Reduced order observer design for discrete-time nonlinear systems

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Abstract

This work is a geometric study of reduced order observer design for discrete-time nonlinear systems. Our reduced order observer design is applicable for Lyapunov stable discrete-time nonlinear systems with a linear output equation and is a generalization of Luenberger’s reduced order observer design for linear systems. We establish the error convergence for the reduced order estimator for discrete-time nonlinear systems using the center manifold theory for maps. We illustrate our reduced order observer construction for discrete-time nonlinear systems with an example.

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1. Introduction

The nonlinear observer design problem was introduced by Thau [1]. Over the past three decades, many significant works have been carried out on the construction of observers for nonlinear systems in the control systems literature [2–10]. This work is an extension of our recent work [8–11] on the full order observer design for nonlinear control systems.

The reduced order observer design for discrete-time nonlinear systems presented in this work is a generalization of the construction of reduced order observers for linear systems devised by Luenberger [12].

To explain the concept of reduced order observers, consider the discrete-time nonlinear system modelled by the equations

\[ \begin{align*}
  x(k+1) &= f(x(k)) \\
  y(k) &= Cx(k)
\end{align*} \]

(1)

where \( x \in \mathbb{R}^n \) is the state and \( y \in \mathbb{R}^p \) is the output of the discrete-time nonlinear system (1). For all practical situations, \( p \leq n \). Suppose that \( C \) has full rank, i.e. \( \text{rank}(C) = p \). Then we can make a linear change of coordinates

\[ \xi = \begin{bmatrix} \xi_m \\ \xi_u \end{bmatrix} = \Lambda x = \begin{bmatrix} C \\ Q \end{bmatrix} x \]

(2)

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where \( Q \) is chosen so that \( A \) is an invertible matrix. Note that \( \xi_m \in \mathbb{R}^p \) and \( \xi_u \in \mathbb{R}^{n-p} \). Such a choice of \( Q \) is made possible by the assumption that \( C \) has full rank.

Under the coordinates transformation (2), the plant (1) takes the form

\[
\begin{bmatrix}
\xi_m(k+1) \\
\xi_u(k+1)
\end{bmatrix} = \begin{bmatrix}
F_1(\xi_m(k), \xi_u(k)) \\
F_2(\xi_m(k), \xi_u(k))
\end{bmatrix} \\
y(k) = \xi_m(k)
\]

(3)

where

\[ F(\xi) = f(A^{-1}\xi). \]

The motivation for the reduced order state estimator or observer stems from the fact that in the plant model (3), the state \( \xi_m \) is directly available for measurement and hence it suffices to build an observer that estimates only the unmeasured state \( \xi_u \). The order of such an observer will correspond to the dimension of the unmeasured state, namely \( n - p \leq n \). This type of observer is called a reduced order observer [12] and it has many important applications in design problems.

In this work, we present a reduced order exponential observer designed for a Lyapunov stable discrete-time plant of the form (3). We establish that the associated estimation error decays to zero exponentially using the center manifold theory for maps [13].

This work is organized as follows. In Section 2, we give the problem statement for reduced order observer design. In Section 3, we present our main results, viz. reduced order exponential order design for Lyapunov stable discrete-time nonlinear systems. In Section 4, we illustrate our main results with an example.

2. Problem statement

In this work, we consider discrete-time nonlinear plants of the form

\[
\begin{bmatrix}
x_m(k+1) \\
x_u(k+1)
\end{bmatrix} = \begin{bmatrix}
F_1(x_m(k), x_u(k)) \\
F_2(x_m(k), x_u(k))
\end{bmatrix} \\
y(k) = x_m(k)
\]

(4)

where \( x_m \in \mathbb{R}^p \) is the measured state, \( x_u \in \mathbb{R}^{n-p} \) the unmeasured state and \( y \in \mathbb{R}^p \) the output of the plant (4). We assume that the state vector

\[ x = \begin{bmatrix} x_m \\ x_u \end{bmatrix} \]

is defined in a neighborhood \( X \) of the origin of \( \mathbb{R}^n \) and \( F : X \to \mathbb{R}^n \) is a \( C^1 \) map vanishing at the origin.

We can define the reduced order exponential observers for the discrete-time plant (4) as follows.

**Definition 1.** Consider a nonlinear difference system defined by

\[ z_u(k+1) = G(z_u(k), y(k)) \]

(5)

where \( z_u \in \mathbb{R}^{n-p} \) and \( G : \mathbb{R}^{n-p} \times \mathbb{R}^p \to \mathbb{R}^{n-p} \) is a locally \( C^1 \) mapping with \( G(0,0) = 0 \). Then the system (5) is called a reduced order exponential observer for the discrete-time plant (4) if the following conditions are satisfied:

**(O1)** If \( z_u(0) = x_u(0) \), then \( z_u(k) = x_u(k) \) for all \( k \in \mathbb{Z}_+ \), where \( \mathbb{Z}_+ \) is the set of all positive integers. (Basically, this requirement states that if the initial estimation error is zero, then the estimation error stays zero for all future time.)

**(O2)** For any given \( \epsilon \)-ball, \( B_\epsilon(0) \), around the origin of \( \mathbb{R}^{n-p} \), there exists a \( \delta \)-ball, \( B_\delta(0) \), around the origin of \( \mathbb{R}^{n-p} \) such that

\[ z_u(0) - x_u(0) \in B_\delta(0) \implies z_u(k) - x_u(k) \in B_\epsilon(0) \quad \text{for all } k \in \mathbb{Z}_+ \]

and, moreover,

\[ \|z_u(k) - x_u(k)\| \leq Ma^k \|z_u(0) - x_u(0)\| \quad \text{for all } k \in \mathbb{Z}_+ \]
for some positive constants $M$ and $a$ with $0 < a < 1$. (Basically, this requirement states that if the initial estimation error is sufficiently small, then the future estimation error can be made to stay in any arbitrarily assigned neighborhood of the origin, and in addition, the estimation error decays to zero exponentially with time.) □

In this work, we consider the problem of finding reduced order exponential observers of the form (5) for Lyapunov stable nonlinear plants of the form (4).

3. Main results

Linearizing the plant (4) at $x = 0$, we obtain the following:

$$
\begin{bmatrix}
    x_m(k+1) \\
    x_u(k+1)
\end{bmatrix} =
\begin{bmatrix}
    A_{11} & A_{12} \\
    A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
    x_m(k) \\
    x_u(k)
\end{bmatrix} +
\begin{bmatrix}
    \phi(x_m(k), x_u(k)) \\
    \psi(x_m(k), x_u(k))
\end{bmatrix}
$$

$$
y(k) =
\begin{bmatrix}
    I \\
    0
\end{bmatrix}
\begin{bmatrix}
    x_m(k) \\
    x_u(k)
\end{bmatrix}
$$

(6)

where $\phi, \psi$ are $C^1$ functions vanishing at the origin together with all their first partial derivatives.

First, we prove a basic lemma.

Lemma 1. The system linearization pair of the plant (6) is detectable if, and only if, the pair $(A_{12}, A_{22})$ is detectable.

Proof. By the PBH test for detectability for linear systems [14], the system linearization pair of the discrete-time plant (6) is detectable if, and only if,

$$\text{rank } \begin{bmatrix} C \\ \lambda I - A \end{bmatrix} = n \quad \text{for all } \lambda \text{ with } |\lambda| \leq 1$$

i.e.

$$\text{rank } \begin{bmatrix} I \\ \lambda I - A_{11} & 0 \\ -A_{21} & \lambda I - A_{22} \end{bmatrix} = n \quad \text{for all } \lambda \text{ with } |\lambda| \leq 1$$

i.e.

$$\text{rank } \begin{bmatrix} -A_{12} \\ \lambda I - A_{22} \end{bmatrix} = n - p \quad \text{for all } \lambda \text{ with } |\lambda| \leq 1$$

the pair $(A_{12}, A_{22})$ is detectable. This completes the proof. □

Our reduced order exponential observer design for the discrete-time nonlinear system (4) is based on the following basic assumptions:

(H1) The equilibrium $x = 0$ of the system (4) is Lyapunov stable.

(H2) The pair $(A_{12}, A_{22})$ is detectable.

We can justify the above assumptions as follows. As pointed out in [8], the stability assumption of the plant dynamics is because of the conceptual problem, viz. what does the existence of a local exponential observer mean in terms of the nonlinear dynamics of the system to be observed? For example, it must mean that the trajectories do not have finite escape time, but what does local existence mean for unbounded trajectories? In view of this crucial factor, we have focused our efforts in treating the local existence of reduced order exponential observers on those nonlinear systems which are Lyapunov stable. This justifies the assumption (H1). By Lemma 1, the assumption (H2) is equivalent to the assumption that the linearization pair of the system (6) is detectable, which is quite standard in nonlinear observer design. In fact, detectability is a necessary condition for the existence of local exponential observers (see [8–10]).

Like in the full order observer case for local exponential observers presented in [8–10], we may consider the generalization of the Luenberger observer given by

$$z_u(k+1) = G(z_u(k), y(k)) \triangleq F_2(y(k), z_u(k)) + L \text{ [correction term]}$$
where $L$ is the observer gain matrix and correction term $= y(k) - x_m(k) = 0$. (Note that $x_m$ is the measured state.) Thus, as in the reduced order observer design for linear systems [11], we can construct a suitable correction term using $\Delta y$, the forward difference of $y$, and the state equation for the dynamics of $x_m$. If the measurement $y$ is subject to noise, then the $\Delta$ operation amplifies the noise, and so this approach may not look very effective. However, in the actual implementation of the reduced order nonlinear estimator, we can avoid the appearance of $\Delta y$ in its equation via a suitable change of coordinates as in [12].

Note that
\[
\Delta y(k) = x_m(k + 1) - x_m(k) = F_1(x_m(k), x_u(k)) - x_m(k).
\]
Therefore,
\[
\Delta y(k) = (A_{11} - I)x_m(k) + A_{12}x_u(k) + \phi(x_m(k), x_u(k)).
\]
Hence, it follows that
\[
y(k + 1) = \Delta y(k) + y(k) = A_{11}y(k) + A_{12}x_u(k) + \phi(y(k), x_u(k)).
\]  
(7)

Note that in Eq. (7), the L.H.S. consists of terms that are available for measurement. Hence, we may view (7) as the new output equation, and accordingly, a correction term can be constructed.

Hence, we consider the following candidate observer:
\[
z_u(k + 1) = F_2(y(k), z_u(k)) + L[\Delta y(k) + y(k) - F_1(y(k), z_u(k))]
\]  
(8)
or equivalently,
\[
z_u(k + 1) = A_{21}y(k) + A_{22}z_u(k) + \psi(y(k), z_u(k))
\]
\[
+ L[y(k + 1) - A_{11}y(k) - A_{12}z_u(k) - \phi(y(k), z_u(k))]
\]  
(9)
where $z_u \in \mathbb{R}^{n-p}$ and the observer gain matrix $L$ is chosen such that $A_{22} - LA_{12}$ is convergent. (Note that a matrix $P$ is called convergent if all the eigenvalues of $P$ lie inside the open unit circle of the complex plane. By assumption (H2), we can always find a matrix $L$ such that $A_{22} - LA_{12}$ is convergent.)

Since $y = x_m$, from the plant dynamics in (6), it follows that
\[
x_u(k + 1) = A_{21}y(k) + A_{22}x_u(k) + \psi(y(k), x_u(k)).
\]  
(10)
The estimation error $e$ is defined by
\[
e = z_u - x_u.
\]
From (9) and (10) and (7), it follows that
\[
e(k + 1) = (A_{22} - LA_{12})e(k) + \psi(y(k), e(k) + x_u(k)) - \psi(y(k), x_u(k))
\]
\[
- L[\phi(y(k), x_u(k) + e(k)) - \phi(y(k), x_u(k))].
\]  
(11)

Note that by construction, the linearization matrix $A_{22} - LA_{12}$ in the error dynamics (11) is convergent, and also $e = 0$ is an invariant manifold for the discrete-time composite system consisting of the plant dynamics (6) and (11). Note also that by assumption (H1), $x = 0$ is a Lyapunov stable equilibrium of the plant dynamics in (6). Hence, by an argument using center manifold theory for maps [13] similar to the proof of Theorem 3 in [11], it can be established that the candidate observer defined by (8) is an exponential observer that estimates the unmeasured state $x_u$ of the plant dynamics in (4).

Next, we note that the implementation of the reduced order estimator given in the formula (8) may pose a problem as it involves the derivative of the measurement vector $y$. It is known that differentiation amplifies noise, so if $y$ is noisy, the use of $\Delta y$ is unacceptable. To get around this difficulty, we define the new estimator state to be
\[
\zeta_u = z_u - Ly.
\]

Then it can be easily shown that
\[
\zeta_u(k + 1) = F_2(y(k), Ly(k) + \zeta_u(k)) - LF_1(y(k), Ly(k) + \zeta_u(k)).
\]
We may summarize the above results in the following main theorem.

**Theorem 1.** Consider the plant (4) that satisfies the assumptions (H1)–(H2). Linearizing the plant equations in (4) at the origin, we obtain the equivalent form for the plant given by (6). Let \( L \) be any matrix (observer gain) such that \( A_{12} - LA_{22} \) is convergent. Then a reduced order state estimator (or observer) for the plant (4) is given by the estimator dynamics (8) having the estimator state \( z_u \) which can be implemented via the following equations:

\[
\begin{align*}
\zeta_2(k+1) &= F_2(y(k), \zeta_2(k) + Ly(k)) - LF_1(y(k), \zeta_u(k) + Ly(k)) \\
\zeta_u(k) &= \zeta_u(k) + Ly(k) \\
z_u(k) &= \zeta_u(k) + Ly(k).
\end{align*}
\] (12)

If the pair \( (A_{12}, A_{22}) \) is observable, then we can construct a reduced order estimator of the form (12) with the steady-state error dictated by error poles, which are the eigenvalues of the error matrix \( A_{22} - LA_{12} \), and which can be arbitrarily placed in the complex plane (subject to conjugate symmetry). □

4. An example

In this section, we will present a simple example to illustrate the construction of reduced order exponential state estimators for discrete-time nonlinear systems.

Consider the discrete-time nonlinear system described by

\[
\begin{align*}
x_1(k+1) &= x_1(k) + x_2(k) - x_1^3(k) \\
x_2(k+1) &= x_2(k) \\
y(k) &= x_1(k)
\end{align*}
\] (13)

where \( x_1 \) is the measured state and \( x_2 \) is the unmeasured state.

We shall construct a reduced order exponential state estimator that yields dead-zone steady-state error, i.e. the error matrix, which is a scalar, has zero eigenvalue. For this purpose, we shall first show that the hypotheses (H1) and (H2) of Theorem 1 are satisfied.

Note that the plant dynamics in (13) has a triangular structure. The second equation, \( x_2(k+1) = x_2(k) \), essentially indicates that \( x_2 \) is a constant. Hence, it is trivially stable at \( x_2 = 0 \). Also, if we set \( x_2 = 0 \) in the first equation, the dynamics

\[
x_1(k+1) = x_1(k) - x_1^3(k)
\]

is asymptotically stable at \( x_1 = 0 \). Hence, by a total stability result in Lyapunov stability theory, it follows that the equilibrium \( (x_1, x_2) = (0, 0) \) is a Lyapunov stable equilibrium of the dynamics in (13). Thus, the assumption (H1) holds.

Next, note that linearizing the plant (13) at \( x = 0 \), we obtain the system matrices

\[
C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}
\]

from which it is clear that the pair \( (C, A) \) is observable. By Lemma 1, it is immediate that the pair \( (A_{12}, A_{22}) \) is observable. Thus, the assumption (H2) also holds.

To meet the dead-zone steady-state error pole specification, we choose \( L \) so that

\[
A_{12} - LA_{22} = 0.
\]

Since \( A_{12} = 1 \) and \( A_{22} = 1 \), it is immediate that \( L = 1 \).

A simple application of Theorem 1 shows that the required reduced order exponential state observer is given by the estimation equation

\[
\zeta_2(k+1) = -y(k) + y^3(k)
\]

and the estimator state \( z_2 \) is given by the equation

\[
z_2(k) = \zeta_2(k) + y(k). \quad \Box
\]
References