fixes a skeleton, which implies that $\theta$ is the identity. This shows that $N_{1}'$ acts regularly on the set $V'$ of points incident with $L_{2}$ but distinct from $p_{2}$ and $p_{3}$. Hence $H_{1}$ acts transitively on $V'$ and $N_{1} \trianglelefteq H_{1}$ partitions $V'$ in equal orbits. But $|V'| = s-1$ is relatively prime to $\pi_{1}$. Since $N_{1}$ is a $\pi_{1}$-group, this implies that $N_{1}$ fixes all elements of $L_{2}$.

Hence $N_{1}$ fixes $L, L_{3}, L_{6}$ and every point on $L_{1}$ and on $L_{2}$. Similarly, the dual result holds.

We now forget about the above notation to derive a geometric property. Consider a point $p$ in $\Gamma$. Consider the geometry $\Gamma_{p}$. Fix a block $K$, a point $x$ on $K$ and a point $y$ off $K$, with $x$ and $y$ non-collinear in $\Gamma$. Remember that $p$ is anti-regular, so every 3 points of $\Gamma_{p}$ which are non-collinear in $\Gamma$ define a unique block. Hence, the number of blocks through $y$ and $x$ meeting $K$ in exactly 2 points is $t-1$. On the other hand, there are in total $s$ blocks through $x$ and $y$, at least one of which meets $K$ in exactly $\{x\}$ (if $K = p^{*}$, then there is a point $w$ on the line joining $u$ and $x \ast u$ at distance...
4 from $y$: $p^u$ and $p^v$ have only $x$ in their intersection, otherwise a pentagon arises), so at most $s-1$ blocks through $x$ and $y$ meet $K$ in a second point. This implies $t \leq s$.

**Step 2.** In this paragraph, we keep our assumption about the regularity of $G$ on the set of ordered heptagons, or equivalently, on the set of skeletons of $\Gamma$. But we handle the case $t<s$, which will be assumed throughout this step. We consider again the notation of the first two paragraphs of Step 1. In particular the subgroup $N_1$ fixes $L_6, L_3$ and $L$, and it fixes $L_1$ and $L_2$ pointwise. In fact, $L$ was essentially arbitrary. So we can define a regular group $N_2$ for a different line $M$ through $p_1$, $L_6 \neq M \neq L_1$ in the same way as $N_1$ was defined for $L$. By the transitivity on $V$ there exists for every $\theta \in N_1$ an element $\theta' \in N_2$ such that $(p_0)^{\theta'} = p_0$. So $\theta \theta'$ fixes $\Sigma$ and all points on $L_1$ or $L_2$, hence $\theta \theta'$ fixes a subhexagon of order $(s, t)$, implying $s \leq t$ or $t = t'$ by Thas [13]. By our assumption, $t = t'$, so $\theta' = \theta^{-1}$ and so $\theta$ fixes both $L$ and $M$. Since $M$ was essentially arbitrary, $\theta$ fixes every line through $p_1$. A similar argument shows that $\theta$ also fixes every line through $p_3$. Now consider a line $X$ through $p_2$, $L_1 \neq X \neq L_2$. The group $H_2$ fixing $\Sigma$ and $X$ has order $s-1$ and acts transitively on the set of points incident with $L_1$ and different from $p_1$ and $p_2$. Suppose any element $\phi \in H_2$ fixes a point $x$ on $L_6$, $p_1 \neq x \neq p_6$. By the Subcase (b) assumption, the point $u$ on $X$, at distance 4 from the point $w$, which is defined by: $w$ is collinear with $x$ and at distance 3 from $L_3$, is different from the point $u'$ at distance 4 from $p_4$ and incident with $X$. Hence $\phi$ fixes the skeleton determined by the apartment containing $p_1$, $p_2$, $u'$, $p_5$ and $p_6$ and furthermore consisting of the line $L_2$ and the point $u$. By the regularity of the action of $G$ on the set of skeletons, $\phi$ must be the identity. Hence $H_3$ acts regularly on the set of points of $L_6$ different from $p_1$ and $p_6$. Since $s > 2$, this group is non-trivial, and letting $p_6$ now vary over $V$, we obtain a group $H_3$ of order $s(s-1)$ acting sharply doubly transitively on $V$ and fixing $p_1$, $p_2$, $p_3$ and $L_6$, $L_1$, $L_2$ and $X$. A similar argument as above shows that in fact $N_1$ is a subgroup of $H_3$ and hence we conclude that $\Gamma$ is $(p_1, L_1, p_2, L_2, p_3)$-transitive.

By transitivity, $\Gamma$ is also $(p_2, L_2, p_3, L_3, p_4)$-transitive with corresponding group $N_3$ (so every element of $N_1$ fixes all elements incident with one of the points $p_2$, $p_3$, $p_4$ or with one of the lines $L_2$, $L_3$). Suppose the commutator $[N_1, N_3]$ is trivial. It is easy to see that this implies that every element of $N_1$ fixes every line through every point of $L_1$ respectively $L_2$. This implies that, with dual notation, $L_1^\perp = L_2^\perp$, where $M_4$ is a line opposite $L_1$, meeting $L_3$ and at distance 4 from $L_6$. This means that the dual $\Gamma^\perp$ of $\Gamma$ satisfies the assumption of Case (i), and hence we may assume that $\Gamma^\perp$ also satisfies the assumption of Subcase (b). So $s \leq t$, a contradiction. Hence $[N_1, N_3]$ is non-trivial. But it is easily seen that
any element $\theta$ of $[H_1, N_2]$ fixes every element incident with one of $L_1$, $p_2$, $L_2$, $p_1$, $L_3$. By conjugating $\theta$ with the subgroup of $G$ fixing $\Sigma$, we see that $\Gamma$ is $(L_1, p_2, L_2, p_1, L_3)$-transitive. Hence $\Gamma$ is Moufang. But this is again a contradiction, because no Moufang hexagon satisfies the assumption of Subcase (b), see e.g., Ronan [10], (5.9). Hence this situation cannot occur.

**Step 3.** In this step, we still assume that $G$ acts regularly on the set of skeletons of $\Gamma$, but by the last paragraph, we necessarily have $s = t$. We consider the notation of the first two paragraphs of Step 1 again, in particular the group $N_1$ fixing $L_0$, $L_3$ and $L$, and fixing $L_1$ and $L_2$ point-wise. Note that this group must fix at least one other line $L'$ through $p_2$, $L_1 \neq L' \neq L_2$, and one other line $L''$ through $p_3$, $L_2 \neq L'' \neq L_3$ (because $N_1$ is a $\pi$-group for some prime $\pi$ and $t$ is a power of $\pi$). Similarly, there is a group $N_2$ of order $s = t$ fixing $p_1$, $p_3$ and $p_4$, all lines through $p_2$ and $p_3$, a point $p$ on $L_1$, $p'$ on $L_2$ and $p''$ on $L_3$, $p, p', p''$ not in $\Sigma$.

We now interrupt our proof for a moment to get back to the notation of the last paragraph of Step 1, in order to derive some more geometric properties. In the geometry $\Gamma_p$, we fix a point $p \ast p'$ with $p'$ a point of $\Gamma$ at distance 4 from $p$. Let $L$ be the line joining $p'$ and $p \ast p'$. Let $x$ be a point collinear with $p'$ and opposite $p$. By the Case (i) assumption, there are exactly $s$ points $y$ opposite $p$ and collinear with $p'$ for which $p \ast y = p'$. Evidently, every other point $a$ collinear with $p'$ and opposite $p$ gives rise to a different block $p'' \neq p \ast y$ of $\Gamma_p$, and moreover $p'' \cap p' = \{p \ast p', p''\}$. So $p'$ defines exactly $s$ blocks of $\Gamma_p$ which meet two by two in $p \ast p'$. Varying $p'$ over $L$ (keeping it at distance 4 from $p$ of course), we see that the set of points opposite $p$ and at distance 3 from $L$ define at most $s^2$ blocks in $\Gamma_p$ ("at most" since some of them could coincide). Now consider two arbitrary lines $L_1$ and $L_2$ through $p$, not incident with $p \ast p'$, and consider arbitrary points $p_1$ and $p_2$ on $L_1$ and $L_2$ respectively, $p_1 \neq p \neq p_2$. Let $p'_1$ be a point not incident with $L$ but collinear with $p'$ and at distance 4 from $p_1$. Then $p'_1$ defines a block in $\mathcal{N}$ through $p_1$, and hence by the Subcase (b) assumption, there is a point $x$ at distance 3 from $L$ such that $\{p_1, p'_2\} \subseteq p''$. Varying $p_1$ and $p_2$ over $L_1$ respectively $L_2$, we see that at least $s^2$ blocks through $p \ast p'$ are defined by points at distance three from $L$, hence exactly $s^2$. It is now easy to see that the incidence structure $\Pi(p, L)$ with point set the set of points collinear with $p$ but not collinear with $p \ast p'$, and line set the set of blocks of the form $p''$ with $x$ at distance 3 from $L$ and opposite $p$, together with the ordinary lines through $p$, forms an affine plane (with the obvious incidence relation). Every point on $L$ different from $p \ast p'$ symbolizes a point at infinity of $\Pi(p, L)$ and also $p$ is a point at infinity of $\Pi(p, L)$ in an obvious way.

We now get back to our previous situation (first paragraph of this Step 3). Let $\theta$ be any non-trivial element of $N_1$. This collineation induces...
in $\Pi(p_3, L_3)$ a non-trivial axial collineation (indeed, all points of the line $L_2$ are fixed), hence $\theta$ is central. Since $\theta$ fixes the lines $L_1, L_3$ and the line at infinity of $\Pi(p_3, L_3)$, the center must be the point $p_3$ at infinity, which is incident with all three fixed lines mentioned. Hence $\theta$ fixes all lines through $p_2$. Similarly $\theta$ induces an axial non-trivial collineation in $\Pi(p_1, L_3)$ (all points at infinity of $\Pi(p_1, L_3)$ are fixed), and hence $\theta$ is central, but as already three lines through the point $p_1$ at infinity are fixed (the line at infinity, $L$ and $L_1$), $p_1$ must be the center, hence $\theta$ fixes all lines through $p_1$ and similarly, also all lines through $p_3$. We conclude that $\Gamma$ is $(p_1, L_1, p_2, L_2, p_3)$-transitive. Considering again the commutator $[N_1, N_3]$ as in the second paragraph of Step 2, we obtain that either $\Gamma^D$ has anti-regular points, in which case $\Gamma$ is dually $(L_1, p_2, L_2, p_3, L_3)$-transitive and hence Moufang; or $[N_1, N_3]$ is non-trivial and $\Gamma$ is Moufang again. But as before, this is a contradiction since a Moufang hexagon cannot satisfy the assumption of Subcase (b).

So we conclude that Case (i), Subcase (b) cannot occur if $G$ acts regularly on the set of ordered heptagons.

Step 4. We now drop every extra assumption on $\Gamma$ and $G$. Consider a certain fixed heptagon in $\Gamma$ and take the intersection of all subhexagons containing this heptagon. This is again a thick generalized hexagon $\Gamma'$ which does not contain strictly any thick subhexagon. Clearly $G$ induces in $\Gamma'$ a group of collineations acting transitively on the set of ordered heptagons, but since $\Gamma'$ does not contain strictly any subhexagon, this action must be regular. It is also clear that $\Gamma'$ satisfies the assumptions of Case (i) and Subcase (b), so by the previous steps, $\Gamma'$ cannot exist. Hence $\Gamma$ cannot exist.

This completes the proof of Case (i).

2.4. Case (ii)

Here we assume that all intersection sets have order $\geq 2$ in both $\Gamma$ and $\Gamma^D$, and that $s = t$. As in the previous case, it suffices to show that this situation cannot occur if $G$ acts regularly on the set of ordered heptagons.

Note that $\Gamma$ does not even contain any subhexagon of order $(1, s)$ or $(s, 1)$ since this would imply that $\Gamma$ satisfies the condition of Case (i). Indeed, if $\Gamma'$ is a subhexagon of order $(1, s)$ containing two points $p, p'$ at distance 4 from each other, then all points $u$ of $\Gamma'$ collinear with $p$ and opposite $p$ determine the same set $pu$ as this set consists of all points of $\Gamma'$ collinear with $p$.

Consider an apartment $\Sigma = (p_1, L_1, \ldots, p_6, L_6, p_1)$ as before, then we again obtain a sharply doubly transitive permutation group and a group $N_1$ fixing $L_6, L_3, L_3$ and $L_2$ pointwise, and fixing some lines $L, L', L''$ through $p_1, p_2, p_3$ respectively and not contained in $\Sigma$; $N_1$ acts regularly
on the set of points incident with $L_a$ but different from $p_1$. By the transitivity property of $G$, we can choose either $L$ or $L'$ or $L''$ (but we obtain possibly a different group $N^*$) arbitrarily (but with the same restrictions).

With every element $\theta$ of $N^*$ corresponds an element $\theta'$ of $N^*$ such that $\theta \theta'$ fixes $\Sigma$ elementwise. But it also fixes every point on $L_1$ and on $L_2$, hence it fixes a subhexagon of order $(s, t')$. By Thas [13], $t' = 1$ or $t' = s$. We already ruled out $t' = 1$, hence $t' = s$ and so $\theta = \theta'$. We conclude that $N^*$ also fixes every line through $p_1$, every line through $p_2$, and every line through $p_3$. So $\Gamma$ is $(p_1, L_1, L_2, p_3)$-transitive. Also the dual transitivity holds here and so $\Gamma$ is Moufang. But this is impossible since every Moufang hexagon of order $(s, s)$ has a subhexagon of order $(1, s)$ or $(s, 1)$, see e.g., Ronan [10], (6.11).

This completes the proof of the Theorem 1.

3. Proof of Theorem 2

Now we suppose that $\Gamma$ is a finite thick generalized octagon of order $(s, t)$ admitting a group $G$ acting transitively on the set of all ordered non-agons of $\Gamma$, or equivalently (as for hexagons), on the set of all skeletons of $\Gamma$. Upon taking the intersection of all suboctagons containing a fixed non-agon, we may suppose that $G$ acts regularly on the above sets (because if we show that this regular situation cannot occur, then also the more general transitive situation is impossible). By the fact that $2st$ must be a perfect square, see Feit & Higman [3], we may assume that $s < t$.

We adopt the following notation throughout: $\Sigma$ is the apartment $(p_1, L_1, p_2, ..., p_8, L_8, p_1)$, where consecutive elements are incident. The distance is again the one inherited from the incidence graph and elements at distance 8 are called opposite.

The reader can easily generalize the definition of Moufang condition to octagons and by Tits [19], all Moufang octagons are Ree-Tits octagons. We again consider some steps.

Step A. In this step, we show the claim that $G$ acts regularly on configurations of the form $(\Sigma, p, L)$, where $L$ is a line incident with $p_1$, $p$ is a point incident with $L_2$, and neither $L$ nor $p$ is in $\Sigma$. The subgroup $H_1$ of $G$ fixing $\Sigma$ elementwise has order $(s-1)(t-1)$ and acts transitively on the set of lines through $p_1$ different from $L_3$ and $L_4$. The stabilizer $H_2$ of $L$ in $H_1$ has therefore order $(s-1)$. We have to show that $H_2$ acts transitively on the set of $s-1$ points on $L_2$ different from $p_5$ and $p_6$. Suppose this is not the case, then there is a collineation $\theta$ in $H_2$ fixing some point $p$ on $L_2$, $p \neq p \neq p_3$. There is a unique line $M$ at distance 4 from $L$ and at distance 3 from $p_5$; there is a unique point $x$ on $M$ at distance 6 from $p$; there

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is a unique point $y$ on $L_3$ (different from $p_2$ and $p_3$) at distance 6 from $x$ and there is a unique point $z$ on $L_3$ at distance 6 from $y$. Obviously, $\theta$ fixes all these elements. Dually, there is a line $Z$ through $p_8$, $L_7 \neq Z \neq L_8$, fixed by $\theta$. But now $\theta$ fixes a skeleton $(\Sigma; Z, y)$, hence $\theta$ is the identity and our claim follows.

**Step B.** We show half of the Moufang condition. As for generalized hexagons (Step I of 2.3.2), we have a sharply doubly transitive group $H_5$ acting on the lines through $p_8$ different from $L_8$, fixing $p_8, p_1, ..., p_4, p$, where $p$ is a certain arbitrarily chosen point on $L_8$ ($p$ not in $\Sigma$). As in 2.3.2 and using Step A above, one shows again that $H_5$ has a regular normal subgroup $N_3$ fixing every line through $p_1, p_2$ and $p_3$. Since we have now $t > s$, and by Thas [14], $\Gamma$ does not contain a suboctagon of order $(s', t)$ unless $s = s'$; we can dualize the argument of Step 2 of 2.3.2 (using also Step A above) to obtain that $\Gamma$ is $(L_8, p_1, ..., L_3)$-transitive.

Note that, dually, we have a group $N_1$ of order $s$ fixing $L_1$, $L_3$ and a certain arbitrary line $L$ through $p_1, L_9 \neq L \neq L_1$, and fixing every point on $L_1, L_2$ and $L_3$. The group $N_1$ acts regularly on the set of points incident with $L_8$ but different from $p_1$.

In particular we deduce that both $s$ and $t$ are powers of a prime (not necessarily the same one, but since $2st$ is a square, at least one of these primes equals 2).

**Step C.** In this step, we determine two geometric properties that $\Gamma$ must have, if it were not Moufang.

(R1) If $\Gamma$ is not Moufang, then the commutator $[N_1, N_3]$ must be trivial ($N_3$ is the group of all $(L_1, p_2, ..., L_3)$-elations). As in Step 2 of 2.3.2, this means that, whenever $x$ is a point collinear with $p_5$ (or $p_4$) and at distance 6 from $p_2$ (or $p_3$), then $p^*_1 = p^*_2$ (where $y^*$ is the set of points collinear with $y$ and at distance 6 from $z$; $y$ and $z$ must be opposite).

(R2) Let $p$ be any point on $L_1$, $p_1 \neq p \neq p_2$. Define the elements $p I M_2 I x_1 I M_2 I x_2 I M_3 I x_1 I L_4$. Let $M$ be any line through $p_4$, $L_3 \neq M \neq L_4$, and let $\theta$ (respectively $\theta'$) be the unique $(L_3, p_1, ..., L_3)$-elation (respectively $(M_2, p, L_1, ..., L_3)$-elation) mapping $L_4$ onto $M$.

First assume that the unique point $x$ on $M$ at distance 6 from $p_8$ is opposite $x_1$. Then the collineation $\theta' \theta^{-1}$ fixes all points of $L_1, L_2$ and $L_3$, it fixes all lines through $p_2$ and $p_3$ and it does not fix all points on $L_4$. By composing with a suitable $(L_1, p_2, ..., L_4)$-elation, we obtain a collineation $\phi$ fixing $L_8$ and $L_4$, every point on $L_1, L_2$ and $L_3$, and every line through $p_2$ and $p_3$. By conjugation, the group $N_3$ of such collineations acts transitively (hence regularly) on the points incident with $L_4$ different from $p_4$. If at least one element $\phi$ of $N_3$ fixes at least one line $L$ through $p_1, L_9 \neq L \neq L_1$, then by conjugation with the subgroup $H_5$ of $G$ fixing $\Sigma$ and $L$,
every element of \( N_5 \) must fix \( L \). By conjugation with the subgroup \( H_1 \), fixing \( \Sigma \) (and acting transitively on the set of lines through \( p_1 \) different from \( L_5 \) and \( L_1 \)), we see that \( N_5 \) must fix every line through \( p_1 \). By a similar argument, \( N_5 \) fixes all lines through \( p_4 \), if it does not act semi-regularly on the set of lines through \( p_4 \) different from \( L_1 \) and \( L_4 \). So if \( \Gamma \) is \( \langle p_1, L_1, \ldots, p_4 \rangle \)-transitive unless \( N_5 \) acts semi-regularly on the set of \( t-1 \) lines through \( p_1 \) different from \( L_5 \) and \( L_1 \), or on the set of \( t-1 \) lines through \( p_4 \) different from \( L_1 \) and \( L_4 \). If this happens, then \( s \) divides \( t-1 \).

We now show that also \( s-1 \) must divide \( t-1 \). Let \( \theta \in N_5 \) be non-trivial. Let \( \theta' \in N_1 \) such that \( \theta \theta^{-1} \) fixes \( \Sigma \). Since \( \theta \) does not fix \( L \), \( \theta \theta^{-1} \) is non-trivial and fixes every point on \( L_1 \), \( L_2 \), and \( L_3 \), hence it fixes a suboctagon of order \( (s,1) \). This suboctagon is easily seen to be Moufang (by the presence of the group \( N_1 \) or \( N_5 \)), and by transitivity, it is also self-dual. Hence it is the double of a symplectic quadrangle of characteristic 2 and, if \( s \neq 2 \), the elations generate the symplectic group \( \text{PSp}_4(s) \) which contains a subgroup \( K \) (of “generalized homologies”) of order \( s-1 \) which fixes \( \Sigma \) and every point on \( L_1 \), and which acts transitively on the remaining points of \( L_5 \) (indeed, this follows from the fact that the symplectic group \( \text{PSp}_4(s) \) is simple in this case). No non-trivial element of \( K \) can fix an additional line through \( p_1 \) since otherwise a non-trivial thick suboctagon is fixed. Hence the claim for \( s \neq 2 \). But if \( s = 2 \), then \( t = 4 \) and the result follows.

Hence \( s(s-1) \) divides \( t-1 \). But this implies that either \( s(s-1) = t-1 \), or \( 2s(s-1) \leq t-1 \). Note that certainly \( t \) is odd, and hence \( t \) is a square.

But if \( s(s-1) = t-1 \), then \( s^2 - s + 1 = t \) and so \( (s-1)^2 < t < s^2 \), a contradiction. Hence \( 2s(s-1) \leq t-1 \leq s^2-1 \) (by the inequality of Higman [6]), implying \( s = 1 \). This shows that our assumption is false. Hence \( x \) is at distance 6 from \( x_1 \), for every choice of \( M \) and \( p \).

From this we derive the following property of \( \Gamma \):

\[(\text{RR}) \quad \text{If } x \text{ and } y \text{ are opposite points of } \Gamma, L \text{ is a line at distance } 3 \text{ from } y \text{ and } 5 \text{ from } x, \text{ and } z \text{ is opposite } x \text{ and at distance } 3 \text{ from } L, \text{ then either } x'y' = x'z', \text{ or } |x'y' \cap x'z'| = 1.\]

Indeed, suppose \( |x'y' \cap x'z'| \geq 2 \). Let \( a \) be the unique point collinear with \( x \) and at distance 3 from \( L \), and let \( b \) be a second point in \( x'y' \cap x'z' \). Let \( L_1 \) and \( L_2 \) be the unique lines through \( b \) at distance 5 from \( y \) and \( z \) respectively, and let \( p_1 \) and \( p_2 \) be the points on \( L \) collinear with \( y \) and \( z \) respectively. If \( p_1 = p_2 \) or \( L_1 = L_2 \), then \( x'y' = x'z' \) by (R1) and (R2) respectively. So suppose \( p_1 \neq p_2 \) and \( L_1 \neq L_2 \). Let \( u \) be the unique point collinear with \( z \) and at distance \( 5 \) from \( L_1 \). By (R1), \( x'y' = x'z' \) and by (R2), \( x'y' = x'z' \). Hence (RR) follows.

The property (RR) expresses a kind of distance-2 regularity. In the next and last step, we will show that this is impossible for any generalized
octagon. The proof is completely the same as the one ruling out distance-2 regular octagons in Van Maldeghem [21], since in fact (RR) is the only thing used in the proof in that paper. For the convenience of the reader, we repeat this proof here.

Step D. Let \( (p_1, p_2, \ldots, p_8, p_{8k}, p_{8k+1}) \) be as above. We assume that property (RR) holds for every pair of opposite points \( x, y \) (which we may by transitivity). Let \( p_1 \) be incident with \( L_4 \) but different from both \( p_1 \) and \( p_8 \) (\( \Gamma \) is thick). Construct the sequence \( (p_1', p_1, p_2', p_2, p_3', p_3, p_4', p_4) \) such that \( p_4 \) is incident with \( L_4 \). Let \( (p_2, p_3, p_4, \ldots, p_{8k-2}, p_{8k-1}) \) be a sequence with \( p_i \neq p_i' \neq p_i \) (again possible by the thickness assumption). Since \( \{p_1, p_3\} \subseteq p_2 \cap p_3 \), property (RR) implies that \( p_1 \) is at distance 6 from \( p_3 \) and so we can define a sequence \( (p_1', p_1, x_1, x_2, x_3, \ldots, x_i, x_j, \ldots) \). Clearly \( x_1 \) is incident with neither \( p_2, p_1, \) nor \( p_2', p_1' \).

Suppose first that \( x_2 \) is not incident with \( p_2, p_3 \). Let \( x_3 \) be the unique point on \( p_2, p_3 \) at distance 6 from \( p_4 \). Clearly \( p_2 \neq x_3 \neq p_4 \). But \( \{x_2, p_3\} \subseteq (p_2) \cap (p_3) \), hence the distance between \( p_2 \) and \( x_3 \) is 6, so \( p_2, p_3 \) are opposite. Now \( p_2 \) and \( p_3 \) share the points \( x_3 \) and \( p_4 \), and so there is a sequence \( (x_3, p_4, x_1, x_2, y_1, y_2, y_3, x_4, x_5, p_6, p_7) \). Clearly \( y_1 \) is incident with neither \( p_2, p_1 \) nor \( p_2, p_1' \). And if \( y_2 \) were incident with \( p_2, p_1' \), then the distance between \( p_2 \) and \( p_2' \) would be 6, contradicting the fact that they are opposite. Also, \( p_2, x_3 \neq p_2, y_1 \) (otherwise a cycle of length 14 or 12 via \( x_3 \) and \( p_3 \) arises). Clearly \( p_2 \) and \( y_1 \) are opposite, but this contradicts \( \{p_1, p_1', y_1\} \subseteq (p_2) \) and \( \{p_1, p_1'\} \subseteq (p_2) \) and property (RR). We conclude that \( x_2 \) must be incident with \( p_2, p_3 \).

So we may suppose that \( x_2 \) is incident with \( p_2, p_3 \). By symmetry, \( y_2 \) (defined as in the previous paragraph) must be incident with \( p_2, p_3 \). But then \( (p_2, x_1, y_2, x_2, x_3, p_4, p_5, p_6) \) forms a cycle of length 14 in \( \Gamma \), a contradiction.

This completes the proof of Theorem 2.

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