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Vibration of fast trains, palindromic eigenvalue problems and structure-preserving doubling algorithms

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Abstract

The vibration of fast trains is governed by a quadratic palindromic eigenvalue problem $(\lambda^2 A_1^T + \lambda A_0 + A_1)x = 0$, where $A_0, A_1 \in \mathbb{C}^{n \times n}$ and $A_0^T = A_0$. Accurate and efficient solution can only be obtained using algorithms which preserve the structure of the eigenvalue problem. This paper reports on the successful application of the structure-preserving doubling algorithms.

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1. Introduction

Railway travel was the first form of mass transport. After World War II, improvements in automobiles, highways and aircrafts made them practical for a greater portion of the population, especially in the US. However, in the densely populated areas of Europe and Japan, emphasis was given to rebuilding the railways. Due to the rise of the price of petroleum and environmental concerns, railway travel is back in favour. All over the world, railways are being upgraded or built for modern trains running on higher speed.

Consequently, there is tremendous interest in vibration analysis of fast trains, as indicated by the amount of publicity for the associated palindromic eigenvalue problem [14]. The problem was first raised in a study in Germany [10,11], associated with the company SFE GmbH in Berlin. Existing fast train systems, like the Japanese Shinkansen, the French TGV and the German ICE, are being modernized and expanded. Based on these systems, new networks are being planned or built in Europe, the US and Asia.

Vibration is produced from the interaction between the wheels of trains and the rails underneath. Due to the ever increasing speed (currently up to 300 km/h) of modern trains, it is important to study this vibration. This research does not only contribute to the increased comfort of passengers, in terms of lower noise and vibration levels. More

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importantly, the safety in the operation of the trains will be improved and the operational and construction costs optimized [11,14,16,17]. In addition, innovative designs of railway bridges, embedded rail structures (ERS) and train suspension systems require accurate resolution of the vibration.

From the finite element model of the rail sections (see details in Section 2), we need to solve the palindromic eigenvalue problem

$$\mathcal{P}(\lambda)x = 0, \quad x \neq 0, \quad (1)$$

with the matrix quadratic

$$\mathcal{P}(\lambda) \equiv \lambda^2 A_1 + \lambda A_0 + A_1^T, \quad (2)$$

where $A_i \in \mathbb{C}^{n \times n}$ ($i = 0, 1$) and $A_0^T = A_0$. Here A_0 and A_1 are dependent on some parameter ω associated with the speed of the train, the eigenvalues λ are related to the vibration frequencies and the corresponding eigenvectors x reflect the shape of the vibration. The problem is described as *palindromic*¹ as A_1 appears at both ends of (2) and A_0 is symmetric. Consequently, transposition of (1) shows that the set of eigenvalues demonstrates a “symplectic” behaviour (i.e., a symmetry with respect to the unit circle), containing both an eigenvalue λ and its reciprocal.

For the finite element model to work reasonably accurately, n (proportional to the number of elements) has to be large, up to several hundred thousand. To optimize the design of the rail and the train, we need to calculate all $2n$ eigenvalues and their corresponding eigenvectors, for a wide range of speeds (or ω). In addition, (1) is badly scaled, with A_1 being singular in general and λ spreading out evenly. All these conspire to make our problem very difficult and computing-intensive, even for modest values of n .

Previously, the problem (for small values of n up to several thousand) was tackled by existing finite element packages, producing useless answers with very poor accuracy. The only available approach was to convert (or *linearize*) (1) to a problem twice as large. Available algorithms typically transform the larger linearized problem to simpler forms. However, the symplectic structure in (1) is not preserved by these transformations, producing large numerical errors (and no significant digits in the answers; see [11,14] for details).

Recently, a linearization in the form $\lambda Z \pm Z^T$, which preserves symplecticity to some extent, was discovered [11,16,17]. This linearization, coupled with standard software, leads to better but still very low accuracy. This represents a vast improvement over previous attempts but the results are still not accurate enough for the optimization of design parameters [14].

Note that we can transform $\lambda Z - Z^T$ back to the form $v(-Z) + (-Z)^T$ with $v = -\lambda$. Similarly, $\lambda^2 A_1^T + \lambda A_0 + A_1$ and $v^2 A_1^T - v A_0 + A_1$ define the same palindromic eigenvalue problem.

A great foundation for the solution of palindromic eigenvalue problems has been laid in [11,16,17]. There has been much recent interest in quadratic eigenvalue problems [20]. For the vibration analysis of fast trains, see [11,14] for general introductions and [10] for details. For general perturbation of eigenvalues for polynomial eigenvalue problems, see [1]. Some perturbation of palindromic eigenvalue problems can be found in [5]. On results for general matrix polynomials, see the masterpiece [6].

This paper is organized as follows. A finite element model for the vibration analysis of fast trains will be described in Section 2. The deflation of zero and infinite eigenvalues is discussed in Section 3. The structure-preserving doubling algorithms (SDAs) are developed and analysed in Sections 4 and 5. The associated numerical examples will be presented in Section 6 and the paper is concluded in Section 7.

This paper is a preliminary report on the successful application of the SDAs on the quadratic palindromic eigenvalue problem. The problem is difficult, far from being solved (see the concluding comments in Section 7). Our numerical results show much promise but there are theoretical gaps we are aiming to fill. Nevertheless, we feel this report is warranted, in view of the tremendous interest shown to date.

2. Finite element model for vibration of fast trains

We shall study the resonance phenomena of the track under high frequent excitation forces. Research in this area not only contributes to the safety of the operations of high-speed trains but also new designs of train bridges, ERS

¹ Literally “running back again” in Greek, with “palin” meaning “again” and “drom” meaning “run”; as in the palindromes “madam” or “nurses run”.

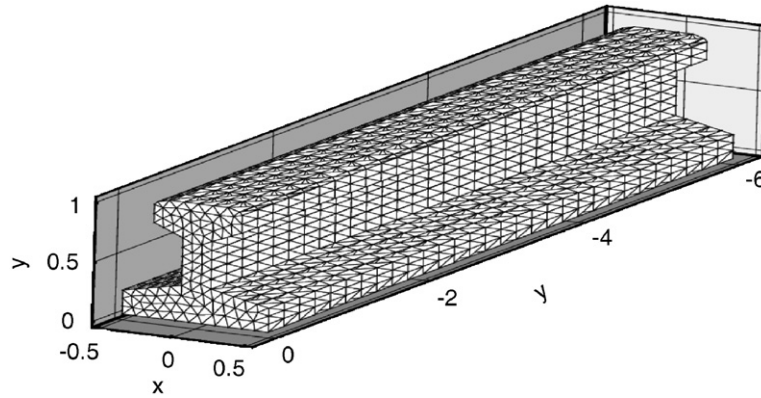


Fig. 1. A 3D rail model.

and train suspension systems. Recently, the dynamic response of the vehicle–rails–bridge interaction system under different train speeds has been studied in [21] and a procedure for designing an optimal ERS is proposed in [18]. An accurate numerical estimation to the resonance frequencies of the rail plays an important role in both works. However, as mentioned in [14], the classic finite element packages fail to deliver correct resonance frequencies for such problems [14]. Here, we would like to compare the method proposed in [16], with the generalized SDA methods proposed in [3,15], in solving the palindromic eigenvalue problems arising from spectral modal analysis of the resonance of the rail under a periodic excitation force.

We assume that the rail sections between consecutive sleeper bays are identical, distances between consecutive wheels are the same and the wheel loads are equal. Fig. 1 shows an example of the rail section we consider here.

Based on our assumptions, we model the rail under cargo wheel loads by a section of rail between two sleepers. The external force is assumed to be periodic and the displacements of two boundary cross sections of the modelled rail are assumed to have a ratio κ , which is dependent on the excitation frequency of the external force. In the following, we consider the rail as a 3D isotropic elastic solid and a 3D finite element model of the solid with linear isoparametric tetrahedron elements is introduced.

From the element virtual work principle, the equilibrium state of the solid element e under external body forces satisfies the following equation:

$$\int_e (\delta \varepsilon^T) C \varepsilon dV + \int_e (\delta q^T) \rho \ddot{q} dV = \int_e (\delta q^T) f dV. \tag{3}$$

Here, ρ is the mass density, f is the time-dependent body force, $q = [u, v, w]$ is the displacement vector, $\varepsilon = [\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}, \frac{\partial w}{\partial z} + \frac{\partial u}{\partial y}]$, and δq^T and $\delta \varepsilon^T$ are the virtual displacement and the corresponding virtual strain vectors, and

$$C = \frac{E}{(1 + \nu)(1 - 2\nu)} \text{diag}(C_1, C_2)$$

is the well-known strain–stress relationship, where E is the Young’s modulus, ν is the Poisson ratio and

$$C_1 = \begin{bmatrix} 1 - \nu & \nu & \nu \\ \nu & 1 - \nu & \nu \\ \nu & \nu & 1 - \nu \end{bmatrix}, \quad C_2 = \left(\frac{1 - 2\nu}{2} \right) I_3.$$

Let ϕ_i and $[u_i, v_i, w_i]^T$ ($i = 1, \dots, 4$) be the linear nodal basis function and the nodal displacement vector associated with the i th node of the element e , respectively, and let $X_e = [X_1^T, X_2^T, X_3^T, X_4^T]^T$, $B_e = [B_1, B_2, B_3, B_4]$ and

$N_e = [N_1, N_2, N_3, N_4]$, where

$$X_i = \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix}, \quad N_i = \begin{bmatrix} \phi_i & 0 & 0 \\ 0 & \phi_i & 0 \\ 0 & 0 & \phi_i \end{bmatrix}, \quad B_i = \begin{bmatrix} \frac{\partial \phi_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial \phi_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial \phi_i}{\partial z} \\ \frac{\partial \phi_i}{\partial y} & \frac{\partial \phi_i}{\partial x} & 0 \\ \frac{\partial \phi_i}{\partial z} & 0 & \frac{\partial \phi_i}{\partial x} \\ 0 & \frac{\partial \phi_i}{\partial z} & \frac{\partial \phi_i}{\partial y} \end{bmatrix}.$$

Eq. (3) can now be discretized into the following linear equations:

$$\sum_e \left(\int_e B_e^T C B_e dV \right) X_e + \rho \left(\int_e N_e^T N_e dV \right) \ddot{X}_e = \sum_e \left(\int_e N_e^T N_e dV \right) F_e, \tag{4}$$

where $F_e = [F_1^T, F_2^T, F_3^T, F_4^T]^T$ and F_i ($i = 1, \dots, 4$) is the i th nodal force vector acting on element e . In the following, we denote $K = \sum_e \int_e B_e^T C B_e dV$, and $M = \sum_e \rho \int_e N_e^T N_e dV$. Eq. (4) can now be written as

$$KX + M\ddot{X} = \rho^{-1}MF. \tag{5}$$

When considering the dynamic response of the solid, dissipative forces such as the force due to frictions have to be considered. Their effect is introduced in the form of the so-called viscous damping $D\dot{X}$ where D is the damping matrix. In this paper, proportional damping proposed in [19] is employed where D is a linear combination of K and M . The equation of motion involving viscous damping can now be written as

$$KX + D\dot{X} + M\ddot{X} = \rho^{-1}MF.$$

Due to the given boundary conditions on a uniform mesh, K , D and M have the following form:

$$\begin{bmatrix} G_{11} & G_{12} & 0 & \cdots & 0 & \frac{1}{\kappa} G_{m,m+1}^T \\ G_{12}^T & G_{22} & G_{23} & 0 & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & G_{m-2,m-1}^T & G_{m-1,m-1} & G_{m-1,m} \\ \kappa G_{m,m+1} & 0 & \cdots & 0 & G_{m-1,m}^T & G_{m,m} \end{bmatrix}$$

with $G_{ii} \in \mathbb{C}^{n_i \times n_i}$ for $i = 1, \dots, m$. Furthermore, from the spectral modal analysis, one considers $X = \hat{X}e^{i\omega t}$ where ω is the frequency of the external excitation force and \hat{X} is the corresponding eigenmode. Consequently, we arrive to a palindromic eigenvalue problem

$$(\kappa A_1 + A_0 + \kappa^{-1}A_1^T)\hat{X} = 0,$$

where $A_0, A_1 \in \mathbb{C}^{n \times n}$ with $n = n_1 + \dots + n_m$ and

$$[A_1]_{ij} = \begin{cases} K_{m,m+1} + i\omega D_{m,m+1} - \omega^2 M_{m,m+1} & (\text{if } i = m \text{ and } j = 1), \\ 0 & (\text{otherwise}), \end{cases} \tag{6}$$

$$[A_0]_{ij} = \begin{cases} K_{i,j} + i\omega D_{i,j} - \omega^2 M_{i,j} & (\text{if } i - 1 \leq j \leq i + 1), \\ 0 & (\text{otherwise}). \end{cases} \tag{7}$$

3. Deflation

We shall consider the deflation of zero and infinite eigenvalues in this section. For the deflation of $\lambda = \pm 1$, consult [16].

From their definitions in (6), A_1 and A_0 can be partitioned as

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ L & 0 & 0 \end{bmatrix} \in \mathbb{C}^{n \times n}, \quad A_0 = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12}^T & C_{22} & C_{23} \\ 0 & C_{23}^T & C_{33} \end{bmatrix} \in \mathbb{C}^{n \times n},$$

where $L \in \mathbb{C}^{n_m \times n_1}$, $C_{11} = C_{11}^T \in \mathbb{C}^{n_1 \times n_1}$, $C_{33} = C_{33}^T \in \mathbb{C}^{n_m \times n_m}$ and $C_{22} = C_{22}^T \in \mathbb{C}^{\ell \times \ell}$ with $\ell = n - n_1 - n_m$. Assume that C_{22} is nonsingular. We have observed that this assumption is generically valid from the numerical examples we have encountered. Otherwise, the preprocessing procedure in [11,16] should be applied.

Let

$$\Theta = \begin{bmatrix} I_{n_1} & -C_{12}C_{22}^{-1} & 0 \\ 0 & I_\ell & 0 \\ 0 & -C_{23}^TC_{22}^{-1} & I_{n_m} \end{bmatrix}, \quad \Pi = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & I_{n_m} \\ 0 & I_\ell & 0 \end{bmatrix}.$$

Then, using a similarity transformation, $\mathcal{P}(\lambda)$ can be transferred to the following form:

$$\begin{aligned} \Pi\Theta\mathcal{P}(\lambda)\Theta^T\Pi^T &= \begin{bmatrix} \lambda(C_{11} - C_{12}C_{22}^{-1}C_{12}^T) & L^T - \lambda C_{12}C_{22}^{-1}C_{23} & 0 \\ \lambda(\lambda L - C_{23}^TC_{22}^{-1}C_{12}^T) & \lambda(C_{33} - C_{23}^TC_{22}^{-1}C_{23}) & 0 \\ 0 & 0 & \lambda C_{22} \end{bmatrix} \\ &= \text{diag}(I_{n_1}, \lambda I_{n_m}, I_\ell) \begin{bmatrix} \mathcal{S}(\lambda) & 0 \\ 0 & \lambda C_{22} \end{bmatrix}, \end{aligned} \tag{8}$$

where

$$\mathcal{S}(\lambda) = \begin{bmatrix} \lambda\tilde{C}_{11} & L^T - \lambda\tilde{C}_{12} \\ \lambda L - \tilde{C}_{12}^T & \tilde{C}_{22} \end{bmatrix}$$

with $\tilde{C}_{11} \equiv C_{11} - C_{12}C_{22}^{-1}C_{12}^T$, $\tilde{C}_{12} \equiv C_{12}C_{22}^{-1}C_{23}$ and $\tilde{C}_{22} \equiv C_{33} - C_{23}^TC_{22}^{-1}C_{23}$.

Lemma 3.1. Let $A_{i,i+1} = [A_0]_{i,i+1}$ for $i = 1, \dots, m - 1$ defined in (7) and L has full row rank. Then all of the eigenvalues of $\mathcal{S}(\lambda)$ are nonzero.

Proof. Let u_1 and u_2 be vectors so that

$$0 = \mathcal{S}(0) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} L^Tu_2 \\ -C_{23}^TC_{22}^{-1}C_{12}^Tu_1 + (C_{33} - C_{23}^TC_{22}^{-1}C_{23})u_2 \end{bmatrix}.$$

Since L has full row rank, it implies that $u_2 = 0$ and then

$$C_{23}^TC_{22}^{-1}C_{12}^Tu_1 = 0. \tag{9}$$

Let

$$\begin{bmatrix} v_1 \\ \vdots \\ v_{m-2} \end{bmatrix} \equiv C_{22}^{-1}C_{12}^Tu_1. \tag{10}$$

Substituting $C_{23}^T = [0 \ \cdots \ 0 \ A_{m-1,m}^T]$ and (10) into (9), we get $A_{m-1,m}^T v_{m-2} = 0$. Since $A_{m-1,m}$ has full row rank, it implies that $v_{m-2} = 0$. From (10),

$$\begin{aligned} \begin{bmatrix} A_{12}^T u_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} &= C_{12}^T u_1 = C_{22} \begin{bmatrix} v_1 \\ \vdots \\ v_{m-3} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} A_{22}v_1 + A_{23}v_2 \\ A_{23}^T v_1 + A_{33}v_2 + A_{34}v_3 \\ \vdots \\ A_{m-4,m-3}^T v_{m-5} + A_{m-3,m-3}v_{m-4} + A_{m-3,m-2}v_{m-3} \\ A_{m-3,m-2}^T v_{m-4} + A_{m-2,m-2}v_{m-3} \\ A_{m-2,m-1}^T v_{m-3} \end{bmatrix}, \end{aligned}$$

where $A_{ii} = [A_0]_{ii}$ for $i = 1, \dots, m$. Comparing the coefficients of above vectors, we have

$$0 = A_{m-2,m-1}^T v_{m-3}, \tag{11a}$$

$$0 = A_{m-3,m-2}^T v_{m-4} + A_{m-2,m-2}v_{m-3}, \tag{11b}$$

$$0 = A_{m-4,m-3}^T v_{m-5} + A_{m-3,m-3}v_{m-4} + A_{m-3,m-2}v_{m-3}, \tag{11c}$$

\vdots

$$0 = A_{23}^T v_1 + A_{33}v_2 + A_{34}v_3, \tag{11d}$$

$$A_{12}^T u_1 = A_{22}v_1 + A_{23}v_2. \tag{11e}$$

By the assumption that $A_{i,i+1}$ for $i = 2, \dots, m - 2$ has full row rank and from (11a)–(11d), we have $v_{m-3} = \dots = v_1 = 0$. Substituting $v_1 = v_2 = 0$ into (11e) and using the assumption that A_{12} has full row rank, $u_1 = 0$ which implies that $\mathcal{S}(0)$ is nonsingular without zero eigenvalue. \square

From (8), we have the following result for the relationship between eigenpairs of $\mathcal{S}(\lambda)$ and those of $\mathcal{P}(\lambda)$.

Lemma 3.2. *Let $[x^T, y^T]^T$ be an eigenvector of $\mathcal{S}(\lambda)$. Then*

$$\Theta_1^T \Pi_2^T \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ -C_{22}^{-1}(C_{12}^T x + C_{23}y) \\ y \end{bmatrix}$$

is an eigenvector of $\mathcal{P}(\lambda)$. Furthermore, $\sigma(\mathcal{P}(\lambda)) = \sigma(\mathcal{S}(\lambda)) \cup \{0, \infty\}$.

4. Structure-preservation algorithms

4.1. SDAI

After swapping the row-blocks, the pencil $\mathcal{S}(\lambda)$ is equivalent to

$$\lambda \begin{bmatrix} L & 0 \\ \tilde{C}_{11} & -\tilde{C}_{12} \end{bmatrix} + \begin{bmatrix} -\tilde{C}_{12}^T & \tilde{C}_{22} \\ 0 & L^T \end{bmatrix},$$

which is in a generalized standard symplectic form (GSSF) [12]. The structure-preserving doubling algorithm (SDA1) in [12] can then be applied to solve the corresponding eigenvalue problem. During the iteration, some matrices are required to be well-conditioned. When this does not hold, Cayley transforms can be applied to transform the corresponding symplectic matrix pair to an associated Hamiltonian matrix and then back, introducing free two parameters against which this condition can be optimized. For details, see [2].

In terms of accuracy and speed of convergence, SDA1 behaves similarly as SDA2 below. However, the operation count for SDA1 doubles that of SDA2. As a result, we shall not discuss SDA1 further. However, SDA1 is a weapon in reserve against difficult palindromic eigenvalue problems, when some assumptions for SDA2 are not satisfied.

4.2. Palindromic linearization and QZ

Assume that \tilde{C}_{22} is invertible. Define a new λ -matrix $\tilde{\mathcal{P}}(\lambda)$ as follows:

$$\begin{aligned} \tilde{\mathcal{P}}(\lambda) &\equiv \begin{bmatrix} I_{n_1} & -L^T \tilde{C}_{22}^{-1} \\ 0 & I_{n_3} \end{bmatrix} \mathcal{P}(\lambda) \begin{bmatrix} I_{n_1} & 0 \\ \tilde{C}_{22}^{-1} \tilde{C}_{12}^T & I_{n_3} \end{bmatrix} \\ &= \begin{bmatrix} \lambda(\tilde{C}_{11} - L^T \tilde{C}_{22}^{-1} L - \tilde{C}_{12} \tilde{C}_{22}^{-1} \tilde{C}_{12}^T) + L^T \tilde{C}_{22}^{-1} \tilde{C}_{12}^T & -\lambda \tilde{C}_{12} \\ & \lambda L \\ & & \tilde{C}_{22} \end{bmatrix} \end{aligned} \tag{12}$$

and let $[\tilde{x}^T, \tilde{y}^T]^T$ be an eigenvector of $\tilde{\mathcal{P}}(\lambda)$; i.e.,

$$\lambda[(\tilde{C}_{11} - L^T \tilde{C}_{22}^{-1} L - \tilde{C}_{12} \tilde{C}_{22}^{-1} \tilde{C}_{12}^T)\tilde{x} - \tilde{C}_{12}\tilde{y}] + L^T \tilde{C}_{22}^{-1} \tilde{C}_{12}^T \tilde{x} = 0, \tag{13a}$$

$$\lambda L \tilde{x} + \tilde{C}_{22} \tilde{y} = 0. \tag{13b}$$

Since \tilde{C}_{22} is invertible, from (13b) \tilde{y} can be represented as

$$\tilde{y} = -\lambda \tilde{C}_{22}^{-1} L \tilde{x}. \tag{14}$$

Substituting (14) into (13a), we get the following new small size palindromic quadratic eigenvalue problem:

$$\mathcal{P}_d(\lambda)\tilde{x} \equiv (\lambda^2 A_{d1} + \lambda A_{d0} + A_{d1}^T)\tilde{x} = 0, \tag{15}$$

where

$$A_{d1} = \tilde{C}_{12} \tilde{C}_{22}^{-1} L \in \mathbb{C}^{n_1 \times n_1}, \tag{16}$$

$$A_{d0} = \tilde{C}_{11} - L^T \tilde{C}_{22}^{-1} L - \tilde{C}_{12} \tilde{C}_{22}^{-1} \tilde{C}_{12}^T \in \mathbb{C}^{n_1 \times n_1}. \tag{17}$$

Assume that $\lambda \neq -1$ (or the eigenvalue can be deflated as in [16]). From (15), we get

$$\begin{bmatrix} A_{d1}^T & A_{d0} \\ 0 & A_{d1}^T \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \lambda \tilde{x} \end{bmatrix} = \lambda \begin{bmatrix} 0 & -A_{d1} \\ A_{d1}^T & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \lambda \tilde{x} \end{bmatrix}, \tag{18}$$

$$\begin{bmatrix} A_{d1} & 0 \\ A_{d0} & A_{d1} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \lambda \tilde{x} \end{bmatrix} = -\frac{1}{\lambda} \begin{bmatrix} 0 & -A_{d1} \\ A_{d1}^T & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \lambda \tilde{x} \end{bmatrix}. \tag{19}$$

Adding (18) and subtracting (19) with the following equation:

$$\begin{bmatrix} 0 & -A_{d1} \\ A_{d1}^T & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \lambda \tilde{x} \end{bmatrix} = \begin{bmatrix} 0 & -A_{d1} \\ A_{d1}^T & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \lambda \tilde{x} \end{bmatrix},$$

we get

$$\begin{bmatrix} A_{d1}^T & A_{d0} - A_{d1} \\ A_{d1}^T & A_{d1}^T \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \lambda \tilde{x} \end{bmatrix} = (\lambda + 1) \begin{bmatrix} 0 & -A_{d1} \\ A_{d1}^T & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \lambda \tilde{x} \end{bmatrix}, \tag{20}$$

$$\begin{bmatrix} A_{d1} & A_{d1} \\ A_{d0} - A_{d1}^T & A_{d1} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \lambda \tilde{x} \end{bmatrix} = -\frac{\lambda + 1}{\lambda} \begin{bmatrix} 0 & -A_{d1} \\ A_{d1}^T & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \lambda \tilde{x} \end{bmatrix}. \tag{21}$$

Substituting (20) into (21), then the palindromic quadratic eigenvalue problem in (15) can be linearized as the following generalized palindromic eigenvalue problem:

$$\begin{bmatrix} A_{d1}^T & A_{d0} - A_{d1} \\ A_{d1}^T & A_{d1}^T \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \lambda \tilde{x} \end{bmatrix} + \lambda \begin{bmatrix} A_{d1} & A_{d1} \\ A_{d0} - A_{d1}^T & A_{d1} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \lambda \tilde{x} \end{bmatrix} = 0. \tag{22}$$

The standard QZ algorithm [7] can then be applied to the above palindromic linearization.

Substituting \tilde{x} into (14), eigenvector $[\tilde{x}^T, \tilde{y}^T]^T$ of $\tilde{\mathcal{P}}(\lambda)$ can be obtained. From (12), the following result is easily obtained.

Lemma 4.1. *Let $[\tilde{x}^T, \tilde{y}^T]^T$ be an eigenvector of $\tilde{\mathcal{P}}(\lambda)$. Then*

$$\begin{bmatrix} x \\ y \end{bmatrix} \equiv \begin{bmatrix} I_{n_1} & 0 \\ \tilde{C}_{22}^{-1} \tilde{C}_{12}^T & I_{n_m} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \tilde{x} \\ \tilde{C}_{22}^{-1} \tilde{C}_{12}^T \tilde{x} + \tilde{y} \end{bmatrix}$$

is an eigenvector of $\mathcal{P}(\lambda)$.

Now, we summary above precesses with Lemmas 3.2 and 4.1 in the following algorithm.

Algorithm 4.1 (QZ for palindromic QEP).

Input: $C_{11}, C_{22}, C_{33}, C_{12}, C_{23}, L; \tau$ (a small tolerance);

Output: an eigenpair $(\lambda, [x^T, z^T, y^T]^T)$ of Palindromic QEP.

Compute $\tilde{C}_{11} = C_{11} - C_{12} C_{22}^{-1} C_{12}^T$,

$\tilde{C}_{12} = C_{12} C_{22}^{-1} C_{23}$,

$\tilde{C}_{22} = C_{33} - C_{23}^T C_{22}^{-1} C_{23}$,

$A_{d1} = \tilde{C}_{12} \tilde{C}_{22}^{-1} L$,

$A_{d0} = \tilde{C}_{11} - L^T \tilde{C}_{22}^{-1} L - \tilde{C}_{12} \tilde{C}_{22}^{-1} \tilde{C}_{12}^T$;

Set $Z = \begin{bmatrix} A_{d1}^T & A_{d0} - A_{d1} \\ A_{d1}^T & A_{d1}^T \end{bmatrix}$;

Compute the eigenpair (λ, \hat{x}) of $Z\hat{x} + \lambda Z^T \hat{x} = 0$;

Set $\tilde{x} = \hat{x}(1 : n)$; Solve $\tilde{C}_{22} \tilde{y} = -\lambda L \tilde{x}$;

Set $x = \tilde{x}$; Compute $y = \tilde{C}_{22}^{-1} \tilde{C}_{12}^T \tilde{x} + \tilde{y}$, $z = -C_{22}^{-1} (C_{12}^T x + C_{23} y)$;

4.3. SDA2

Suppose that X is nonsingular. Rewrite $\mathcal{P}_d(\lambda)$ in (15) as

$$\mathcal{P}_d(\lambda) = (\lambda A_{d1} - X) X^{-1} (\lambda X - A_{d1}^T) + \lambda (A_{d1} X^{-1} A_{d1}^T + X + A_{d0}).$$

It follows that $\mathcal{P}_d(\lambda)$ can be factorized (or square-rooted) as

$$\mathcal{P}_d(\lambda) = (\lambda A_{d1} - X) X^{-1} (\lambda X - A_{d1}^T)$$

for some nonsingular X if and only if X is satisfied the following nonlinear matrix equation with the plus sign (NME):

$$A_{d1} X^{-1} A_{d1}^T + X + A_{d0} = 0. \tag{23}$$

We can easily prove the following lemma on the existence of the solutions of the NME:

Lemma 4.2. *Let $(A_1 \oplus A_2, [Y_1, Y_2])$ be an eigenpair of $\mathcal{P}_d(\lambda)$ in the sense that*

$$A_{d1} Y_i A_i^2 + A_{d0} Y_i A_i + A_{d1}^T Y_i = 0 \quad (i = 1, 2),$$

where $Y_i \in \mathbb{C}^{n_1 \times n_1}$ for $i = 1, 2$. Suppose that A_{d1} and Y_i ($i = 1, 2$) are invertible. Then the corresponding NME (23) has the solutions $X = A_{d1}^T Y_i A_i^{-1} Y_i^{-1}$ ($i = 1, 2$).

Evidently, there are many solutions to the NME, each will facilitate the factorization of $\mathcal{P}_d(\lambda)$ we aim for. Assume that there are no eigenvalues on the unit circle. Consequently, we can partition the spectrum into $A_s \oplus A_s^{-1}$, with A_s containing the stable eigenvalues (inside the unit circle). The SDA will seek a stable solution $X_s \equiv A_{d1}^T Y_s A_s^{-1} Y_s^{-1}$, where Y_s contains the eigenvectors corresponding to A_s . Note that X_s is unique as it is independent of the order of the eigenvalues in A_s .

The structure-preserving doubling algorithm (SDA2) in [15] can then be applied to solve the NME, and subsequently the palindromic eigenvalue problem.

Algorithm 4.2 (SDA for palindromic QEP).

Input: $C_{11}, C_{22}, C_{33}, C_{12}, C_{23}, L; \tau$ (a small tolerance);

Output: an eigenpair $(\lambda, [x^T, z^T, y^T]^T)$ of Palindromic QEP.

Compute $\tilde{C}_{11} = C_{11} - C_{12}C_{22}^{-1}C_{12}^T$,
 $\tilde{C}_{12} = C_{12}C_{22}^{-1}C_{23}$,
 $\tilde{C}_{22} = C_{33} - C_{23}^T C_{22}^{-1}C_{23}$,
 $A_{d1} = \tilde{C}_{12}\tilde{C}_{22}^{-1}L$,
 $A_{d0} = \tilde{C}_{11} - L^T\tilde{C}_{22}^{-1}L - \tilde{C}_{12}\tilde{C}_{22}^{-1}\tilde{C}_{12}^T$;

Set $k = 0, R_k = A_{d1}^T, Q_k = -A_{d0}$ and $P_k = 0$;

Do until convergence:

 Compute $R_{k+1} = R_k(Q_k - P_k)^{-1}R_k$,
 $Q_{k+1} = Q_k - R_k^T(Q_k - P_k)^{-1}R_k$,
 $P_{k+1} = P_k + R_k(Q_k - P_k)^{-1}R_k^T, k = k + 1$;

 If $\|Q_k - Q_{k-1}\| \leq \tau\|Q_k\|$, Stop;

End;

Compute the left/right eigenpairs $(\lambda_u, \tilde{x}_s), (\lambda_u, \hat{x}_r)$ of $Q_k \hat{x} = \lambda A_{d1} \hat{x}$;

Solve $(\lambda_u Q_k - A_{d1}^T)\tilde{x}_u = Q_k \hat{x}_r$; Set $\lambda_s = \lambda_u^{-1}$;

Solve $\tilde{C}_{22}\tilde{y} = -\lambda L\tilde{x}$ with $(\lambda, \tilde{x}) = (\lambda_s, \tilde{x}_s)$ or $(\lambda, \tilde{x}) = (\lambda_u, \tilde{x}_u)$;

Set $x = \tilde{x}$; Compute $y = \tilde{C}_{22}^{-1}\tilde{C}_{12}^T\tilde{x} + \tilde{y}, z = -C_{22}^{-1}(C_{12}^T x + C_{23}y)$;

In the above algorithm, we require the invertibility of the matrices $Q_k - P_k$. This is the case for large values of k , as indicated by Corollary 1. We have so far encountered no difficulties in our numerical experiments.

We would also like to point out that the SDA can be shown to be equivalent to applying cyclic reduction [8] to the resolvent equation $A_{d1}X^2 + A_{d0}X + A_{d1}^T = 0$.

Finally, equations similar to (23), for real matrices or with the transpose replaced by Hermitian, have been studied and existing results do not apply. The lack of positivity of matrices or an associated inner product make our NME difficult to study.

5. Convergence of algorithms

The behaviour of the SDAs is well-documented in [2,3,9,12,13,15]. However, these results are mostly written for real problem with real variables and have to be modified for our situation. Following the development in [15], let $\mathcal{M} - \lambda\mathcal{L} \in \mathbb{C}^{2n \times 2n}$ be a *T-symplectic pencil*, in the sense that

$$\mathcal{M}J\mathcal{M}^T = \mathcal{L}J\mathcal{L}^T, \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}. \tag{24}$$

Define the nonempty null set

$$\mathcal{N}(\mathcal{M}, \mathcal{L}) \equiv \left\{ [\mathcal{M}_*, \mathcal{L}_*] : \mathcal{M}_*, \mathcal{L}_* \in \mathbb{C}^{2n \times 2n}, \text{rank}[\mathcal{M}_*, \mathcal{L}_*] = 2n, [\mathcal{M}_*, \mathcal{L}_*] \begin{bmatrix} \mathcal{L} \\ -\mathcal{M} \end{bmatrix} = 0 \right\}.$$

For any given $[\mathcal{M}_*, \mathcal{L}_*] \in \mathcal{N}(\mathcal{M}, \mathcal{L})$, define

$$\widehat{\mathcal{M}} = \mathcal{M}_* \mathcal{M}, \quad \widehat{\mathcal{L}} = \mathcal{L}_* \mathcal{L}.$$

The transformation $\mathcal{M} - \lambda\mathcal{L} \rightarrow \widehat{\mathcal{M}} - \lambda\widehat{\mathcal{L}}$ is a doubling transformation. Below is an adaptation of [15, Theorem 2.1]:

Theorem 5.1. *Let $\widehat{\mathcal{M}} - \lambda\widehat{\mathcal{L}}$ be a doubling transformation of a T-symplectic pencil $\mathcal{M} - \lambda\mathcal{L}$. Then we have:*

- (a) *The pencil $\widehat{\mathcal{M}} - \lambda\widehat{\mathcal{L}}$ is still T-symplectic.*
- (b) *If $\mathcal{M} \begin{bmatrix} U \\ V \end{bmatrix} = \mathcal{L} \begin{bmatrix} U \\ V \end{bmatrix} S$, where $U, V \in \mathbb{C}^{n \times m}$ and $S \in \mathbb{C}^{m \times m}$, then*

$$\widehat{\mathcal{M}} \begin{bmatrix} U \\ V \end{bmatrix} = \widehat{\mathcal{L}} \begin{bmatrix} U \\ V \end{bmatrix} S^2.$$

- (c) *If the pencil $\mathcal{M} - \lambda\mathcal{L}$ has the Kronecker canonical form*

$$W\mathcal{M}Z = \begin{bmatrix} J_r & 0 \\ 0 & I_{2n-r} \end{bmatrix}, \quad W\mathcal{L}Z = \begin{bmatrix} I_r & 0 \\ 0 & N_{2n-r} \end{bmatrix}, \tag{25}$$

where W, Z are nonsingular, J_r a Jordan matrix corresponding to the finite eigenvalues of $\mathcal{M} - \lambda\mathcal{L}$ and N_{2n-r} a nilpotent Jordan matrix corresponding to the infinite eigenvalues of $\mathcal{M} - \lambda\mathcal{L}$, then there exists a nonsingular matrix \widehat{W} such that

$$\widehat{W}\widehat{\mathcal{M}}Z = \begin{bmatrix} J_r^2 & 0 \\ 0 & I_{2n-r} \end{bmatrix}, \quad \widehat{W}\widehat{\mathcal{L}}Z = \begin{bmatrix} I_r & 0 \\ 0 & N_{2n-r}^2 \end{bmatrix}. \tag{26}$$

Proof. (a) Since $[\mathcal{M}_*, \mathcal{L}_*] \in \mathcal{N}(\mathcal{M}, \mathcal{L})$ implies that $\mathcal{M}_*\mathcal{L} = \mathcal{L}_*\mathcal{M}$, it follows from (24) that

$$\begin{aligned} \widehat{\mathcal{M}}J\widehat{\mathcal{M}}^T &= \mathcal{M}_*J\mathcal{M}^T\mathcal{M}_*^T = \mathcal{M}_*\mathcal{L}J\mathcal{L}^T\mathcal{M}_*^T \\ &= (\mathcal{L}_*\mathcal{M})J(\mathcal{M}^T\mathcal{L}_*^T) = \mathcal{L}_*\mathcal{L}J\mathcal{L}^T\mathcal{L}_*^T = \widehat{\mathcal{L}}J\widehat{\mathcal{L}}^T \end{aligned}$$

implying that $\widehat{\mathcal{M}} - \lambda\widehat{\mathcal{L}}$ is T-symplectic.

- (b) Again using $\mathcal{M}_*\mathcal{L} = \mathcal{L}_*\mathcal{M}$, we have

$$\mathcal{M}_*\mathcal{L} \begin{bmatrix} U \\ V \end{bmatrix} = \mathcal{L}_*\mathcal{M} \begin{bmatrix} U \\ V \end{bmatrix} S,$$

and hence

$$\begin{aligned} \widehat{\mathcal{M}} \begin{bmatrix} U \\ V \end{bmatrix} &= \mathcal{M}_*\mathcal{M} \begin{bmatrix} U \\ V \end{bmatrix} = \mathcal{M}_*\mathcal{L} \begin{bmatrix} U \\ V \end{bmatrix} S \\ &= \mathcal{L}_*\mathcal{M} \begin{bmatrix} U \\ V \end{bmatrix} S = \mathcal{L}_*\mathcal{L} \begin{bmatrix} U \\ V \end{bmatrix} S^2 = \widehat{\mathcal{L}} \begin{bmatrix} U \\ V \end{bmatrix} S^2. \end{aligned}$$

- (c) Let

$$\widetilde{\mathcal{M}}_* = \begin{bmatrix} J_r & 0 \\ 0 & I_{2n-r} \end{bmatrix} W, \quad \widetilde{\mathcal{L}}_* = \begin{bmatrix} I_r & 0 \\ 0 & N_{2n-r} \end{bmatrix} W.$$

We then have

$$\begin{aligned} \widetilde{\mathcal{M}}_*\mathcal{L}Z &= \begin{bmatrix} J_r & 0 \\ 0 & I_{2n-r} \end{bmatrix} W\mathcal{L}Z = \begin{bmatrix} J_r & 0 \\ 0 & I_{2n-r} \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & N_{2n-r} \end{bmatrix} = \begin{bmatrix} J_r & 0 \\ 0 & N_{2n-r} \end{bmatrix}, \\ \widetilde{\mathcal{L}}_*\mathcal{M}Z &= \begin{bmatrix} I_r & 0 \\ 0 & N_{2n-r} \end{bmatrix} W\mathcal{M}Z = \begin{bmatrix} I_r & 0 \\ 0 & N_{2n-r} \end{bmatrix} \begin{bmatrix} J_r & 0 \\ 0 & I_{2n-r} \end{bmatrix} = \begin{bmatrix} J_r & 0 \\ 0 & N_{2n-r} \end{bmatrix}. \end{aligned}$$

As Z is nonsingular, this implies that $[\widetilde{\mathcal{M}}_*, \widetilde{\mathcal{L}}_*] \in \mathcal{N}(\mathcal{M}, \mathcal{L})$. Notice that (25) implies that $\mathcal{M} - \lambda\mathcal{L}$ is regular and $\text{rank} \begin{bmatrix} \mathcal{M} \\ \mathcal{L} \end{bmatrix} = 2n$. Thus $[\mathcal{M}_*, \mathcal{L}_*]$ and $[\widetilde{\mathcal{M}}_*, \widetilde{\mathcal{L}}_*]$ form two different bases of the null set $\mathcal{N}(\mathcal{M}, \mathcal{L})$. Consequently, there exists a nonsingular matrix \widetilde{W} such that $[\widetilde{\mathcal{M}}_*, \widetilde{\mathcal{L}}_*] = \widetilde{W}[\mathcal{M}_*, \mathcal{L}_*]$. It can be verified easily that \widetilde{W} satisfies (26). \square

It is easy to verify that NME (23) has a symmetric nonsingular solution X if and only if X satisfies

$$\mathcal{M} \begin{bmatrix} I \\ X \end{bmatrix} = \mathcal{L} \begin{bmatrix} I \\ X \end{bmatrix} S$$

for some $S \in \mathbb{C}^{n \times n}$, where

$$\mathcal{M} \equiv \begin{bmatrix} A_{d1}^T & 0 \\ -A_{d0} & -I \end{bmatrix}, \quad \mathcal{L} \equiv \begin{bmatrix} 0 & I \\ A_{d1} & 0 \end{bmatrix}.$$

Note that $\mathcal{M} - \lambda \mathcal{L}$ is in the second standard symplectic form (SSF-2) [15].

For the convergence of the SDA, we have the following adaptation of [15, Theorem 4.1]:

Theorem 5.2. *Let X be a symmetric invertible solution of (23) and let $S = X^{-1} A_{d1}$. Then the matrix sequences $\{R_k\}$, $\{Q_k\}$ and $\{P_k\}$ generated by Algorithm 4.2 satisfy*

- (a) $R_k = (X - P_k)S^{2^k}$;
- (b) $Q_k - P_k = (X - P_k) + R_k^T(X - P_k)^{-1}R_k$;
- (c) $Q_k - X = (S^T)^{2^k}(X - P_k)S^{2^k}$;

provided that all the required inverses of $Q_k - P_k$ exist.

Proof. We shall apply mathematical induction. Denote

$$\mathcal{M}_k \equiv \begin{bmatrix} R_k & 0 \\ Q_k & -I \end{bmatrix}, \quad \mathcal{L}_k \equiv \begin{bmatrix} -P_k & I \\ R_k^T & 0 \end{bmatrix},$$

with $R_0 = A_{d1}^T$, $Q_0 = -A_{d0}$ and $P_0 = 0$.

For $k = 1$, the NME in (23) implies the invertibility of

$$\begin{bmatrix} X & R_0 \\ R_0^T & Q_0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ R_0^T X^{-1} & I \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} I & X^{-1}R_0 \\ 0 & I \end{bmatrix}.$$

Further computation yields

$$\begin{bmatrix} I & -R_0 Q_0^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} X & R_0 \\ R_0^T & Q_0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -Q_0^{-1} R_0^T & I \end{bmatrix} = \begin{bmatrix} X - R_0 Q_0^{-1} R_0^T & 0 \\ 0 & Q_0 \end{bmatrix}.$$

Consequently,

$$X - P_1 = X - R_0 Q_0^{-1} R_0^T \tag{27}$$

is invertible, as required in (b).

From (23), it is easy to verify that X satisfies

$$\mathcal{M}_0 \begin{bmatrix} I \\ X \end{bmatrix} = \mathcal{L}_0 \begin{bmatrix} I \\ X \end{bmatrix} S$$

with $S = X^{-1} R_0$. Since $\mathcal{M}_1 - \lambda \mathcal{L}_1$ is a doubling transformation of $\mathcal{M}_0 - \lambda \mathcal{L}_0$. Part (b) in Theorem 5.1 implies

$$\mathcal{M}_1 \begin{bmatrix} I \\ X \end{bmatrix} = \mathcal{L}_1 \begin{bmatrix} I \\ X \end{bmatrix} S^2.$$

The blocks in the above equation yield

$$R_1 = (X - P_1)S^2, \quad Q_1 - X = R_1^T S^2.$$

Together with (27), these imply the invertibility of

$$Q_1 - P_1 = (X - P_1) + R_1^T(X - P_1)^{-1}R_1, \quad Q_1 - X = (S^T)^2(X - P_1)S^2. \tag{28}$$

We have proved the theorem for $k = 1$.

With the theorem holding for all positive integers up to k , we shall prove the case for $k + 1$. Since $Q_k - P_k$ is assumed to be invertible, it follows that R_{k+1} , P_{k+1} and Q_{k+1} are well defined. Similar to the proof of (27), the equality in (b) implies the invertibility of

$$X - P_{k+1} = (X - P_k) - R_k(Q_k - P_k)^{-1}R_k^T.$$

On the other hand, since $\mathcal{M}_{j+1} - \lambda\mathcal{L}_{j+1}$ is a doubling transformation of $\mathcal{M}_j - \lambda\mathcal{L}_j$ for $j = 0, 1, \dots, k$, repeat application of Part (b) in Theorem 5.1 implies

$$\mathcal{M}_{k+1} \begin{bmatrix} I \\ X \end{bmatrix} = \mathcal{L}_{k+1} \begin{bmatrix} I \\ X \end{bmatrix} S^{2^{k+1}}.$$

This, following the same argument leading to (28), implies the invertibility of

$$Q_{k+1} - P_{k+1} = (X - P_{k+1}) + R_{k+1}^T(X - P_{k+1})^{-1}R_{k+1},$$

$$Q_{k+1} - X = (S^T)^{2^{k+1}}(X - P_{k+1})S^{2^{k+1}}.$$

This completes the proof for the $k + 1$ case, and the induction argument. \square

Note that Theorem 5.2 provides only the algebraic expressions for R_k , $Q_k - P_k$ and $Q_k - X$. Convergence to the unique symmetric stable solution X_s , which the SDA seeks, is summarized in the following corollary.

Corollary 1. *When S is stable, $R_k \rightarrow 0$ and $Q_k \rightarrow X$ quadratically as $k \rightarrow \infty$.*

6. Numerical results

In this section, we test a numerical example to illustrate the convergence behaviours of Algorithms 4.1 and 4.2. All computations were performed in MATLAB R2006b using IEEE double-precision floating-point arithmetic ($\text{eps} \approx 2.22 \times 10^{-16}$) on a linux system.

To measure accuracy of an approximate eigenpair (λ, x) for (1), we use the residual

$$\text{Res} \equiv \begin{cases} \|\lambda^2 A_1 x + \lambda A_0 x + A_1^T x\|_F & \text{if } |\lambda| < 1, \\ \|A_1 x + \lambda^{-1} A_0 x + \lambda^{-2} A_1^T x\|_F & \text{if } |\lambda| \geq 1 \end{cases} \tag{29}$$

and the relative residual

$$\text{RRes} \equiv \frac{\|\lambda^2 A_1 x + \lambda A_0 x + A_1^T x\|_F}{(\lambda^2 \|A_1\|_F + |\lambda| \|A_0\|_F + \|A_1\|_F) \|x\|_F}. \tag{30}$$

In our numerical results, the Poisson ratio μ is 0.3, the Young’s modulus E is 2.068×10^{11} and the material density ρ is 7.9×10^3 . The frequency $\omega = 1180$ and the damping matrix D is taken as the following linear combination of K and M :

$$D = 0.2K + 0.8M.$$

Using the finite element discretization with an uniform mesh, we obtain $L, C_{11}, C_{33} \in \mathbb{C}^{303 \times 303}$ and $C_{22} \in \mathbb{C}^{5151 \times 5151}$. In the corresponding palindromic quadratic eigenvalue problem, there are 11,514 eigenvalues before deflation. After deflation, there were 138 finite (nonzero) eigenvalues with absolute values in $[10^{-14}, 10^{14}]$. A distribution of these eigenvalues is shown in Fig. 2. Here we compare the results produced by Algorithm 4.1 with those by Algorithm 4.2 for computing these eigenvalues. The results produced by Algorithms 4.1 and 4.2 are marked by “ \times ” and “ \circ ”, respectively, in Figs. 4 and 5.

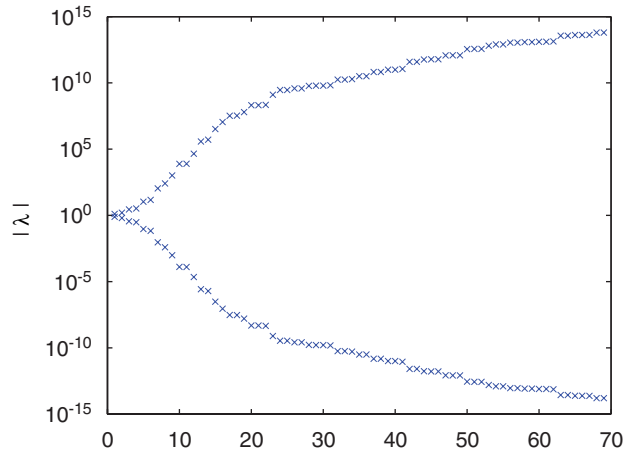


Fig. 2. Eigenvalue distribution.

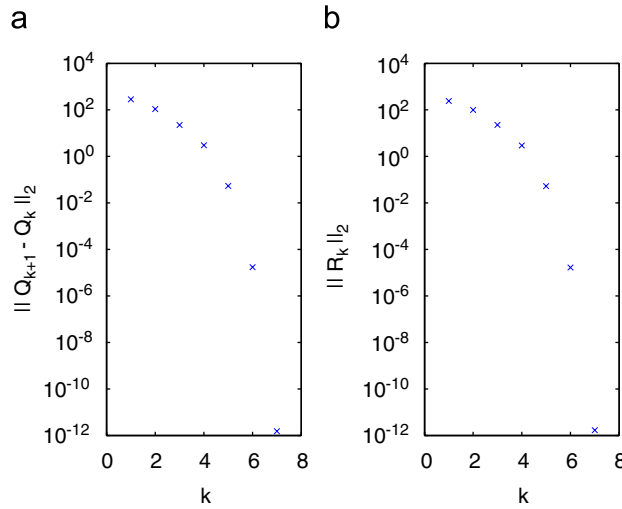


Fig. 3. The norms of $Q_{k+1} - Q_k$ and R_k for each iteration of SDA in Algorithm 4.2.

In [11], Hilliges et al. suggested a structured-preserving method to solve the linearized palindromic eigenvalue problem (22), but the method was still under development. Therefore, we use a classical non-structure-preserving method to solve (22). In our implementation, all the eigenvalue sub-problems in Algorithms 4.1 and 4.2 are computed by MATLAB command “`eig`”. Algorithm 4.2 requires seven iterations to compute approximate solutions of NME (23) by using the SDA. The convergent behaviour of the SDA in Algorithm 4.2, which is matched by the result of Theorem 5.2, is shown in Fig. 3.

In order to compare the accuracy of eigenpairs produced by Algorithms 4.1 and 4.2, the residual “Res” and relative residual “RRes” defined in (29) and (30), respectively, are reported in Fig. 4. From this figure, we see that the eigenpairs produced by Algorithm 4.2 is obviously more accurate than those by Algorithm 4.1 except for the eigenvalues near the unit circle (with similar accuracies). The important reciprocal property of the eigenvalues is reported in Fig. 5. For a computed stable eigenvalue λ_i with $10^{-14} \leq |\lambda_i| \leq 1$, we select a computed unstable eigenvalue $\lambda_{(i)}$ with $1 \leq |\lambda_{(i)}| \leq 10^{14}$ so that $r_i \equiv |\lambda_i \lambda_{(i)} - 1|$ is minimal. In Fig. 5, $\{|\lambda_i|, r_i\}$ are plotted for approximate eigenvalues from both Algorithms 4.1 and 4.2. Because of the structure-preserving nature of the SDA and Algorithm 4.2, the corresponding $\{r_i\}$ are practically zero as expected.

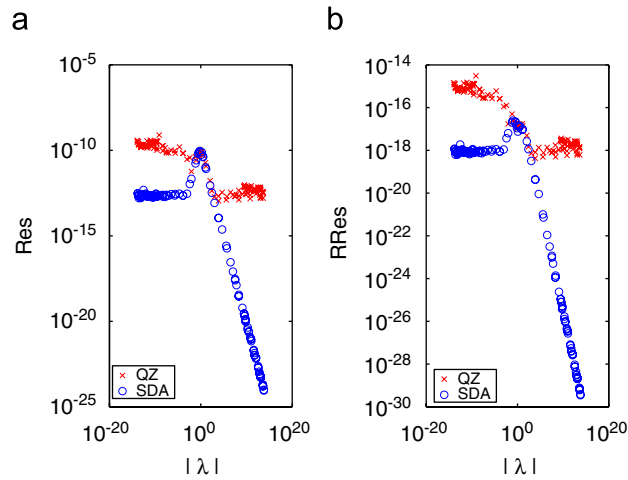


Fig. 4. Residual and relative residual of approximate eigenpair (λ, x) .

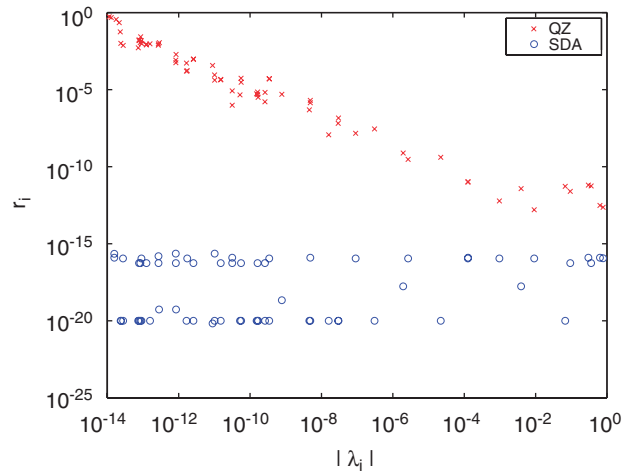


Fig. 5. Reciprocal properties of eigenvalues.

From Fig. 4, the eigenpairs produced by Algorithm 4.2 are of higher accuracy. Furthermore, in Algorithm 4.1, the stable and unstable eigenvalues λ_i and $\lambda_{(i)}$ are computed from the enlarged generalized palindromic eigenvalue problem using the non-structured QZ method [7], whilst Algorithm 4.2 is structure-preserving. Fig. 5 indicates that the reciprocal property is lost in Algorithm 4.1 but preserved in Algorithm 4.2. As shown in Fig. 5, the minimal value $r_i \equiv |\lambda_i \lambda_{(i)} - 1|$ is increasing when $|\lambda_i|$ is decreasing. This means that the eigenvalues are less accurate when they are further away from the unit circle.

In the numerical simulations we reported above, the assumptions that X , C_{22} , L and $Q_k - P_k$ are invertible are satisfied. The condition number of L is in the order of 10^{20} , leading to many eigenvalues away from the unit circle, but somehow the SDA copes rather well. The preprocessing procedure in [11,16] was not applied but may be unavoidable for larger values of n or problems with worse condition.

7. Conclusions

In this paper, we have proposed a structure-preserving doubling algorithm (SDA) for the palindromic eigenvalue problem and vibration analysis of fast trains. For small finite element models (up to several thousands elements), the

numerical results show much promise. Fast quadratic convergence has been achieved in the accurate resolution of the spectrum of the palindromic eigenvalue problems for given parameters ω . The accuracy of the SDA compares well with that of the QZ method (the only other possible method at the moment) for eigenvalue nears the unit circle. Away from the unit circle, the SDA is far superior. The assumptions that the stable symmetric invertible solution X exists and that L and C_{22} (in the deflation process) and $Q_k - P_k$ (in the SDA) are invertible hold in our numerical simulations. Ultimately, we need to solve the palindromic eigenvalue problems for a full spectrum of values in ω . We may be able to avoid a troublesome value of ω_0 , solve the problem for a neighbouring values of ω (possibly with the techniques in [12]) and interpolate to produce the spectrum required for ω_0 . Similarly, the restriction on the absence of eigenvalues on the unit circle can also be relaxed, using the techniques in [4]. If all else fails, the SDA1 in Section 4.1 (with the double-Cayley transform trick) may be called upon.

Recall the lack of positivity of matrices (in the space of complex matrices), a definite inner product (because of the transpose rather than the Hermitian) or the associated canonical forms. These lead to the absence of basic tools and theory, making our task a difficult one. Nevertheless, we have proved some theoretical results for palindromic eigenvalue problems and T-symplectic pencils.

Much work still awaits, and a sample of future tasks is listed below.

- (a) Additional properties from the finite element models and more theory and basic tools for palindromic eigenvalue problems and T-symplectic pencils.
- (b) Problems with more finite elements and larger values of n .
- (c) Efficient implementation of SDA1 with the double-Cayley transform, when some $Q_k - P_k$ are singular.
- (d) Efficient preprocessing of problems when L or C_{22} are ill-conditioned.
- (e) Alternative methods when the NME has no symmetric invertible solution X .
- (f) Correction of solutions, incorporating discretization errors in the finite element model.
- (g) Implementation of algorithms to benefit from the continuous nature of ω , sparseness and other structure of various matrices, possibly on parallel computers.
- (h) Higher order palindromic eigenvalue problems.

The vibration analysis of fast trains and palindromic eigenvalue problems are far from being conquered.

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