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ADDING ACTIVITIES TO THE DUAL INSTEAD OF CUTS TO THE PRIMAL PROBLEM

Søren HOLM
Department of Management, Odense University, Denmark

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When solving a problem by appending cuts the dimension of the corresponding simplex tableau and the basic inverse oscillates, which makes it difficult to implement a cutting plane algorithm based on a standard LP code. Moreover, it is complicated to express a cut in the original variables. In this paper we show that by formulating the dual to the problem and adding activities, these adverse effects can be circumvented. It is shown that the set of activities which can be added is the same as the set of cuts which can be appended and that it is easy to exhibit an activity in the original primal variables. As a consequence of this a new formulation of a cut in the original primal variables is given.

1. Introduction

We will in this paper study some of the adverse properties which are inherent to a solution procedure when a problem is solved by appending cuts, and show how a dual formulation can circumvent these problems.

Consider the primal problem

$$P \qquad \begin{array}{ll} \max & cx \\ \text{s.t.} & Ax \le b, \\ & x \ge 0, \\ & x \in X \end{array}$$

where A is $m \times n$ and c, x and b are of appropriate dimensions. P without the condition $x \in X$ is called the corresponding LP. Here $x \in X$ confines the solutions for the corresponding LP to be any class of problems which can be solved by appending cuts, such as all-integer, mixed integer or cardinality constrained problems, to name just a few.

Let us assume that we have reached a stage where there are k cuts appended:

$$P_k \qquad \begin{array}{ll} \max & cx \\ \text{s.t.} & Ax \le b, \\ & A_k x \le b_k, \\ & x \ge 0, \\ & x \in X \end{array}$$

and assume, without loss of generality, that a cut is discarded when its slack becomes basic, i.e., when the cut becomes nonbinding. Apart from the fact that cutting plane algorithms have not shown themselves to be consistently successful (Crowder, Johnson & Padberg [1]), the solution procedure has the following adverse properties:

(i) When we solve the corresponding LP to P_k, then the dimension of the simplex tableau, excluding the r.h.s., oscillates between $m\times(n+m)$ and $(m+n+1)\times(n+m+n+1)$. Consequently, the sizes of a basis B and its inverse B^{-1} vary between $m\times n$ and $(m+n+1)\times(m+n+1)$. These dimensions follow from the fact that if at any instance the solution to the corresponding LP to P_k is an interior point of P, then all the original variables x, and the slack variables associated with P will be basic, so when a further cut is appended to P_k at least one slack variable corresponding to an appended cut, will be basic (Garfinkel & Nemhauser [2]). If we had not discarded cuts the oscillation would have been even greater.

(ii) In order to strengthen a cut it is necessary to express it in the original x variables. However, this is often a very involved operation. E.g., for the all-integer problem the $(k+1)$-st cut is

$$\left[\{B_{i\cdot}^{-1}\}\begin{pmatrix}A\\A_k\end{pmatrix}\right]x\le\left[\{B_{i\cdot}^{-1}\}\begin{pmatrix}b\\b_k\end{pmatrix}\right],$$

where $[\cdot]$ and $\{\cdot\}$ are the integer and fractional part operators, respectively, and $B_{i\cdot}^{-1}$ is the basic inverse of the source row i on which the cut is generated (Holm & Klein [3]).

(iii) It would be advantageous to be able to make use of a commercial LP code, such as IBM's MPSX or SPERRY's FMPS, as the basis for a cutting plane algorithm, since these codes are very fast, precise, robust, and have a host of inbuilt facilities. However, it is virtually impossible for the normal user of these codes to append a cut to a current solution, mainly since it requires an update of the current B^{-1}, in addition to access to internal files, so the user has to refrain from utilizing these codes.

We will in the following sections show that if we formulate the dual to the problem and solve it by a modified cutting plane algorithm, then all of the adverse properties can be circumvented. Moreover, as a result of the dual formulation it turns out that there exists, for the primal formulation, an alternative way of expressing a cut in the original variables x, which is extremely easy to exhibit.

2. Dual formulations

Consider P_k written in standard form

$$P_k'\qquad \begin{array}{ll} \max & cx+0s+0s_c \\ \text{s.t.} & Ax+Is \qquad\quad = b, \\ & A_kx \qquad + Is_c = b_k, \\ & x,s,s_c\ge 0, \\ & x\in X. \end{array}$$

Consider also the dual formulation of the corresponding LP of P_k written in standard form

$$D'_k \qquad \begin{array}{ll} \min & 0y_s + by + b_k y_c \\ \text{s.t.} & -Iy_s + Ay + A_k y_c = c, \\ & y_s, y, y_c \geq 0 \end{array}$$

where the transpose sign has been omitted.

If B is an optimal basis for the corresponding LP of P'_k and $\tilde{c} = (c, 0, 0)$, then from duality theory we have the following relationships:

$$\begin{aligned} \tilde{c}_B B^{-1} \begin{pmatrix} A \\ A_k \end{pmatrix} - c &= y_s^*, \\ \tilde{c}_B B^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} &= y^*, \\ \tilde{c}_B B^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} &= y_c^* \end{aligned}$$

where (y_s^*, y^*, y_c^*) is an optimal solution to D'_k.

Conversely, if we solved D'_k to optimality with basis $\boldsymbol{B}$, we would have the following relationships:

$$\begin{aligned} \tilde{b}_B \boldsymbol{B}^{-1}(-I) &= -x^*, \\ \tilde{b}_B \boldsymbol{B}^{-1} A - b &= -s^*, \\ \tilde{b}_B \boldsymbol{B}^{-1} A_k - b_k &= -s_c^* \end{aligned}$$

where $\tilde{b} = (0, b, b_k)$ and (x^*, s^*, s_c^*) is an optimal dual solution of the corresponding LP of P'_k.

Thus, instead of solving a sequence of primal problems P'_k, where at each iteration a cut is appended, we could solve a sequence of dual problems D'_k. At each iteration we would solve D'_k and check whether $x^* = \tilde{b}_B \boldsymbol{B}^{-1} \in X$. If this would be the case, then x^* is an optimal solution to the primal problem. If $x^* \notin X$, then we would add an activity which would exclude the current solution but not any solution for which $x \in X$.

3. Properties of the dual formulation

We know from duality theory that if B is an optimal basis to the corresponding LP for P'_k, then there exists a corresponding optimal basis $\boldsymbol{B}$ to its dual D'_k, such that if $i \in B$, then $i \in \boldsymbol{N}$ and for $i \in N$ we have $i \in \boldsymbol{B}$, where N and $\boldsymbol{N}$ are the sets of nonbasic variables for the two problems. When we make a cut from the current tableau in the primal formulation

$$B^{-1}(B \;\; N) \begin{pmatrix} \tilde{x}_B \\ \tilde{x}_N \end{pmatrix} = B^{-1} \begin{pmatrix} b \\ b_k \end{pmatrix}$$

where $\tilde{x}=(x,s,s_c)$, we look for a *basic* variable x_i, which does not fulfill the condition $x_i \in X$, and then we perform some operation $f(\cdot)$ over the *nonbasic* variables and the r.h.s. So the $(k+1)$-st cut to be appended to the current tableau has the form

$$f(B_{i\cdot}^{-1}N)\tilde{x}_N \geq f\left(B_{i\cdot}^{-1}\begin{pmatrix} b \\ b_k \end{pmatrix}\right). \tag{1}$$

E.g., in the case of an integer problem we look at a basic variable $\tilde{x}_{B_i}$ which is not integer:

$$\tilde{x}_{B_i} = B_{i\cdot}^{-1}\begin{pmatrix} b \\ b_k \end{pmatrix} - B_{i\cdot}^{-1}N\tilde{x}_N$$

with $\{\tilde{x}_{B_i}\}>0$, and we append the cut (1) to the current tableau and reoptimize. If we use Gomory's fractional cut, then $f(\cdot)=\{\cdot\}$ and the appended cut becomes

$$-\{B_{i\cdot}^{-1}N\}\tilde{x}_N + s_{c_{k+1}} = -\left\{B_{i\cdot}^{-1}\begin{pmatrix} b \\ b_k \end{pmatrix}\right\}.$$

We have now described how a cut is made and how it is appended to the current solution. This holds true of any cut. What is important to notice is the fact that a cut in the current tableau is a function of the nonbasic variables and the r.h.s. only.

Conversely, when we make an activity from the current tableau in the dual formulation

$$\boldsymbol{B}^{-1}(-I \;\; A \;\; A_k)\begin{bmatrix} y_s \\ y \\ y_c \end{bmatrix} = \boldsymbol{B}^{-1}c$$

with corresponding set of dual variables

$$(\tilde{b}_{\boldsymbol{B}}\boldsymbol{B}^{-1}(-I),\ \tilde{b}_{\boldsymbol{B}}\boldsymbol{B}^{-1}A-b,\ \tilde{b}_{\boldsymbol{B}}\boldsymbol{B}^{-1}A_k-b_k), \tag{2}$$

then we seek to fulfill $x=\tilde{b}_{\boldsymbol{B}}\boldsymbol{B}^{-1}\in X$, so we look for a *nonbasic* variable y_{s_i} which does not fulfill this condition and then we perform some operation $g(\cdot)$ over the corresponding transformed vector and the entry in the $(z-b)$ row (2). So the $(k+1)$-st activity, $y_{c_{k+1}}$, to be added to the current tableau has the form $g(\boldsymbol{B}_{\cdot i}^{-1}(-I))$ and for the entry in the $(z-b)$ row $g(\tilde{b}_{\boldsymbol{B}}\boldsymbol{B}_{\cdot i}^{-1}(-I))$. E.g., in the case of an integer problem we look for a nonbasic variable y_{s_i} for which its corresponding dual variable $x_i = \tilde{b}_{\boldsymbol{B}}\boldsymbol{B}_{\cdot i}^{-1}$ is not integer.

If, moreover, we use Gomory's fractional cut, then $g(\cdot)=\{-\cdot\}$ and the added activity in the current tableau becomes $\{\boldsymbol{B}_{\cdot i}^{-1}\}$ and the entry in the $(z-b)$ row becomes $\{\tilde{b}_{\boldsymbol{B}}\boldsymbol{B}_{\cdot i}^{-1}\}$.

Let us compare the two constructions. From duality theory we have $B^{-1}N = -(\boldsymbol{B}^{-1}\boldsymbol{N})^{\mathrm{t}}$, which together with the observation that

$$B^{-1}\begin{pmatrix} b \\ b_k \end{pmatrix} = -(\tilde{b}_{\boldsymbol{B}}^{\mathrm{t}}\boldsymbol{B}^{-1}\boldsymbol{N}-\tilde{b}_{\boldsymbol{N}}^{\mathrm{t}})^{\mathrm{t}}$$

shows the following equivalence.

Proposition 1. *Let B be the optimal basis to the corresponding LP for P'_k and let* $\boldsymbol{B}$ *be the corresponding optimal basis for D'_k. Then the set of all cuts which can be appended to the primal problem and the set of all activities which can be added to the dual problem are the same.*

This shows that the information contained in the transformed nonbasic vectors for the primal problem, and which is the information needed to make any cut, is the same information which is present in the dual formulation, and the cuts and activities which therefore can be made are exactly the same.

However, the two formulations have different properties. With respect to the adverse properties (i)–(iii) for the primal formulation we observe the following for the dual formulation (we discard an added activity when it becomes nonbasic):

(i′) When we solve D'_k, then the dimension of the simplex tableau, excluding the r.h.s., oscillates between $n \times (m+n)$ and $n \times (m+n+n+1)$. Consequently, $\boldsymbol{B}$ and $\boldsymbol{B}^{-1}$ always have the dimension $n \times n$.

(ii′) The activity expressed in the original primal variables is

$$x^{\mathrm{t}} \boldsymbol{B} g(\boldsymbol{B}^{-1}_{\cdot i}) \leq \bar{b}^{\mathrm{t}}_{\boldsymbol{B}} g(\boldsymbol{B}^{-1}_{\cdot i}) - g(\bar{b}^{\mathrm{t}}_{\boldsymbol{B}} \boldsymbol{B}^{-1}_{\cdot i}). \tag{3}$$

This follows from the fact that $g(\boldsymbol{B}^{-1}_{\cdot i})$, which is added to the current tableau, must be the transformed activity; so $\boldsymbol{B} g(\boldsymbol{B}^{-1}_{\cdot i})$ is the activity in the original dual formulation and by premultiplying this by x^{t} we obtain the primal formulation.

With respect to the r.h.s., the entry in the $(z-b)$ row in the current dual tableau is by definition

$$g(\bar{b}^{\mathrm{t}}_{\boldsymbol{B}} \boldsymbol{B}^{-1}_{\cdot i}) = \bar{b}^{\mathrm{t}}_{\boldsymbol{B}} g(\boldsymbol{B}^{-1}_{\cdot i}) - \bar{b}_{k+1}$$

where $\bar{b}_{k+1}$ is the cost coefficient for the new activity, so the inequality (3) follows.

Observe, that this formulation only requires inner products.

(iii′) It is possible to implement the dual formulation on IBM's MPSX or SPERRY's FMPS, since at each iteration $\boldsymbol{B}^{-1}$ remains constant, when one or more activities are added.

Remark. Although the dual formulation does not have any of the adverse properties the primal formulation exhibits there are cases where the primal formulation may still be the preferred one, namely where the expected number of cuts needed to solve the problem is much less than $n-m$. This is true for many combinatorial problems, such as the travelling salesman problem, the matching problem, etc. These problems are characterized by the fact that $n \gg m$ and that an optimal solution is an extreme point of P, rather than an interior point. For these types of problems the primal formulation may be the preferred one in spite of its adverse properties.

However, from our study of the dual formulation it follows that there is one of the adverse properties which can be alleviated for the primal problem. The dual to D'_k is

$$\begin{aligned} \max \quad & cx \\ \text{s.t.} \quad & -Ix \le 0, \\ & Ax \le b, \\ & A_k x \le b_k, \\ & x \ge 0 \end{aligned}$$

which we will write as

$$\begin{aligned} \max \quad & cx \\ \text{s.t.} \quad & \tilde{A}x \le \tilde{b}, \\ & x \ge 0. \end{aligned}$$

From Proposition 1 we have that $B^{-1}N = -(\boldsymbol{B}^{-1}\boldsymbol{N})^{\mathrm{t}}$.

Let the source row be x_i and therefore the source column is y_i. This implies that

$$B_{i.}^{-1}N = -(\boldsymbol{B}_{.i}^{-1}(-I))^{\mathrm{t}} = (\boldsymbol{B}_{.i}^{-1})^{\mathrm{t}}$$

Moreover, $g^{\mathrm{t}}(\boldsymbol{B}_{i.}^{-1}) = -f(B_{.i}^{-1}N)$, $\boldsymbol{B}^{\mathrm{t}} = \tilde{A}_{N.}$, $\tilde{b}_{\boldsymbol{B}} = \tilde{b}_N$. Therefore, from the dual cut (3) expressed in the original variables it follows that

$$x^{\mathrm{t}}\boldsymbol{B}g(\boldsymbol{B}_{.i}^{-1}) \le \tilde{b}_{\boldsymbol{B}}^{\mathrm{t}} g(\boldsymbol{B}_{.i}^{-1}) - g(\tilde{b}_{\boldsymbol{B}}^{\mathrm{t}}\boldsymbol{B}_{.i}^{-1})$$

so

$$g^{\mathrm{t}}(\boldsymbol{B}_{.i}^{-1})\boldsymbol{B}^{\mathrm{t}}x \le g^{\mathrm{t}}(\boldsymbol{B}_{.i}^{-1})\tilde{b}_{\boldsymbol{B}} - g^{\mathrm{t}}(\tilde{b}_{\boldsymbol{B}}^{\mathrm{t}}\boldsymbol{B}_{.i}^{-1})$$

now, since $\tilde{b}_{\boldsymbol{B}}^{\mathrm{t}}\boldsymbol{B}_{.i}^{-1} = x_i = B_{i.}^{-1}\tilde{b}$ we have proven:

Proposition 2. *Let* $f(B_{i.}^{-1}N)\tilde{x}_N \ge f(B_{i.}^{-1}\binom{b}{b_k})$ *be the* $(k+1)$*-st cut made on the source row* x_i *in the current primal tableau with basis B. Then this cut, expressed in the original primal variables x, is*

$$-f(B_{i.}^{-1}N)\tilde{A}_{N.}\, x \le -f(B_{i.}^{-1}N)\tilde{b}_N + f(B_{i.}^{-1}\tilde{b}).$$

Observe, that in contrast to the hitherto known formulation as given in (ii), this formulation does not involve any other operation than inner products.

The results in this paper have been given for the case where $x \in X$ and can easily be extended to the case where $\tilde{x} \in X$. Moreover, we have also for expository reasons only cut on a single variable x_i. Again the results can easily be extended to the case where we cut on more variables, as is the case for cardinality cuts (Holm [4]).

4. A numerical example

We will in this section compare the primal and dual approaches by going through the first iteration of a small example in detail.

Consider the primal problem P_0' written in standard form

$$
\begin{array}{lll}
P_0' & \max & 2x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 \\
 & \text{s.t.} & x_1 + x_2 + x_3 = 5, \\
 & & -x_1 + x_2 + x_4 = 0, \\
 & & 6x_1 + 2x_2 + x_5 = 21, \\
 & & x \geq 0;\ x_1,\ x_2 \text{ integer.}
\end{array}
$$

The dual formulation D_0' is

$$
\begin{array}{lll}
D_0' & \min & 0y_1 + 0y_2 + 5y_3 + 0y_4 + 21y_5 \\
 & \text{s.t.} & -y_1 + y_3 - y_4 + 6y_5 = 2, \\
 & & -y_2 + y_3 + y_4 + 2y_5 = 1, \\
 & & y \geq 0.
\end{array}
$$

Solving the corresponding LP to P_0' gives the following simplex tableau

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	$-\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{11}{4}$
x_2	0	1	$\frac{3}{2}$	0	$-\frac{1}{4}$	$\frac{9}{4}$
x_4	0	0	-2	1	$\frac{1}{2}$	$\frac{1}{2}$
$z-c$	0	0	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{31}{4}$
	y_1	y_2	y_3	y_4	y_5	

Neither x_1 nor x_2 are integer. We now make Gomory's fractional cut with x_1 as source row:

$$f(B_{1\cdot}^{-1}N)\tilde{x}_N \geq f(B_{1\cdot}^{-1}b)$$

or, since $f(\cdot) = \{\cdot\}$

$$-\{-\tfrac{1}{2} \quad \tfrac{1}{4}\}\begin{pmatrix} x_3 \\ x_5 \end{pmatrix} \leq -\{\tfrac{11}{4}\},$$

so we append the cut

$$-\tfrac{1}{2}x_3 - \tfrac{1}{4}x_5 + x_6 = -\tfrac{3}{4}$$

to the current tableau with $x_6 = s_{c_1}$.

This cut, expressed in the original variables x_1 and x_2, can be derived in two ways:

$$[\{B_{1\cdot}^{-1}\}x \leq [\{B_{1\cdot}^{-1}\}b] \quad \text{or} \quad -f(B_{1\cdot}^{-1}N)\tilde{A}_{N\cdot}x \leq -f(B_{1\cdot}^{-1}N)\tilde{b}_N + f(B_{1\cdot}^{-1}\tilde{b})$$

For the first formulation (Holm & Klein [3]) we get

$$\left[\{-\tfrac{1}{2} \quad 0 \quad \tfrac{1}{4}\}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 6 & 2 \end{bmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \left[\{-\tfrac{1}{2} \quad 0 \quad \tfrac{1}{4}\}\begin{bmatrix} 5 \\ 0 \\ 21 \end{bmatrix}\right]$$

or $2x_1 + x_2 \leq 7$ and the second formulation gives

$$\{-\tfrac{1}{2} \quad \tfrac{1}{4}\}\begin{pmatrix}1 & 1\\ 6 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \le \{-\tfrac{1}{2} \quad \tfrac{1}{4}\}\begin{pmatrix}5\\ 21\end{pmatrix} - \{\tfrac{11}{4}\}$$

or $2x_1 + x_2 \le 7$.

Observe that the second formulation is by far the simplest, since it does not require any other operations than inner products when $f(\cdot)$ is given.

When we solve D_0' to optimality we get

	y_1	y_2	y_3	y_4	y_5	
y_3	$\frac{1}{2}$	$-\frac{3}{2}$	1	2	0	$\frac{1}{2}$
y_5	$-\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{1}{2}$	1	$\frac{1}{4}$
$z-b$	$-\frac{11}{4}$	$-\frac{9}{4}$	0	$-\frac{1}{2}$	0	$\frac{31}{4}$
	$-x_1$	$-x_2$	$-x_3$	$-x_4$	$-x_5$	

We observe that neither x_1 nor x_2 are integer and we make a cut on y_1, corresponding to x_1.

$$g(B_{\cdot 1}^{-1}) = \{-\tfrac{1}{2} \quad \tfrac{1}{4}\}^{\mathrm{t}} = (\tfrac{1}{2} \quad \tfrac{1}{4})^{\mathrm{t}},$$
$$g(\tilde{b}_B^{\mathrm{t}} B_{\cdot 1}^{-1}) = \{\tfrac{11}{4}\}^{\mathrm{t}} = \tfrac{3}{4}$$

which is to be added to the current tableau as the transformed activity $y_{c_1} = y_6$.

This new activity is the dual formulation of the cut appended to the primal problem since

$$x^{\mathrm{t}} B g(B_{\cdot 1}^{-1}) \le \tilde{b}_B^{\mathrm{t}} g(B_{\cdot 1}^{-1}) - g(\tilde{b}_B B_{\cdot 1}^{-1})$$

or

$$(x_1 \quad x_2)\begin{pmatrix}1 & 6\\ 1 & 2\end{pmatrix}\begin{Bmatrix}-\tfrac{1}{2}\\ \tfrac{1}{4}\end{Bmatrix} \le (5 \quad 21)\begin{Bmatrix}-\tfrac{1}{2}\\ \tfrac{1}{4}\end{Bmatrix} - \{\tfrac{11}{4}\},$$

so $2x_1 + x_2 \le 7$.

The activity which is added to the dual formulation has $b_6 = 7$, so the new dual problem becomes

$$\begin{aligned}\min \quad & 0y_1 + 0y_2 + 5y_3 + 0y_4 + 21y_5 + 7y_6\\ \text{s.t.} \quad & -y_1 + y_3 - y_4 + 6y_5 + 2y_6 = 2,\\ & -y_2 + y_3 + y_4 + 2y_5 + y_6 = 1,\\ & y \ge 0.\end{aligned}$$

The two formulations are now reoptimized and cuts and activities are appended as needed to obtain the optimal integer solution.

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