Representation of General and Polyhedral Subsemilattices and Sublattices of Product Spaces

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To Alan J. Hoffman, who has inspired me by the beauty and ingenuity of his mathematics and the warmth of his friendship, on the occasion of his 65th birthday.

Submitted by Uriel G. Rothblum

ABSTRACT

It is shown that each element of the lattice of meet (resp., join) sublattices of a product S of n chains has a representation as the intersection of n subsets of S, the ith of which is decreasing (resp., increasing) for each fixed value of the ith coordinate for each i. This result is applied to show that an arbitrary element of the lattice of sublattices of S has a representation as the intersection of n^2 subsets, the ijth of which is decreasing for each fixed value of the ith and increasing for each fixed value of the jth coordinate for each i, j. Irreducible representations are given in each case, providing an alternative proof of an instance of Hashimoto's (1952) representation of sublattices of a distributive lattice. Moreover, irreducible representations are given for the polyhedral members of the lattice of closed convex subsets of n-dimensional Euclidean space that are at once subsemilattices or sublattices. It is also shown that the polyhedral subsemilattices and sublattices can be represented as duals respectively of pre-Leontief substitution systems and generalized network-flow problems. Finally, the problems of checking whether a polyhedral set is a subsemilattice or sublattice are reduced to that of solving a system of linear inequalities, thereby showing that these recognition problems can be solved in polynomial time.

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1. INTRODUCTION

Subsemilattices[1] and sublattices have applications in many areas of the mathematical sciences. For that and other reasons, it is useful to understand their structure and to be able to recognize them. A fundamental way to do this is to study representations of subsemilattices (resp., sublattices) in the lattice of all such subsets of a semilattice (resp., lattice).

A *(meet) representation* of an element s of a lattice S is a nonempty subset L of S of which s is a meet, written $s = \wedge L$. If L is a finite set, the representation is called *finite*. It is often useful to restrict representations in a lattice S to elements s that are *irreducible*, i.e., $s = t \wedge u$ for some $t, u \in S$ only if $s = t$ or $s = u$. Elements of S that are not irreducible are called *reducible*. A representation is called *irreducible* if each of its elements is irreducible. It is also natural to seek representations that do not contain inessential elements. An element r of a representation L is called *redundant* if $\wedge L = \wedge(L \setminus \{r\})$. A representation is called *irredundant* if none of its elements is redundant. Any finite representation can be refined to make it irredundant by successively deleting redundant elements. For a more complete discussion of these concepts see [2], [3], and [4].

The purpose of this paper is to develop finite and irreducible representations for the elements of several lattices and to discuss their structure. We do this in Section 2 for the lattice of meet (resp., join) sublattices of a finite product S of chains, and in Section 4 for the lattice of sublattices of S. We also develop irreducible representations for the polyhedral meet (resp., join) sublattices of $\Re^n$ in Section 3 and show them to be precisely the duals of pre-Leontief substitution systems. We do this for polyhedral sublattices of $\Re^n$ in Section 5 and show them to be precisely the duals of generalized network-flow problems. Moreover, we show in Section 3 (resp., Section 5) that the problem of checking whether a polyhedral set is a subsemilattice (resp., sublattice) can be reduced to determining whether or not an associated polyhedral set is nonempty, and so can be done in polynomial time. For brevity we limit the discussion of this paper mainly to subsets of finite

[1] A subset L of a poset S is called a meet (resp., join) sublattice of S if each pair r, s of elements of L has a greatest lower bound (resp., least upper bound) in S, denoted $r \wedge s$ and called their meet (resp., denoted $r \vee s$ and called their join), that is also in L. Call L a sublattice (resp., subsemilattice) of S if L is both (resp., either) a meet and (resp., or) join sublattice of S. Call S a lattice (resp., semilattice, meet lattice, join lattice) if it is a sublattice (resp., subsemilattice, meet sublattice, join sublattice) of itself. Call a lattice S complete if every nonempty subset L of S has a greatest lower bound $\wedge L$, called its meet, and a least upper bound $\vee L$, called its join, in S. A chain is a linearly ordered set.

products of chains and to polyhedral subsets of $\mathfrak{R}^n$. We remove these restrictions elsewhere.

The results in this paper were obtained largely during the period 1965–1973 and were presented in a number of forums[2] at that time. In particular, early forms of the results characterizing the polyhedral subsemilattices and sublattices were obtained while I was on sabbatical leave during the academic year 1968–1969 and had the good fortune to be in Alan Hoffman's group at the IBM Research Center. Much of the general theory presented here was developed while I enjoyed the hospitality of Yale University's Department of Administrative Sciences on sabbatical leave during the academic year 1972–1973. Donald Topkis [9, 8] collaborated in the development of the finite representation of sublattices as discussed in footnote 11. The irreducible representations of subsemilattices and sublattices were obtained during 1973–1976, the latter representation being a sharper form of an instance of one found by Hashimoto [5, p. 183]. These and other contributions will be discussed more fully in the sequel.

In order to develop the desired representations, we require a few definitions. Let 2^S be the set of all subsets of an arbitrary set S that is partially ordered by set inclusion $\subseteq$. Then 2^S is a complete lattice in which meets are intersections and joins are unions. Frequently one is interested in the family of all subsets of S that have some property $\mathscr{P}$. If that family includes S among its members and is closed under intersections, then each subset L of S has a *$\mathscr{P}$-hull*, viz., the intersection of all subsets of S that contain L and have the property $\mathscr{P}$. (For example, if S is a lattice, the *sublattice hull* of L is the intersection of all sublattices of S that contain L.) In that event, the family is a meet sublattice of 2^S and a complete lattice, called a *Moore lattice*, in which the join of any subfamily $\mathscr{F}$ of the family is the $\mathscr{P}$-hull of the union of the members of $\mathscr{F}$. The lattice of meet (resp., join) sublattices of a meet (resp., join) lattice is a Moore lattice, as is the lattice of sublattices of a lattice. And so is the lattice of closed convex[3] subsets of $\mathfrak{R}^n$, as well as its intersection with the lattice of meet (resp., join) sublattices and the lattice of sublattices.

[2]I presented this work in a seminar at Yale University in April 1973, the 1973 International Symposium on Mathematical Programming [9], a Seminar in Dynamic and Lattice Programming at Stanford University in the first quarter of 1974, and a series of invited Lectures at IRIA, France in September 1974. I also gave talks on lattice programming—including the representation of sublattices—in the spring of 1973 at Columbia University, Cornell University, University of Montreal, and University of Rochester; a Special Invited Lecture at a regional meeting of the IMS in 1973; and a Short Course for the AMS in January 1975.

[3]Convex is used throughout in the sense of vector spaces, not lattices.

2. REPRESENTATION OF SUBSEMILATTICES

Let $S = \times_N S_k$ be a product of posets S_k, $k \in N$. A subset L of S is called *i-decreasing* (resp., *i-increasing*) if $i \in N$ and for each $r = (r_k) \in L$ and $s = (s_k) \in S$ with $r \geqslant s$ (resp., $r \leqslant s$) and $r_i = s_i$, one has $s \in L$. Since for each fixed $i \in N$, the set of i-decreasing (resp., i-increasing) subsets of S is a Moore lattice, each subset L of S has an *i-decreasing* (resp., *i-increasing*) *hull* denoted $L_i^{\downarrow}$ (resp., $L_i^{\uparrow}$). It is easy to see that, for each i, both the i-decreasing and the i-increasing hulls of L have useful internal representations as projections $\pi_S K$ of subsets K of $L \times S$ on S, viz.,

$$L_i^{\downarrow} \equiv \pi_S\{(r, s) \in L \times S : r \geqslant s,\ r_i = s_i\} \tag{1}$$

and

$$L_i^{\uparrow} \equiv \pi_S\{(r, s) \in L \times S : r \leqslant s,\ r_i = s_i\}. \tag{2}$$

THEOREM 1 (Representation of subsemilattices of products of chains). *The following properties of a subset L of a finite*[4] *product $S = \times_N S_k$ of chains*[5] *are equivalent*:

(a) *L is a meet (resp., join) sublattice of S.*

(b) *L is the intersection of its $|N|$ i-decreasing (resp., i-increasing) hulls for all $i \in N$.*

(c) *L is the intersection of i-decreasing (resp., i-increasing) subsets of S for all $i \in N$.*

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[4] If N is not finite, the implications (b) $\Rightarrow$ (c) $\Rightarrow$ (a) and their proofs nevertheless remain valid. The implication (a) $\Rightarrow$ (b) and its proof also remain valid if L is conditionally meet (resp., join) subcomplete, i.e., for each nonempty subset K of L that is bounded below (resp., above) in S, $\wedge K$ (resp., $\vee K$) exists in S and is in L.

[5] Actually, the representation of meet (resp., join) sublattices in Theorem 1—i.e., (a) implies (b)—and its proof remain valid under the weaker hypothesis that the S_k are meet (resp., join) lattices. Moreover, the i-decreasing (resp., i-increasing) hull (1) (resp., (2)) is then a meet (resp., join) sublattice because the class of meet (resp., join) sublattices is closed under projections. However, under the weaker hypothesis, an arbitrary i-decreasing (resp., i-increasing) set need not be a meet (resp., join) sublattice, so (c) does not imply (a). For this reason, the more general representation does not seem as useful as the one presented.

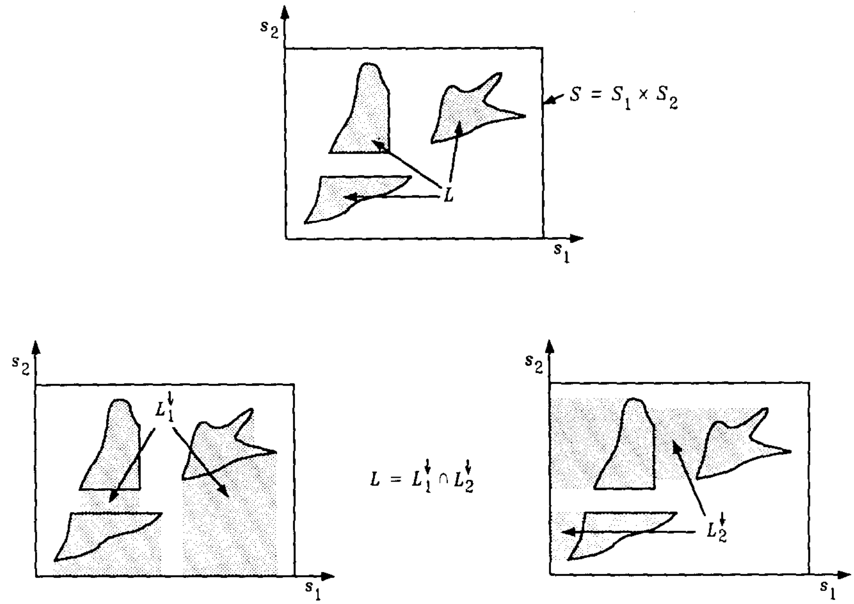

FIG. 1. Representation of a meet sublattice L of a product $S_1 \times S_2$ of two chains.

Proof. By duality, it suffices to prove the result reading without parentheses.

(a) $\Rightarrow$ (b): Since $L_i^{\downarrow}$ is a hull of L, $L \subseteq \bigcap_N L_i^{\downarrow}$. If $s \in \bigcap_N L_i^{\downarrow}$, there exist $r^i \in L$ with $r^i \geqslant s$ and $r_i^i = s_i$ for all $i \in N$. Thus $s = \bigwedge_N r^i \in L$ because L is a meet sublattice of S.

(b) $\Rightarrow$ (c): The i-decreasing hulls of L are i-decreasing subsets of S for all $i \in N$.

(c) $\Rightarrow$ (a): Since intersections of meet sublattices are meet sublattices, it suffices to show that each i-decreasing subset L of S is a meet sublattice. To that end, suppose $r, s \in L$. By possibly relabeling, we can assume that $r_i \leqslant s_i$ since S_i is a chain. Then $r \wedge s \in S$, $r \wedge s \leqslant r$ and $(r \wedge s)_i = r_i$. Thus because L is i-decreasing, $r \wedge s \in L$. ■

The 1- and 2-decreasing hulls $L_1^{\downarrow}$ and $L_2^{\downarrow}$ respectively of a meet sublattice L of a product $S = S_1 \times S_2$ of two chains are illustrated in Figure 1. Observe that, as Theorem 1 asserts, $L = L_1^{\downarrow} \cap L_2^{\downarrow}$.

Representation of Subsemilattice Hulls of Subsets of Finite Products of Chains

For any subset L of the $|N|$-chain product S with finite $|N|$, let $L^\wedge$ (resp., $L^\vee$) be the *meet-sublattice* (resp., *join-sublattice*) hull of L. Of course $L^\wedge$ (resp., $L^\vee$) is the set of all finite meets (resp., joins) of elements of L. The next result expresses both the meet- and join-sublattice hulls of L in terms of their i-decreasing and i-increasing hulls.

COROLLARY 2 (Representation of subsemilattice hulls of subsets of finite products of chains). *If L is a subset of a finite product $S = \times_N S_k$ of chains, then the meet-sublattice (resp., join-sublattice) hull of L is the intersection of its i-decreasing (resp., i-increasing) hulls for all $i \in N$.*

Proof. By duality, it is enough to prove the result reading without parentheses. By the representation-of-subsemilattices Theorem 1, $\bigcap_N L_i^\downarrow \subseteq \bigcap_N (L^\wedge)_i^\downarrow = L^\wedge \subseteq \bigcap_N L_i^\downarrow$, so equality occurs throughout. ∎

Irreducible Representations of Subsemilattices

We now extend the representation-of-subsemilattices Theorem 1 by developing irreducible representations and characterizing the elements thereof. In order to do this, we shall generally have to give up the finiteness and even (finite) irredundance of the representation. The key to developing irreducible representations of subsemilattices is to observe first that L is i-decreasing if and only if its complement L^c is i-increasing. Incidentally, this fact and Theorem 1 imply that meet (resp., join) sublattices of finite products of chains have representations in meet (resp., join) sublattices whose complements are join (resp., meet) sublattices.

THEOREM 3 (Irreducible representation of subsemilattices of finite products of chains). *Every meet (resp., join) sublattice of a finite product $S = \times_N S_k$ of chains has a representation as an intersection of sets, each of which is the complement of an i-increasing (resp., i-decreasing) hull of a singleton set $\{s\}$ for some $i \in N$ and $s \in S$, and is irreducible,[6] the last being so if and only if either $s = \bigvee S$ (resp., $s = \bigwedge S$) or $s_i \neq \bigvee S_i$ (resp., $s_i \neq \bigwedge S_i$).[7]*

[6] Since the lattice of meet (resp., join) sublattices is compactly generated, the existence of the irreducible representations follows from the theory of such lattices [3, pp. 43–44]. However, the present approach has the advantages of also describing the qualitative structure of the elements of the representation and of not relying on Zorn's lemma.

[7] I am greatly indebted to Margarida Mello for sharpening my earlier version of Theorem 3 by characterizing precisely which of the complements of i-decreasing (resp., i-increasing) hulls of singleton sets are irreducible. Her characterization and a slight modification of her proof thereof are included here with her kind permission.

Proof. By duality, it is enough to prove the result for a meet sublattice L of S. We claim first that for each $s \in \mathrm{S} \setminus L$, there is an $i \in N$ such that $\{s\}_i^{\uparrow} \subseteq \mathrm{S} \setminus L$ and either $s = \vee \mathrm{S}$ or $s_i \neq \vee \mathrm{S}_i$. To see this, observe from the representation-of-subsemilattices Theorem 1 that there is a $j \in N$ such that $s \notin L_j^{\downarrow}$. Since $L_j^{\downarrow}$ is j-decreasing, $L_j^{\downarrow c}$ is j-increasing, so $\{s\}_j^{\uparrow} \subseteq L_j^{\downarrow c} \subseteq \mathrm{S} \setminus L$. If either $s = \vee \mathrm{S}$ or $s_j \neq \vee \mathrm{S}_j$, the claim is proved by setting $i = j$. In the contrary event, $s_j = \vee \mathrm{S}_j$ and there is an $i \neq j$ such that $s_i \neq \vee \mathrm{S}_i$. Then $\{s\}_i^{\uparrow} \subseteq \{s\}_j^{\uparrow} \subseteq \mathrm{S} \setminus L$, establishing the claim.

It remains to justify the claimed characterization of irreducible $\{s\}_i^{\uparrow c}$. If $s = \vee \mathrm{S}$, then $\{s\}_i^{\uparrow c} = \mathrm{S} \setminus \{\vee \mathrm{S}\}$ is trivially irreducible. If $s_i \neq \vee \mathrm{S}_i$, then $\{s\}_i^{\uparrow c}$ is also irreducible. To see this, it suffices to show that each meet sublattice $P \supset \{s\}_i^{\uparrow c}$ also contains $s \notin \{s\}_i^{\uparrow c}$. Now there is an $r \in P \setminus \{s\}_i^{\uparrow c}$, so $r_i = s_i$ and $r \geqslant s$. Choose $t \in \mathrm{S}$ such that $s_i < t_i$, which is possible because $s_i \neq \vee \mathrm{S}_i$, and $t_j = s_j$ for $j \neq i$. Then $t \in \{s\}_i^{\uparrow c} \subset P$ and so $s = r \wedge t \in P$.

If $s_i = \vee \mathrm{S}_i$ and $s_j \neq \vee \mathrm{S}_j$ for some $j \neq i$, then $\{s\}_i^{\uparrow c}$ is reducible. For choose $r \in \mathrm{S}$ so that $s_j < r_j$, which is possible because $s_j \neq \vee \mathrm{S}_j$, and $r_k = s_k$ for $k \neq j$. Then $P = \{s\}_i^{\uparrow c} \cup \{r\}$ and $Q = \{s\}_i^{\uparrow c} \cup \{s\}$ are meet sublattices of S that are distinct from $\{s\}_i^{\uparrow c}$, and $\{s\}_i^{\uparrow c} = P \cap Q$. ∎

We remark that if $\vee \mathrm{S}_i$ (resp., $\wedge \mathrm{S}_i$) does not exist, then $\{s\}_i^{\uparrow c}$ (resp., $\{s\}_i^{\downarrow c}$) is an irreducible meet (resp., join) sublattice for every $s \in \mathrm{S}$, because then trivially $s_i \neq \vee \mathrm{S}_i$ (resp., $s_i \neq \wedge \mathrm{S}_i$).

In Figure 2 we illustrate an element of the irreducible representation of Theorem 3 for the meet sublattice L of a product $\mathrm{S} = \mathrm{S}_1 \times \mathrm{S}_2$ of two chains given in Figure 1. Notice that the 1-increasing hull $\{s\}_1^{\uparrow}$ of the element $s \in \mathrm{S} \setminus L$ given there is contained in $\mathrm{S} \setminus L$, so the complement of $\{s\}_1^{\uparrow}$ contains L but not s, and so separates L from s. By contrast, the complement of $\{s\}_2^{\uparrow}$ does not contain L.

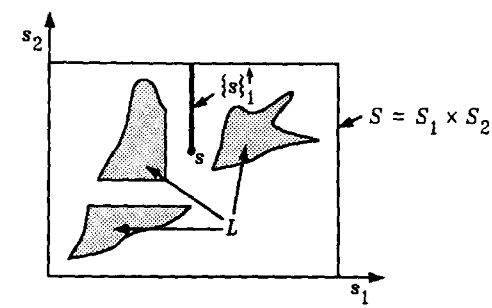

FIG. 2. Element of irreducible representation of a meet sublattice L of a product $\mathrm{S}_1 \times \mathrm{S}_2$ of two chains.

Observe that Theorem 3 assures that each meet sublattice of S can be expressed as an intersection of sets, each of which is a set of $r \in S$ that do not satisfy both of the inequalities

$$r \geqslant s \quad \text{and} \quad r_i \leqslant s_i \tag{3}$$

for some fixed $s \in S$ and $i \in N$. An equivalent formulation is that r satisfies at least one of the $|N|+1$ (strict) inequalities

$$r_j < s_j \quad \text{for some } j \in N \quad \text{or} \quad r_i > s_i.$$

Analogous formulations for join sublattices may be found by reversing the above inequalities.

Moreover, Theorem 3 implies that each meet (resp., join) sublattice of a finite product of chains has a coordinate-free irreducible representation as an intersection of sets of the form $D^c \cup I^c$ where D is a principal dual ideal (resp., prime dual ideal) and I is a prime ideal (resp., principal ideal).[8] To see this for meet sublattices, let D be the first of the sets in (3) and I be the second.[9]

Finally, note that Theorem 3 implies the main assertion of Theorem 1, viz., (a) $\Rightarrow$ (c), since complements of i-increasing (resp., i-decreasing) hulls of singleton sets are i-decreasing (resp., i-increasing).

[8] Let S be a poset. A subset L of S is decreasing (resp., increasing) if for each $r \in L$ and $s \in S$ with $s \leqslant r$ (resp., $s \geqslant r$), one has $s \in L$. A subset L of S is an ideal (resp., dual ideal) if it is a decreasing join (resp., increasing meet) sublattice. An ideal (resp., dual ideal) is prime if its complement is a dual ideal (resp., ideal). A set of the form $\{s \in S : s \leqslant r\}$ (resp., $\{s \in S : s \geqslant r\}$) is the principal ideal (resp., principal dual ideal) generated by $r \in S$.

[9] Garrett Birkhoff kindly pointed out to me in 1973 that representations of subsemilattices of products of chains can be used to provide representations of subsemilattices of quite general semilattices. To see this, use the mapping f that sends each element s of a meet (resp., join) semilattice S into the principal ideal (resp., principal dual ideal) generated by s. Then f is a meet (resp., join) isomorphism of S into the lattice of meet (resp., join) sublattices of S. Since that lattice is a Moore lattice of subsets of S, and 2^S can be considered to be—more precisely, is isomorphic to—a product of $|S|$ two-element chains, the representations given here can be used in 2^S and then carried back into S.

3. REPRESENTATION AND RECOGNITION OF POLYHEDRAL SUBSEMILATTICES

Next we apply the representation-of-subsemilattices Theorem 1 to give irreducible representations of the polyhedral[10] elements of the Moore lattice of closed convex meet (resp., join) sublattices of $\Re^n$. To that end, call a matrix *pre-Leontief* if each of its columns has at most one positive element.

THEOREM 4 (Irreducible representations of polyhedral subsemilattices as dual pre-Leontief substitution polyhedra). *The following are equivalent*:

(a) *L is a polyhedral meet (resp., join) sublattice of* $\Re^n$.

(b) $L = \{s \in \Re^n : As \leqslant b\}$ *for some matrix* $(A\ b)$ *with* $-A$ (*resp.*, A) *having pre-Leontief transpose.*

(c) *L is the intersection of finitely many closed half-spaces, each of which is an irreducible element of the lattice of closed convex meet (resp., join) sublattices of* $\Re^n$.

Proof. By duality, it is enough to prove the result reading without parentheses.

(a) $\Rightarrow$ (c): Since L is polyhedral, it follows from (1) that $L_i^{\downarrow}$ is the projection of a polyhedral set and so is itself polyhedral. Thus, there is a matrix $(A^i\ b^i)$ with $L_i^{\downarrow} = \{s \in \Re^n : A^i s \leqslant b^i\}$ for each i. Since $L_i^{\downarrow}$ is i-decreasing, each column of A^i, except possibly the ith, is nonnegative. Now let $A = (A^i)$ and $b = (b^i)$. By the representation-of-subsemilattices Theorem 1, $L = \bigcap_1^n L_i^{\downarrow} = \{s \in \Re^n : As \leqslant b\}$ and $-A$ has pre-Leontief transpose.

(b) $\Rightarrow$ (c): Let $(a_j\ b_j)$ denote the jth row of $(A\ b)$, and put $H_j \equiv \{s \in \Re^n : a_j s \leqslant b_j\}$. From (2), there is an i, depending on j, such that every element of a_j, except possibly the ith, is nonnegative. Thus, H_j is i-decreasing and so, by Theorem 1, is a meet sublattice of $\Re^n$. Moreover, the half-space H_j is also irreducible, since that is so of the closed half-spaces in the lattice of closed convex subsets of $\Re^n$. Thus, since $L = \bigcap_j H_j$, (c) holds.

(c) $\Rightarrow$ (a): Finite intersections of polyhedral meet sublattices are polyhedral meet sublattices. ∎

It should be emphasized that Theorem 4 does not assert that each half-space in every irredundant irreducible representation of a polyhedral

[10] Elsewhere we establish similar (but countable) representations for arbitrary elements of the lattices of closed convex meet sublattices, join sublattices, and sublattices.

meet (resp., join) sublattice is a meet (resp., join) sublattice, but merely that there is such a representation.

EXAMPLE. Suppose that L is the intersection of the three half-spaces $s - t \leqslant 0$, $-s + t \leqslant 0$, and $\alpha s + \beta t \leqslant 0$ in the plane, where α, β are constants satisfying $\alpha + \beta < 0$. Then L is a meet sublattice (indeed, a sublattice) not depending on α, β, and each such representation is irredundant. However, the third half-space is a meet sublattice (and a sublattice) if and only if $\alpha\beta \leqslant 0$. Thus, there are infinitely many irredundant irreducible representations some of whose members are not meet sublattices.

The proof of Theorem 4 suggests a constructive method for finding a finite representation of a polyhedral meet (resp., join) sublattice L in closed half-spaces, each of which is a meet (resp., join) sublattice. The method is to use (1) (resp., (2)) to compute the projections $L_i^{\downarrow}$ (resp., $L_i^{\uparrow}$) by, say, the Fourier–Motzkin elimination procedure. As the proof that (a) implies (b) of Theorem 4 shows, this produces the desired representation of L as a dual pre-Leontief substitution system.

As can be seen from the above example, the representation given in Theorem 4 is not generally unique. However, as we now show, the representation is unique when the polyhedron has full dimension. In order to see this, we require a definition. A closed half-space H in $\Re^n$ is called *tangent* to a closed convex subset L of $\Re^n$ at a point $s \in L$ if $L \subseteq H$ and if the hyperplane bounding H is the unique supporting hyperplane to L at s. It is known [7, p. 172] that if the dimension of a polyhedron L in $\Re^n$ is n, then L has a unique irredundant irreducible representation, viz., its tangent closed half-spaces.

If L is a closed half-space $\{s \in \Re^n : as \leqslant b\}$ and a meet (resp., join) sublattice of $\Re^n$, then every finite irredundant representation of L in closed half-spaces consists solely of the tangent closed half-space L itself. Thus by Theorem 4, $-a$ (resp., a) has at most one positive element.

COROLLARY 5 (Unique irredundant irreducible representation of polyhedral subsemilattices of full dimension). *If L is a polyhedral meet (resp., join) sublattice of $\Re^n$, then each tangent closed half-space of L is a meet (resp., join) sublattice of $\Re^n$. If also the dimension of L is n, then L has a unique irredundant irreducible representation, viz., its tangent closed half-spaces.*

Proof. It follows from Theorem 4 that L has a finite irredundant irreducible representation in closed half-spaces that are at once meet (resp., join) sublattices. Moreover, that representation must contain all the tangent closed half-spaces. If also the dimension of L is n, that representation contains only the tangent closed half-spaces. ∎

Characterization of Polyhedral Subsemilattices by Linear Inequalities

The next result characterizes when a polyhedral set is a subsemilattice by means of linear inequalities.

THEOREM 6 (Characterization of polyhedral subsemilattices by linear inequalities). *If* $L = \{s \in \Re^n : As \leqslant b\}$ *(resp.,* $L = \{s \in \Re^n : -As \leqslant b\}$*) is a nonempty polyhedron for some* $m \times (n+1)$ *matrix* $(A\ b)$*, the following are equivalent:*

(a) *L is a meet (resp., join) sublattice of* $\Re^n$.

(b) *There exist* $m \times m$ *matrices* λ^i *and* $m \times n$ *matrices* $\mu^i = (\mu^i_{jk})$ *for* $i = 1, \ldots, n$ *that satisfy the linear inequalities*

$$
\begin{aligned}
-\mu^i + \lambda^i A &= 0, \qquad i = 1, \ldots, n,\\
\sum_1^n \mu^l &= A,\\
\sum_1^n \lambda^l b &\leqslant b,
\end{aligned}
\tag{4}
$$

$$\mu^i_{jk} \geqslant 0, \quad \lambda^i \geqslant 0 \qquad \textit{for all } i, j, k \textit{ with } i \neq k.$$

Proof. By duality, it suffices to prove the result reading without parentheses. And by the representation-of-subsemilattices Theorem 1, L is a meet sublattice of $\Re^n$ if and only if $L \supseteq \bigcap_1^n L_i^{\downarrow}$, i.e., if and only if every sequence $s, r^1, \ldots, r^n \in \Re^n$ that satisfies the system

$$
\begin{aligned}
Ar^i &\leqslant b,\\
r^i &\geqslant s, \qquad (i = 1, \ldots, n)\\
r_i^i &= s_i
\end{aligned}
\tag{5}
$$

has the property that $As \leqslant b$. This is so if and only if for each $1 \leqslant j \leqslant n$, the maximum of $a_j s$ subject to (5) does not exceed b_j, where $(a_j\ b_j)$ is the jth row of $(A\ b)$. By the duality theorem of linear programming, this is so if and only if the dual of this linear program has a feasible solution whose objective function does not exceed b_j for each j. The system (4) expresses this fact. ∎

Polynomial-Time Recognition of Polyhedral Subsemilattices

One immediate implication of Theorem 6 is the following.

COROLLARY 7 (Polynomial-time recognition of polyhedral subsemilattices). *There is a polynomial-time algorithm for testing whether a polyhedron is a meet (resp., join) sublattice.*

Proof. This follows from Theorem 6, the fact that the size of the linear inequality system (4) is polynomial in that of the system $As \leqslant b$, and the fact that there exist algorithms for solving systems of linear inequalities that run in time that is polynomial in the size thereof, e.g., those of Khachiyan and Karmarkar. ∎

Computations

One way of testing whether or not a nonempty polyhedron $L = \{s \in \Re^n : As \leqslant b\}$ is a meet sublattice is to proceed as follows. First permute and partition the rows of the matrix $(A \quad b)$ so that

$$(A \quad b) = \begin{bmatrix} A' & b' \\ A'' & b'' \end{bmatrix},$$

where A' is the submatrix of A whose rows contain at most one negative element. Then check whether every solution of $A's \leqslant b'$ is a solution of $A''s \leqslant b''$, i.e., whether the latter inequalities are redundant. This is so if and only if the matrix linear inequalities $\lambda A' = A''$, $\lambda b' \leqslant b''$, and $\lambda \geqslant 0$ have a matrix solution λ with the same number of rows as A'' and columns as A'^{T}.

If the double-primed inequalities are redundant, then the polyhedron is a meet sublattice. If the double-primed inequalities are not redundant, then check whether or not the polyhedron has dimension n, i.e., whether or not there is a positive number t and an $s \in \Re^n$ such that $As + 1t \leqslant b$, where 1 is an m-element column vector of ones. In view of Corollary 5, if the polyhedron L has dimension n, then L is not a meet sublattice.

If the double-primed inequalities are not redundant and the polyhedron L has dimension less than n, then check whether or not the inequalities $As \leqslant b$ are implied by the inequalities (5). This is so if and only if the inequalities (4) have a solution, or by Theorem 6, if and only if L is a meet sublattice.

In each of the above cases, existence of a solution to the system of inequalities can be checked by using any algorithm for solving linear programs, e.g., the simplex method. Also one can check whether L is a join sublattice by simply replacing A by $-A$ in the above development.

4. REPRESENTATION OF SUBLATTICES

Let $S = \times_N S_k$ be a product of lattices S_k, $k \in N$. Each i-decreasing j-increasing subset of S is evidently a cylinder with base in $S_i \times S_j$ for $i \neq j$ and base in S_i for $i = j$. Since for each fixed $i, j \in N$ the set of i-decreasing j-increasing subsets of S is a Moore lattice, each subset L of S has an *i-decreasing j-increasing hull*, denoted $L_{ij}^{\downarrow\uparrow}$. Evidently, for each $i, j \in N$, $L_{ij}^{\downarrow\uparrow} \equiv L_{ji}^{\uparrow\downarrow}$ has an internal representation as a projection of a subset of $L \times S$ on S, viz.,

$$L_{ij}^{\downarrow\uparrow} = \pi_S\{(r,s) \in L \times S : r_i \leqslant s_i,\ r_j \geqslant s_j\}. \tag{6}$$

The next result shows that the i-decreasing j-increasing hull can be formed in either of two equivalent ways, viz., by taking the i-decreasing hull and then the j-increasing hull, or by taking the j-increasing hull and then the i-decreasing hull.

LEMMA 8 (i-Decreasing j-increasing hulls). *If L is a subset of a product $S = \times_N S_k$ of lattices, then $(L_i^{\downarrow})_j^{\uparrow} = L_{ij}^{\downarrow\uparrow} = (L_j^{\uparrow})_i^{\downarrow}$ for all $i, j \in N$.*

Proof. By duality, it is enough to prove the first equality. To that end, observe from the projection formulas (1), (2), and (6) on putting $T = S$ that

$$\begin{aligned}(L_i^{\downarrow})_j^{\uparrow} &= \pi_T\{(s,t) \in L_i^{\downarrow} \times T : s \leqslant t,\ s_j = t_j\}\\ &= \pi_T\{(r,s,t) \in L \times S \times T : s \leqslant r,\ s_i = r_i,\ s \leqslant t,\ s_j = t_j\}\\ &= \pi_T\{(r,t) \in L \times T : r_i \leqslant t_i,\ r_j \geqslant t_j\} = L_{ij}^{\downarrow\uparrow}.\end{aligned}$$ ∎

THEOREM 9 (Representation of sublattices of products of chains).[11] *The following properties of a subset L of a finite*[12] *product* $S = \times_N S_k$ *of chains*[13] *are equivalent*:

(a) *L is a sublattice of S.*
(b) *L is the intersection of its* $|N|^2$ *i-decreasing j-increasing hulls for all* $i, j \in N$.
(c) *L is the intersection of i-decreasing j-increasing subsets of S for all* $i, j \in N$.

Proof. (a) $\Rightarrow$ (b): By (1), $L_i^{\downarrow}$ is the projection of a sublattice of the lattice $L \times S$ on S and so is a sublattice of S. Thus, by the representation-of-subsemilattices Theorem 1 and Lemma 8, $L = \bigcap_i L_i^{\downarrow} = \bigcap_i [\bigcap_j (L_i^{\downarrow})_j^{\uparrow}] = \bigcap_{i,j} L_{ij}^{\downarrow\uparrow}$.

(b) $\Rightarrow$ (c): The i-decreasing j-increasing hulls of L are i-decreasing j-increasing subsets of S for all i, j.

(c) $\Rightarrow$ (a): By the representation-of-subsemilattices Theorem 1, i-decreasing j-increasing subsets are sublattices, and intersections of sublattices are sublattices. ■

We illustrate the representation presented in Theorem 9 for the sublattice L of a product $S = S_1 \times S_2$ of two chains given in Figure 3. There we exhibit the i-decreasing j-increasing hulls $L_{ij}^{\downarrow\uparrow}$ of L for each $i, j = 1,2$. Observe that as Theorem 9 asserts, $L = L_{11}^{\downarrow\uparrow} \cap L_{22}^{\downarrow\uparrow} \cap L_{12}^{\downarrow\uparrow} \cap L_{21}^{\downarrow\uparrow}$.

---

[11] I established that (c) implies (a) in 1965. The converse, viz., that (a) implies (c), has a more complex history. I established the converse where S is n-dimensional Euclidean space and L is a closed convex subset of full dimension in stages between the autumn of 1968 and the winter of 1970–1971. Armed with these and other results of mine, D. M. Topkis made a crucial advance in August 1971 by establishing Corollary 11 below and deriving a second result from which the general converse could have been, but was not at that time, derived. In attempting to understand Topkis' long proof, I formulated and proved the converse in the winter of 1972–1973 essentially as given here. About two years later, Topkis developed a short proof that Corollary 11 implies the converse and published it in [8].

[12] If N is not finite, the implications (b) $\Rightarrow$ (c) $\Rightarrow$ (a) and their proofs nevertheless remain valid. The implication (a) $\Rightarrow$ (b) and its proof also remain valid if S is conditionally meet and join subcomplete relative to itself and L is subcomplete, i.e., every nonempty subset of L has both a meet and join in S that are also in L.

[13] Actually, the representation of sublattices in Theorem 9, i.e., (a) implies (b), and its proof remain valid under the weaker hypothesis that the S_k are lattices. Moreover, the i-decreasing j-increasing hull (6) is then a sublattice, because the class of sublattices is closed under projections. However, under the weaker hypothesis, an arbitrary i-decreasing j-increasing set need not be a sublattice, so (c) does not imply (a). For this reason, the more general representation does not seem as useful as the one presented.

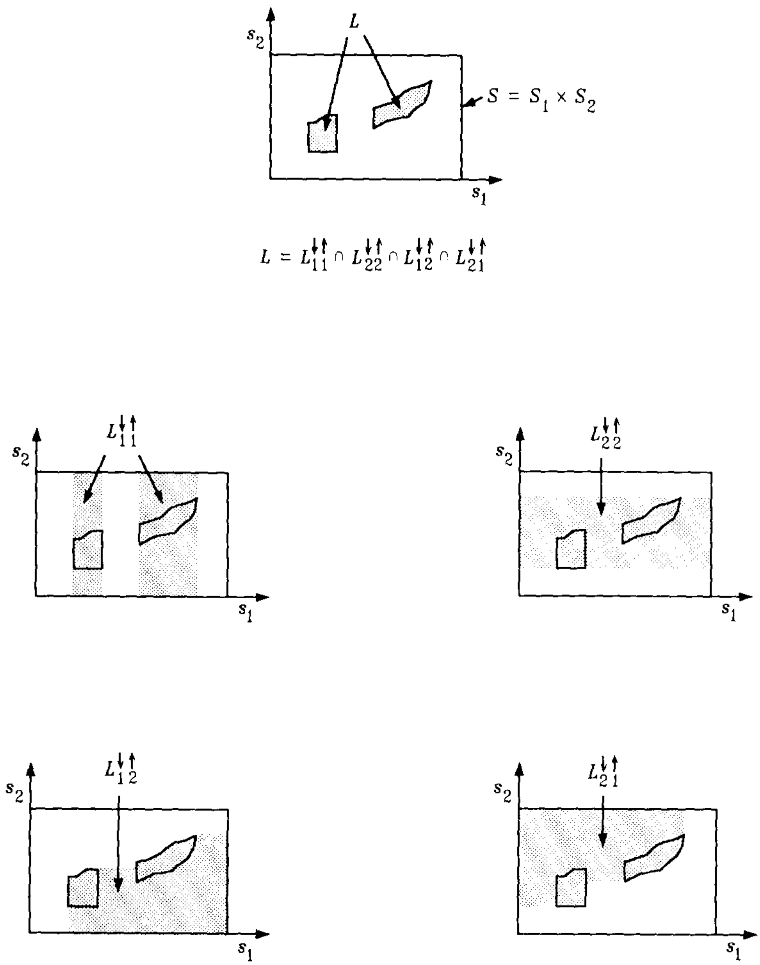

FIG. 3. Representation of a sublattice L of a product $S_1 \times S_2$ of two chains.

Representation of Sublattices of Finite Products of Chains in Projections

It is possible to give representations of sublattices of S in their projections. The next two results accomplish this.

COROLLARY 10 (Representation of projections of sublattices of finite products of chains). *If L is a sublattice of a finite product $S = \times_N S_k$ of chains and $Q \subseteq N$, then the cylinder in S whose base is the projection of L on $\times_Q S_k$ is $\bigcap_{i,j \in Q} L_{ij}^{\downarrow\uparrow}$.*

Proof. Let $\pi_Q K$ be the projection of $K \subseteq S$ on $\times_Q S_k$. Since each i-decreasing j-increasing subset of S with $i, j \in Q$ is a cylinder with base in $\times_Q S_k$, it follows that $\pi_Q L_{ij}^{\downarrow\uparrow}$ is the i-decreasing j-increasing hull of the sublattice $\pi_Q L$ of $\times_Q S_k$. Thus, by the representation-of-sublattices Theorem 9, $\pi_Q^{-1}\pi_Q L = \pi_Q^{-1}(\bigcap_{i,j\in Q}\pi_Q L_{ij}^{\downarrow\uparrow}) = \bigcap_{i,j\in Q} L_{ij}^{\downarrow\uparrow}$. ∎

The next result was discovered independently by G. M. Bergman in 1967 (see [1]) and by D. M. Topkis [8] in 1971.

Corollary 11 (Representation of sublattices of finite products of chains in cylinders whose bases are two-dimensional projections). *If L is a sublattice of a finite product $S = \times_N S_k$ of chains, then L is the intersection of the cylinders whose bases are the projections of L on $S_i \times S_j$ for all $i \neq j \in N$.*

Proof. This is immediate from the representation-of-sublattices Theorem 9 and Corollary 10, where Q ranges over the pairs $i, j \in N$ with $i \neq j$. ∎

Representation of Sublattice Hulls of Subsets of Finite Products of Chains

For each subset L of the finite product $S = \times_N S_k$ of chains, let $L^{\wedge\vee}$ be the *sublattice hull* of L. Clearly, $L^{\wedge\vee}$ is the set of finite meets and joins of elements of L. An easy application of the distributivity of S shows that the sublattice hull can be generated sequentially, i.e., $(L^{\wedge})^{\vee} = L^{\wedge\vee} = (L^{\vee})^{\wedge}$.

Corollary 12 (Representation of sublattice hulls of subsets of finite products of chains). *If L is a subset of a finite product $S = \times_N S_k$ of chains, then the sublattice hull of L is the intersection of its i-decreasing j-increasing hulls for all $i, j \in N$.*

Proof. Since $L_{ij}^{\downarrow\uparrow} \supseteq L$ is a sublattice of S for all i, j, it follows from the representation-of-sublattices Theorem 9 that $\bigcap_{i,j} L_{ij}^{\downarrow\uparrow} \subseteq \bigcap_{i,j}(L^{\wedge\vee})_{ij}^{\downarrow\uparrow} = L^{\wedge\vee} \subseteq \bigcap_{i,j} L_{ij}^{\downarrow\uparrow}$, so equality occurs throughout. ∎

Irreducible Representations of Sublattices of Finite Products of Chains

We now extend the representation-of-sublattices Theorem 9 by developing irreducible representations of the type established by Hashimoto [5]. In order to do this, we shall generally have to give up the finiteness and irredundance of the representation. The key to developing irreducible representations of sublattices is to observe first that L is i-decreasing j-increasing if and only if its complement L^c is i-increasing j-decreasing. Thus complements of these sublattices are also sublattices. Incidentally, this fact, together with Theorem 9, implies that each sublattice of a finite product of chains has a representation in sublattices whose complements are sublattices, which is an instance of a result of Koh [6].

THEOREM 13 (Irreducible representation of sublattices of finite products of chains). *Every nonempty sublattice of a finite product* $S = \times_N S_k$ *of chains has a representation as an intersection of sets, each of which is the complement of an i-increasing j-decreasing hull of a singleton set* $\{s\}$ *for some* $i, j \in N$ *and* $s \in S$, *and is irreducible,*[14] *the last being so if and only if either* (a) $s_i \neq \vee S_i$ *and* $s_j \neq \wedge S_j$ *or* (b) (s_i, s_j) *is the least or greatest element of* $S_i \times S_j$.[15]

Proof. Observe that if (a) or (b) does not hold for some $i, j \in N$ and $s \in S$, then either (c) $s_i = \vee S_i$ and $s_j \neq \vee S_j$ or (d) $s_i \neq \wedge S_i$ and $s_j = \wedge S_j$. Also, $s \notin \{s\}_{ij}^{\uparrow\downarrow c}$ for all $i, j \in N$ and $s \in S$. Now let L be a nonempty sublattice of S.

We claim first that for each $s \in S \setminus L$, there exist $i, j \in N$ for which $\{s\}_{ij}^{\uparrow\downarrow c}$ contains L and either (a) or (b) holds. To that end, notice from the representation-of-sublattices Theorem 9 that there exist $k, l \in N$ such that $s \notin L_{kl}^{\downarrow\uparrow}$. Since $L_{kl}^{\downarrow\uparrow}$ is k-decreasing l-increasing, $L_{kl}^{\downarrow\uparrow c}$ is k-increasing l-decreasing, so $\{s\}_{kl}^{\uparrow\downarrow} \subseteq L_{kl}^{\downarrow\uparrow c} \subseteq S \setminus L$. Hence, $\{s\}_{kl}^{\uparrow\downarrow c}$ contains L. Now on setting $i = k$ and $j = l$, it follows that one of (a), (b), (c), or (d) holds. If (c) holds, then $\{s\}_{ll}^{\uparrow\downarrow c} \supseteq \{s\}_{kl}^{\uparrow\downarrow c} \supseteq L$, so instead on setting $i = j = l$, we see that (a) or (b) holds. Similarly, if (d) holds, then $\{s\}_{kk}^{\uparrow\downarrow c} \supseteq \{s\}_{kl}^{\uparrow\downarrow c} \supseteq L$, so instead on setting $i = j = k$, we see again that (a) or (b) holds, which establishes the claim.

It remains to show that $\{s\}_{ij}^{\uparrow\downarrow c}$ is irreducible if (a) or (b) holds and reducible if (c) or (d) holds. To that end, if (a) holds, then since $s \notin \{s\}_{ij}^{\uparrow\downarrow c}$, $\{s\}_{ij}^{\uparrow\downarrow c}$ is irreducible if every sublattice $P \supset \{s\}_{ij}^{\uparrow\downarrow c}$ also contains s. To see that $s \in P$, observe first that there exists $r \in P \setminus \{s\}_{ij}^{\uparrow\downarrow c}$, so $r_i \leqslant s_i$ and $r_j \geqslant s_j$. Now there exist $t_i > s_i$ in S_i and $u_j < s_j$ in S_j. Put $t_k = s_k$ for $k \neq i$, $u_k = s_k$ for $k \neq j$, $t = (t_l)$, and $u = (u_l)$. Then $t, u \in \{s\}_{ij}^{\uparrow\downarrow c} \subset P$, so $s = (r \wedge t) \vee u \in P$.

[14] Since the lattice of sublattices is compactly generated, the existence of the irreducible representations follows from the theory of such lattices [3, pp. 43–44]. However, the present approach has the advantages of also describing the qualitative structure of the elements of the representation and of not relying on Zorn's lemma.

[15] I am greatly indebted to Margarida Mello for her collaboration in sharpening my earlier version of Theorem 13 to characterize precisely which of the nonempty complements of i-increasing j-decreasing hulls of singleton sets are irreducible. The characterization and a slight modification of her proof thereof are included here with her kind permission.

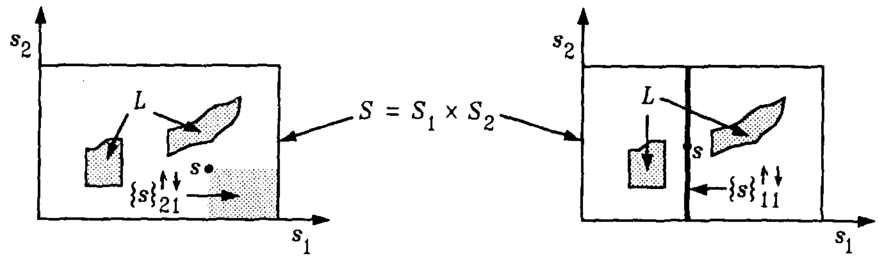

FIG. 4. Elements of irreducible representation of a sublattice L of a product $S_1 \times S_2$ of two chains.

If (b) holds, then either $(s_i, s_j) = (\wedge S_i, \wedge S_j)$ or $(s_i, s_j) = (\vee S_i, \vee S_j)$. In the former event, we show that $\{s\}_{ij}^{\uparrow\downarrow c}$ is irreducible by contradiction. Thus suppose that $\{s\}_{ij}^{\uparrow\downarrow c}$ is reducible, whence $\{s\}_{ij}^{\uparrow\downarrow c} = P \cap Q$ for some sublattices P, Q distinct from $\{s\}_{ij}^{\uparrow\downarrow c}$. Then there exist $r \in P \setminus \{s\}_{ij}^{\uparrow\downarrow c}$ and $t \in Q \setminus \{s\}_{ij}^{\uparrow\downarrow c}$. Thus, $r_i = t_i = s_i$ and $r_j, t_j \geqslant s_j$. Let $u_k = r_k \wedge t_k$ for $k \neq i$, and choose $u_i > s_i$ in S_i. The last is possible, for if not, $S_i = \{s_i\}$, whence $\{s\}_{ij}^{\uparrow\downarrow c} = \varnothing \subset L \subseteq \{s\}_{ij}^{\uparrow\downarrow c}$, which is a contradiction. Now $u_j \geqslant s_j$, $u = (u_l) \in \{s\}_{ij}^{\uparrow\downarrow c} = P \cap Q$, and $\sigma = (\sigma_l) \equiv r \wedge u = t \wedge u \in P \cap Q = \{s\}_{ij}^{\uparrow\downarrow c} = \{v \in S : v_i > s_i\}$, contradicting the fact that $\sigma_i = s_i$. If instead $(s_i, s_j) = (\vee S_i, \vee S_j)$, then $\{s\}_{ij}^{\uparrow\downarrow c}$ is irreducible by duality on interchanging the roles of i and j and applying what was just shown.

If (c) holds, there is an $r_j > s_j$ in S_j. Then $P = \{t \in S : t_j \neq r_j\}$ and $Q = \{t \in S : t_j < s_j \text{ or } t_j = r_j\}$ are sublattices distinct from $\{s\}_{ij}^{\uparrow\downarrow c}$, and $\{s\}_{ij}^{\uparrow\downarrow c} = P \cap Q$. If instead (d) holds, there is an $r_i < s_i$ in S_i. Then $P = \{t \in S : t_i \neq r_i\}$ and $Q = \{t \in S : t_i > s_i \text{ or } t_i = r_i\}$ are sublattices distinct from $\{s\}_{ij}^{\uparrow\downarrow c}$, and $\{s\}_{ij}^{\uparrow\downarrow c} = P \cap Q$. Thus, in both cases, $\{s\}_{ij}^{\uparrow\downarrow c}$ is reducible. ■

We remark that if $\vee S_i$ and $\wedge S_j$ do not exist, then $\{s\}_{ij}^{\uparrow\downarrow c}$ is an irreducible sublattice for every $s \in S$, because then trivially $s_i \neq \vee S_i$ and $s_j \neq \wedge S_j$.

In Figure 4 we illustrate two elements of the irreducible representation of Theorem 13 for the sublattice L of a product $S = S_1 \times S_2$ of two chains given in Figure 3. Observe that the 2-increasing 1-decreasing hull of the element $s \in S \setminus L$ in the left side of the figure is contained in $S \setminus L$, so the complement of $\{s\}_{21}^{\uparrow\downarrow}$ contains L, but not s, and so separates L from s. Similarly, the 1-increasing 1-decreasing hull of the element $s \in S \setminus L$ in the right side of the figure is contained in $S \setminus L$, so the complement of $\{s\}_{11}^{\uparrow\downarrow}$ also contains L, but not s, and so separates L from s. Moreover, it is apparent from both the left and right sides of the figure that the complement of no other i-increasing j-decreasing hull of s contains L in either case.

Observe that Theorem 13 assures that each nonempty sublattice of S can be expressed as an intersection of sets, each of which is a set of $r \in S$, that do not satisfy both of the inequalities

$$r_j \geqslant s_j \quad \text{and} \quad r_i \leqslant s_i$$

for some fixed $s \in S$ and $i, j \in N$. An equivalent formulation of this condition is that r satisfies at least one of the two (strict) inequalities

$$r_i > s_i \quad \text{or} \quad r_j < s_j. \tag{7}$$

Moreover, Theorem 13 implies that each nonempty sublattice of a finite product of chains has a coordinate-free representation as an intersection of sets of the form $D \cup I$ where D is a prime dual ideal and I is a prime ideal. To see this, let D be the first of the sets in (7) and I be the second. This is an instance of a result of Hashimoto [5, p. 183], because a product of chains is a distributive lattice.

Finally, note that Theorem 13 implies the main assertion of Theorem 9, viz., (a) $\Rightarrow$ (c), because complements of i-increasing j-decreasing hulls of singleton sets are i-decreasing j-increasing.

5. REPRESENTATION AND RECOGNITION OF POLYHEDRAL SUBLATTICES

We now apply the representation-of-sublattices Theorem 9 to give irreducible representations of the polyhedral elements of the Moore lattice of closed convex sublattices of $\mathfrak{R}^n$. To that end, recall that a *generalized node-arc incidence matrix* is a matrix in which each column has at most one positive and one negative element.

THEOREM 14 (Irreducible representations of polyhedral sublattices as dual generalized network-flow polyhedra). *The following are equivalent*:

(a) *L is a polyhedral sublattice of $\mathfrak{R}^n$.*

(b) $L = \{s \in \mathfrak{R}^n : As \leqslant b\}$ *for some matrix* $(A \ b)$ *with A the transpose of a generalized node-arc incidence matrix.*

(c) *L is the intersection of finitely many closed half-spaces, each of which is an irreducible element of the lattice of closed convex sublattices of $\mathfrak{R}^n$.*

Proof. (a) $\Rightarrow$ (b): Since L is polyhedral, $L_{ij}^{\downarrow\uparrow}$ is polyhedral because by (6) it is a projection of a polyhedral set. Thus there is a matrix $(A^{ij}\ b^{ij})$ with $L_{ij}^{\downarrow\uparrow} = \{s \in \Re^n : A^{ij}s \leqslant b^{ij}\}$. Since $L_{ij}^{\downarrow\uparrow}$ is i-decreasing j-increasing, the ith and jth columns of A^{ij} are respectively nonpositive and nonnegative, and the remaining columns vanish. Let $A = (A^{ij})$ and $b = (b^{ij})$. Then by the representation-of-sublattices Theorem 9, $L = \bigcap_{i,j} L_{ij}^{\downarrow\uparrow} = \{s \in \Re^n : As \leqslant b\}$ and A is the transpose of a generalized node-arc incidence matrix.

(b) $\Rightarrow$ (c): Let $(a_k\ b_k)$ denote the kth row of $(A\ b)$, and put $H_k \equiv \{s \in \Re^n : a_k s \leqslant b_k\}$. From (b), there is a pair i, j such that the ith and jth elements of a_k are respectively nonpositive and nonnegative, and the remaining elements vanish. Thus, H_k is i-decreasing j-increasing, and so by Theorem 9, is a sublattice of $\Re^n$. Moreover, the half-space H_k is also irreducible, since that is so of the closed half-spaces in the lattice of closed convex subsets of $\Re^n$. Thus, since $L = \bigcap_k H_k$, (c) holds.

(c) $\Rightarrow$ (a): Finite intersections of polyhedral sublattices are polyhedral sublattices. ■

Observe that, as the example following Theorem 4 shows, Theorem 14 does not assert that each half-space in every irredundant irreducible representation of a polyhedral sublattice is a sublattice, but merely that there is such a representation. However, the proof of Theorem 14 suggests a constructive method for finding a finite representation of a polyhedral sublattice L in closed half-spaces, each of which is a sublattice. The method is to use (6) to compute the projections $L_{ij}^{\downarrow\uparrow}$ by, say, the Fourier-Motzkin elimination procedure. As the proof that (a) implies (b) of Theorem 14 shows, this produces the desired representation of L as a dual generalized network-flow polyhedron.

If L is a closed half-space $\{s \in \Re^n : as \leqslant b\}$ and a sublattice of $\Re^n$, then every finite irredundant representation of L in closed half-spaces consists solely of the tangent closed half-space L itself. Thus by Theorem 14, a has at most two nonzero elements, and they are of opposite sign.

COROLLARY 15 (Unique irredundant irreducible representation of polyhedral sublattices of full dimension). *If L is a polyhedral sublattice of $\Re^n$, then each tangent closed half-space of L is a sublattice of $\Re^n$. If also the dimension of L is n, then L has a unique irredundant irreducible representation, viz., its tangent closed half-spaces.*

Proof. It follows from Theorem 14 that L has a finite irredundant irreducible representation in closed half-spaces that are at once sublattices. Moreover, that representation must contain all the tangent closed half-spaces. If also the dimension of L is n, that representation contains only the tangent closed half-spaces. ■

Corollary 16 (Characterization of affine subsemilattices and sublattices). *The following are equivalent*:

(a) *L is an affine meet sublattice of* $\mathfrak{R}^n$.
(b) *L is an affine join sublattice of* $\mathfrak{R}^n$.
(c) *L is an affine sublattice of* $\mathfrak{R}^n$.
(d) $L = \{s \in \mathfrak{R}^n : As = b\}$ *for some matrix* $(A \;\; b)$ *with A the transpose of a generalized node-arc incidence matrix.*
(e) *L is the intersection of finitely many hyperplanes, each of which is a sublattice of* $\mathfrak{R}^n$.

Proof. (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c): By duality, it is enough to show (a) implies (b). Suppose $r, s \in L$. Then $r \wedge s \in L$ because L is a meet sublattice. Thus since L is affine, $r \vee s = r + s - r \wedge s \in L$, as claimed.

(c) $\Rightarrow$ (d): By Theorem 14, L has an irredundant irreducible representation $H_1, \ldots, H_m$ with $H_i = \{s \in \mathfrak{R}^n : a_i s \leqslant b_i\}$ for some vector $(a_i \;\; b_i)$ with $a_i \neq 0$ having at most one positive and one negative element for each i. Suppose $s \in L$. We show that s is on the boundary of H_i for each i. If not, s is in the interior of H_i for some i. But H_i is not redundant, so there is an $r \in L$ that is on the boundary of H_i. Thus since L is affine, $2r - s \in L = \bigcap_j H_j \subseteq H_i$, a contradiction. Hence L is the intersection of the hyperplanes bounding $H_1, \ldots, H_m$ as claimed.

(d) $\Rightarrow$ (e): For each i, the ith row $(a_i \;\; b_i)$ of $(A \;\; b)$ and its negative both have at most one positive and at most one negative element. Thus, the half-spaces $\{s \in \mathfrak{R}^n : a_i s \leqslant b_i\}$ and $\{s \in \mathfrak{R}^n : -a_i s \leqslant -b_i\}$ are both sublattices, as is their intersection, the bounding hyperplane $\{s \in \mathfrak{R}^n : a_i s = b_i\}$.

(e) $\Rightarrow$ (a): Intersections of affine sublattices are affine sublattices. ∎

Characterization of Polyhedral Sublattices by Linear Inequalities

The next result characterizes when a polyhedral set is a sublattice by means of linear inequalities.

Theorem 17 (Characterization of polyhedral sublattices by linear inequalities). *If $L = \{s \in \mathfrak{R}^n : As \leqslant b\}$ is a nonempty polyhedron for some $m \times (n+1)$ matrix $(A \;\; b)$, the following are equivalent*:

(a) *L is a sublattice of* $\mathfrak{R}^n$.
(b) *There exist $m \times m$ matrices λ^i and $m \times n$ matrices μ^i, $i = 1, \ldots, n$, that satisfy* (4), *and $m \times m$ matrices $\bar{\lambda}^i$ and $m \times n$ matrices $\bar{\mu}^i$, $i = 1, \ldots, n$, that satisfy* (4) *with $-A$ replacing A.*

(c) *There exist* $m \times m$ *matrices* λ^{ij} *and column m-vectors* μ^{ij}, ν^{ij}, $i, j = 1,\ldots,n$, *that satisfy the linear inequalities*

$$\begin{aligned}
&\mu^{ij}1_i - \nu^{ij}1_j + \lambda^{ij}A = 0, \qquad \text{all } i, j,\\
&\sum_{k,l=1}^{n} \left(-\mu^{kl}1_k + \nu^{kl}1_l \right) = A,\\
&\sum_{k,l=1}^{n} \lambda^{kl}b \leqslant b,\\
&\mu^{ij} \geqslant 0, \quad \nu^{ij} \geqslant 0, \quad \lambda^{ij} \geqslant 0, \qquad \text{all } i, j,
\end{aligned} \tag{8}$$

where 1_i *is the ith unit row n-vector for all* i.

Proof. (a) $\Leftrightarrow$ (b): Apply both parts of Theorem 6.

(a) $\Leftrightarrow$ (c): By the representation-of-sublattices Theorem 9, L is a sublattice of $\Re^n$ if and only if every sequence $s, r^{ij} \in \Re^n$, $i, j = 1,\ldots,n$, that satisfies

$$\begin{aligned}
&Ar^{ij} \leqslant b,\\
&r_i^{ij} \leqslant s_i, \qquad (i, j = 1,\ldots,n)\\
&r_j^{ij} \geqslant s_j
\end{aligned} \tag{9}$$

also satisfies $As \leqslant b$. This is so if and only if for each $1 \leqslant k \leqslant n$, the maximum of $a_k s$ subject to (9) does not exceed b_k, where $(a_k \ \ b_k)$ is the kth row of $(A \ \ b)$. By the duality theorem of linear programming, this is so if and only if the dual of this linear program has a feasible solution whose objective function does not exceed b_k for each k. The system (8) expresses this fact. ∎

Polynomial-Time Recognition of Polyhedral Sublattices

One immediate implication of Theorem 17 is the following.

COROLLARY 18 (Polynomial-time recognition of polyhedral sublattices). *There is a polynomial-time algorithm for testing whether a polyhedron in* $\Re^n$ *is a sublattice thereof.*

Proof. This follows from Theorem 17, the fact that the size of the linear inequality system (8) is polynomial in that of the system $As \leqslant b$, and the fact that there exist algorithms for solving systems of linear inequalities that run in time that is polynomial in the size thereof, e.g., those of Khachiyan and Karmarkar. ■

Computations

One polynomial-time algorithm for testing whether a polyhedron is a sublattice is to use the polynomial-time algorithms for separately testing whether a polyhedron is a meet and a join sublattice as discussed at the end of Section 3. This appears to be more efficient than testing whether a polyhedron is a sublattice by using (c) of Theorem 17.

Dual Network-Flow Polyhedra

Theorem 14 gives a sublattice characterization of dual generalized network-flow polyhedra. The next result does likewise for dual network-flow polyhedra. A *node-arc incidence matrix* is a matrix in which each column has one $+1$, one -1, and zeros elsewhere. Let 1 be a column n-vector of ones.

COROLLARY 19 (Characterization of dual network-flow polyhedra). *The following are equivalent*:

(a) *L is a polyhedral sublattice of $\mathfrak{R}^n$, and $L + \lambda 1 = L$ for all $\lambda \in \mathfrak{R}$.*

(b) *$L = \{s \in \mathfrak{R}^n : As \leqslant b\}$ for some matrix $(A\ \ b)$ with A the transpose of a node-arc incidence matrix.*

Proof. (a) $\Rightarrow$ (b): By Theorem 14, $L = \{s \in \mathfrak{R}^n : As \leqslant b\}$ for some matrix $(A\ \ b)$ for which A is the transpose of a generalized node-arc incidence matrix. We can, of course, assume without loss of generality that no row of A is vacuous, for any such row can be removed. Now we must have $A1 = 0$, for if not, $A1 \neq 0$. Then we find for any fixed $s \in L$, by choosing $|\lambda|$ large enough, that $A(s + \lambda 1) = As + \lambda A1 \nleqslant b$, which contradicts the fact that $L + \lambda 1 = L$. Since $A1 = 0$, each row of A has two nonzero elements, one the negative of the other. Thus, on multiplying each row of $(A\ \ b)$ by the reciprocal of the positive element in the corresponding row of A, we can assume without loss of generality that each row of A has one $+1$, one -1, and zeros elsewhere.

(b) $\Rightarrow$ (a): By Theorem 14, L is a polyhedral sublattice of $\mathfrak{R}^n$. Also, since for each $s \in L$ and $\lambda \in \mathfrak{R}$ we have $A(s + \lambda 1) = As \leqslant b$, it follows that $L + \lambda 1 = L$. ■

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