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Second order differentiability of paths via a generalized $\frac{1}{2}$-variation [☆]

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Abstract

We find an equivalent condition for a continuous vector-valued path to be Lebesgue equivalent to a twice differentiable function. For that purpose, we introduce the notion of a $VBG_{1/2}$ function, which plays an analogous role for the second order differentiability as the classical notion of a VBG_* function for the first order differentiability. In fact, for a function $f:[a,b]\to X$, being Lebesgue equivalent to a twice differentiable function is the same as being Lebesgue equivalent to a differentiable function g with a pointwise Lipschitz derivative such that $g''(x)$ exists whenever $g'(x)\neq 0$. We also consider the case when the first derivative can be taken non-zero almost everywhere.

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1. Introduction

Zahorski [15] and Choquet [1] (see also Tolstov [13]) proved a result characterizing curves ($f:[a,b]\to\mathbb{R}^n$) that allow a differentiable parametrization (respectively a differentiable parametrization with almost everywhere non-zero derivative) as those curves having the VBG_* property (respectively which are also not constant on any interval). Fleissner and Foran [7] reproved this later (for real functions only and not considering the case of a.e. non-zero derivatives) using a different result of Tolstov. The definition of VBG_* is classical; see e.g. [12]. The mentioned results were generalized by L. Zajíček and the author [4] to curves with values in Banach spaces (and also metric spaces using the metric derivative instead of the usual one). Laczkovich, Preiss [10], and Lebedev [11] studied (among other things) the case of C^n-parametrizations of real-valued functions ($n\geqslant 2$). For a nice survey of differentiability of real-valued functions via homeomorphisms, see [8]. L. Zajíček and the author [5] characterized the situation when a Banach space-valued curve admits a C^2-parametrization (for Banach spaces with a C^1 norm) or a parametrization with finite convexity (for arbitrary Banach spaces).

Let X be a normed linear space, and $f:[a,b]\to X$. We say that f is *Lebesgue equivalent* to $g:[a,b]\to X$ provided there exists a homeomorphism h of $[a,b]$ onto itself such that $g=f\circ h$. In the present note, we prove the

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following two theorems characterizing the situation when a vector-valued path allows a twice differentiable parametrization (respectively such a parametrization with almost everywhere non-zero derivative):

Theorem 1. *Let X be a normed linear space, and $f : [a,b] \to X$ be continuous. Then the following are equivalent*:

(i) *f is Lebesgue equivalent to a twice differentiable function g;*
(ii) *f is Lebesgue equivalent to a differentiable function g whose derivative is pointwise-Lipschitz and such that for all $x \in [a,b]$, $g''(x)$ exists whenever $g'(x) \neq 0$;*
(iii) *f is $VBG_{1/2}$.*

Theorem 2. *Let X be a normed linear space, and $f : [a,b] \to X$ be continuous. Then the following are equivalent*:

(i) *f is Lebesgue equivalent to a twice differentiable function g with $g'(x) \neq 0$ for a.e. $x \in [a,b]$;*
(ii) *f is Lebesgue equivalent to a differentiable function g whose derivative is a pointwise Lipschitz function which is non-zero a.e. in $[a,b]$ and such that for all $x \in [a,b]$, $g''(x)$ exists whenever $g'(x) \neq 0$;*
(iii) *f is $VBG_{1/2}$, and f is not constant in any interval.*

As a matter of fact, a definition of a new notion of a $VBG_{1/2}$ function (see Definition 6 below) involving a certain fractional variation, that was inspired by the results of Laczkovich, Preiss, and Lebedev, is necessary to achieve our goal.

The case of n-times differentiable functions for $n \geqslant 3$ is more complicated even in the case $X = \mathbb{R}$, and this case is treated in a separate paper [3] (where we also prove a version of Zahorski lemma for n-times differentiable homeomorphisms). The difficulty in the case of higher order derivatives of paths stems from the fact that although for a curve parametrized by the arc-length, the first derivative (provided it exists) is equal to the tangent unit vector (and thus has norm 1), the magnitude of higher order derivatives is not thus simply bounded. The proof in the real-valued case of $n \geqslant 3$ uses some auxiliary variations and proceeds in a rather indirect way. This is a similar phenomenon as the case of C^1 parametrizations being different from the case of C^n $(n > 1)$ parametrizations; see e.g. [10, p. 405] (since, in some sense, the C^1 case corresponds to the case of twice-differentiable functions).

2. Preliminaries

By λ we will denote the Lebesgue measure on $\mathbb{R}$. By X, we will always denote a normed linear space, and by $B(x,r)$ an open ball with center x and radius $r > 0$. We say that a norm $\|\cdot\|$ on X is *Gâteaux differentiable* provided $Tv := \lim_{t\to 0} t^{-1}(\|x+tv\| - \|x\|)$ (where $v \in X$) is a bounded linear operator for each $x \neq 0$. If X is separable, then it is well known that X admits an equivalent Gâteaux differentiable norm (see e.g. [2, Theorem II.3.1(ii)]). For $f : [a,b] \to X$, we define the derivative $f'(x) := \lim_{t\to 0} t^{-1}(f(x+t) - f(x))$ for $x \in (a,b)$ (at the endpoints, we take the corresponding unilateral derivatives). Similarly, the second derivative $f''(x)$ of f at x is defined as $f''(x) := (f')'(x)$. Note that the property of "being twice differentiable" is preserved under equivalent renormings of X.

We say that f is *pointwise-Lipschitz at* $x \in [a,b]$ provided $\overline{\lim}_{t\to 0} \frac{\|f(x+t)-f(x)\|}{|t|}$ is finite. We say that f is *pointwise-Lipschitz* provided f is pointwise-Lipschitz at each $x \in [a,b]$.

Let $f : [a,b] \to X$ be continuous, and assume that X has a Gâteaux differentiable norm (there is no loss of generality in this assumption since the continuity of f implies that $Y := \overline{\operatorname{span}}(f([a,b]))$ is separable, and we can work with Y instead). By K_f we will denote the set of points $x \in [a,b]$ such that there is no open interval U containing x such that $f|_{\overline{U}}$ is either constant or admits an arc-length parametrization which is twice differentiable.

In the case of $X = \mathbb{R}$, the set K_f coincides with the set of points of varying monotonicity of f (see e.g. [10]). Obviously, K_f is closed and $\{a,b\} \subset K_f$. Since the property of a function being twice differentiable clearly does not depend on the equivalent renorming of X, we see that the set K_f does not depend on the choice of the (equivalent) Gâteaux smooth norm on X.

We have to assume that the norm on X is Gâteaux differentiable in order to have the following desirable property: if $f : [a,b] \to X$ is twice differentiable (respectively f' is pointwise-Lipschitz and $f''(x)$ exists whenever $f'(x) \neq 0$), and $f'(x) \neq 0$ for some $x \in (a,b)$, then $f|_{[x-\delta,x+\delta]}$ admits a twice differentiable arc-length parametrization (respectively an arc-length parametrization with a pointwise Lipschitz derivative which is also twice differentiable).

Let $K \subset [a,b]$ be a closed set with $a, b \in K$. We say that an interval $(c,d) \subset [a,b]$ is *contiguous to K in $[a,b]$* provided $c, d \in K$ and $(c,d) \cap K = \emptyset$ (i.e. it is a maximal open component of $[a,b] \setminus K$ in $[a,b]$).

By $V(f,[a,x])$ we denote the (usual) variation of f on $[a,x]$. We will sometimes use the notation $v_f(x) := V(f,[a,x])$ for $x \in [a,b]$. We say that $\{y_i\}_{i=0}^N$ is a *partition* of $[a,b]$ provided $a = y_0 < y_1 < \cdots < y_N = b$.

It is well known (see e.g. [9, Theorem 7] together with [6, Theorem 2.10.13]) that if $f : [a,b] \to X$ is Lipschitz, and $f'(x)$ exists for almost all $x \in [a,b]$, then

$$V\big(f,[a,b]\big) = \int_a^b \big\| f'(x) \big\| \, dx. \tag{2.1}$$

We will need the following lemma.

Lemma 3. *If X is a normed linear space with a Gâteaux differentiable norm, $f : [a,b] \to X$ has a pointwise Lipschitz derivative and $f''(x)$ exists whenever $f'(x) \neq 0$ and $x \in [a,b]$, then for each $x \in K_f$ either $x = a$ or $x = b$ or $f'(x) = 0$.*

Proof. If $x \in K_f \setminus \{a,b\}$ and $f'(x) \neq 0$, then there exists $\delta > 0$ such that $\|f'(y)\| > \eta$ for all $y \in B(x,2\delta)$ and some $\eta > 0$. Thus, $f''(y)$ exists whenever $y \in B(x,2\delta)$ by the assumptions since $f'(y) \neq 0$ for all $y \in B(x,2\delta)$. By (2.1), note that $|v_f(s) - v_f(t)| \geqslant \eta|s-t|$ for all $s,t \in B(x,2\delta)$, and also $(v_f^{-1})'(z) = 1/\|f'(v_f^{-1}(z))\|$ for each $z \in v_f(B(x,2\delta))$. For all $z \in v_f(B(x,2\delta))$ (by the chain rule for derivatives) we obtain

$$\big(f \circ v_f^{-1}\big)'(z) = \frac{f'(v_f^{-1}(z))}{\|f'(v_f^{-1}(z))\|},$$

and (write $v = v_f^{-1}$)

$$(f \circ v)''(z) = \frac{f''(v(z)) - \langle D(\|\cdot\|, f'(v(z))), f''(v(z))\rangle \cdot f'(v(z))/\|f'(v(z))\|}{\|f'(v(z))\|^2}.$$

Since $f \circ v|_{v_f(\overline{B}(x,\delta))}$ is an arc-length parametrization of $f|_{\overline{B}(x,\delta)}$, we have a contradiction with $x \in K_f$. □

We shall need the following lemma. For a proof, see e.g. [4, Lemma 2.7].

Lemma 4. *Let $\{a,b\} \subset B \subset [a,b]$ be closed, and $f : [a,b] \to \mathbb{R}$ be continuous. If $\lambda(f(B)) = 0$, then we have $V(f,[a,b]) = \sum_{i \in \mathcal{I}} V(f,[c_i,d_i])$, where $I_i = (c_i,d_i)$, $(i \in \mathcal{I} \subset \mathbb{N})$ are all intervals contiguous to B in $[a,b]$.*

As in [10], for $g : [a,b] \to \mathbb{R}$, $\alpha \in (0,1)$, and $K \subset [a,b]$, we will define $V_\alpha(g,K)$ as a supremum of sums

$$\sum_{i=1}^m \big|g(b_i) - g(a_i)\big|^\alpha,$$

where the supremum is taken over all collections $\{[a_i,b_i]\}_{i=1}^m$ of non-overlapping intervals in $[a,b]$ with $a_i, b_i \in K$ for $i = 1,\dots,m$.

We will need the following auxiliary lemma:

Lemma 5. *Let $\alpha \in (0,1)$, $A \subset \mathbb{R}$ be bounded, $f : A \to \mathbb{R}$ be uniformly continuous with $V_\alpha(f,A) < \infty$. Then $\lambda(\overline{f(A)}) = 0$.*

Proof. By [10, Theorem 2.10], it follows that $SV_\alpha(f,A) = 0$ (see [10] for the definition of SV_α). The inequality $t \leqslant t^\alpha$ for $t \in [0,1]$ shows that if $SV_\alpha(f,A) = 0$, then $SV_1(f,A) = 0$, and thus [10, Theorem 2.9] yields $\lambda(\overline{f(A)}) = 0$. □

We will need the following notion, which plays the role of VBG_* for the second order differentiability.

Definition 6. We say that a continuous $f : [a, b] \to X$ is $VBG_{1/2}$ provided f has bounded variation, and there exist closed sets $A_m \subset [a, b]$ ($m \in \mathcal{M} \subset \mathbb{N}$) such that $K_f = \bigcup_{m\in\mathcal{M}} A_m$, and $V_{1/2}(v_f, A_m) < \infty$ for each $m \in \mathcal{M}$.

If h is a homeomorphism of $[a, b]$ onto itself such that $g = f \circ h$, then clearly

$$K_g = K_{f\circ h} = h^{-1}(K_f). \tag{2.2}$$

Using this fact, the equality $v_g = v_{f\circ h} = v_f \circ h$ (since h is a homeomorphism), and the implied equality $V_{1/2}(v_g, h^{-1}(A_m)) = V_{1/2}(v_f, A_m)$, it follows that if f is $VBG_{1/2}$ and g is Lebesgue equivalent to f, then g is $VBG_{1/2}$. Also, since the definition of the class $VBG_{1/2}$ involves only the variation of a given function f, the class of $VBG_{1/2}$ functions does not depend on the (equivalent) norm of X since if $\|\cdot\|_i$, $i = 1, 2$, are two norms on X with $C_1\|x\|_1 \leqslant \|x\|_2 \leqslant C_2\|x\|_1$ for all $x \in X$, then

$$C_1^{\frac{1}{2}} \cdot V_{\frac{1}{2}}\big(v_f^{\|\cdot\|_1}, K\big) \leqslant V_{\frac{1}{2}}\big(v_f^{\|\cdot\|_2}, K\big) \leqslant C_2^{\frac{1}{2}} \cdot V_{\frac{1}{2}}\big(v_f^{\|\cdot\|_1}, K\big)$$

for any closed $K \subset [a, b]$ (here, $v_f^{\|\cdot\|_i}$ is a variation of f with respect to $\|\cdot\|_i$, $i = 1, 2$).

The following example shows that we cannot equivalently replace v_f by f in Definition 6 (even in the case $X = \mathbb{R}$).

Example 7. There exists a continuous function $f : [0, 1] \to \mathbb{R}$ with bounded variation such that f is not $VBG_{1/2}$, but there exist closed $A_m \subset K_f$ such that $K_f = \bigcup_m A_m$, and $V_{1/2}(f, A_m) < \infty$.

Proof. Let $C \subset [0, 1]$ be the standard middle-thirds Cantor set. By $\mathcal{I}_n$ we will denote the collection of all intervals contiguous to C such that $\lambda(I) < 3^{-n}$ for $I \in \mathcal{I}_n$, and by K_i^n, where $i = 1, \dots, 2^n$, $n \in \mathbb{N}$, denote the closed intervals at level $n + 1$ of the construction. Note that there exist open intervals $I_{nik} \subset [0, 1]$ and numbers $a_{nik} > 0$, where $n, k \in \mathbb{N}$ and $i = 1, \dots, 2^n$, such that

- (i) $\sum_{n,k\in\mathbb{N}} \sum_{i=1}^{2^n} a_{nik} < \infty$,
- (ii) $\sum_{k\in\mathbb{N}} \sqrt{a_{nik}} = \infty$ whenever $n \in \mathbb{N}$ and $i = 1, \dots, 2^n$,
- (iii) $I_{nik} \cap I_{n'i'k'} = \emptyset$ whenever $(n, i, k) \neq (n', i', k')$,
- (iv) if $k \neq k'$, then there exists $x \in C$ such that either $I_{nik} < x < I_{nik'}$ or $I_{nik'} < x < I_{nik}$,
- (v) $I_{nik} \subset K_i^n$, and $I_{nik} \in \mathcal{I}_n$ for all $n, k \in \mathbb{N}$, and $i = 1, \dots, 2^n$.

To construct the numbers a_{nik} satisfying (i) and (ii), let $a_n > 0$ be any sequence with $\sum_n a_n < \infty$. Let $a_{ni} := a_n/2^n$ for $i = 1, \dots, 2^n$, $c_{ni} > 0$ be chosen so that $a_{ni} = c_{ni} \cdot \sum_k k^{-2}$ for all $n \in \mathbb{N}$, $i = 1, \dots, 2^n$, and $a_{nik} := c_{ni} \cdot k^{-2}$. To construct the intervals I_{nik} satisfying (ii)–(v), let $A = \{(n, i, k)\colon n, k \in \mathbb{N},\ i = 1, \dots, 2^n\}$, and let $b : \mathbb{N} \to A$ be a bijection. We will identify $I_{b(j)}$ with I_{nik} provided $b(j) = (n, i, k)$. Let $b(1) = (n_1, i_1, k_1)$. Choose $I_{b(1)} \in \{I \in \mathcal{I}_{n_1}\colon I \subset K_{i_1}^{n_1}\}$ (this set is clearly non-empty), and we will construct $I_{b(j)}$ by induction. If $I_{b(1)}, \dots, I_{b(j)}$ were constructed, then we denote $b(j + 1) = (n, i, k)$, and choose

$$I_{b(j+1)} \in \big\{I \in \mathcal{I}_n\colon I \subset K_i^n\big\} \setminus \{I_{b(1)}, \dots, I_{b(j)}\} \tag{2.3}$$

(this can be done since the involved set of intervals is clearly infinite).

Since b is a bijection, we defined I_{nik} for each $n, k \in \mathbb{N}$, and $i = 1, \dots, 2^n$. Conditions (iii) and (iv) follow from the fact that all the distinct intervals I_{nik} are intervals contiguous to the Cantor set C. Condition (v) follows from (2.3).

Let $I = (a, b) \subset [0, 1]$ be an open interval. We denote $l(I) = a$, $r(I) = b$, and $c(I) = \frac{a+b}{2}$. We will define $f(x) := 0$ when $x \in [0, 1] \setminus (\bigcup_{n,k\in\mathbb{N}} \bigcup_{i=1}^{2^n} I_{nik})$, $f(c(I_{nik})) := a_{nik}$, and f to be continuous and affine on $[l(I_{nik}), c(I_{nik})]$ and $[c(I_{nik}), r(I_{nik})]$. Then f is a continuous function and by (i) it follows that $V(f, [0, 1]) < \infty$. Index the countable family of closed sets

$$\{C\} \cup \big\{\big\{l(I_{nik}), c(I_{nik}), r(I_{nik})\big\}\colon n, k \in \mathbb{N},\ i = 1, \dots, 2^n\big\}$$

as $(A_m)_{m\in\mathbb{N}}$. It immediately follows that $K_f = \bigcup_{m\in\mathbb{N}} A_m$ and $V_{1/2}(f, A_m) < \infty$ for all $m \in \mathbb{N}$ (since $f|_C \equiv 0$, and all those A_m that satisfy $A_m \neq C$ are finite).

Now we will show that f is not $VBG_{1/2}$. Suppose that $\tilde{A}_m$ satisfy $V_{1/2}(v_f, \tilde{A}_m) < \infty$, and $K_f = \bigcup_m \tilde{A}_m$. Since $C = \bigcup_m (C \cap \tilde{A}_m)$, by the Baire category theorem, there exists $m_0 \in \mathbb{N}$ and an open interval U such that $C \cap U \subset$

$C \cap \tilde{A}_{m_0} \cap U$ and $C \cap U \neq \emptyset$. Thus, there exists $n \in \mathbb{N}$ and $i \in \{1, \ldots, 2^n\}$ such that $K_i^n \subset U$, and conditions (iv), (v), and (ii) imply that

$$V_{\frac{1}{2}}(v_f, \tilde{A}_{m_0}) \geqslant \sum_{\{I \in \mathcal{I}_n:\ I \subset K_i^n\}} \big(V(v_f, I)\big)^{\frac{1}{2}} \geqslant \sum_k \sqrt{a_{nik}} = \infty,$$

which contradicts the choice of the sets $\tilde{A}_m$. Thus, f is not $VBG_{1/2}$. □

3. Lemmata

The following lemma is a sufficient condition for a function to be $VBG_{1/2}$.

Lemma 8. *Let $f : [a,b] \to X$ have a pointwise Lipschitz derivative, and suppose that $f''(x)$ exists whenever $f'(x) \neq 0$ and $x \in [a,b]$. Then f is $VBG_{1/2}$.*

Proof. Because f' is continuous on $[a,b]$ (and thus bounded), by [6, §2.2.7] we see that f is Lipschitz (and thus has finite variation). Without any loss of generality, we can assume that the norm on X is Gâteaux differentiable (see Section 2). By Lemma 3, it follows that

$$f'(x) = 0 \quad \text{whenever } x \in K_f \setminus \{a,b\}. \tag{3.1}$$

For $j \in \mathbb{N}$ define

$$D_j = \left\{ x \in [a,b]:\ \|f'(x) - f'(z)\| \leqslant j|x - z| \text{ for all } z \in B\left(x, \frac{1}{j}\right) \cap [a,b] \right\}.$$

Since f' is pointwise-Lipschitz, it follows that $[a,b] = \bigcup_j D_j$, and D_j is closed for each $j \in \mathbb{N}$. Let $D_j = \bigcup_{k \in \mathbb{N}} D_{jk}$ be such that each D_{jk} is closed, and $\operatorname{diam}(D_{jk}) < 1/j$. We order the doubly-indexed sequence $(K_f \cap D_{jk})_{j,k}$ into a single sequence (while omitting empty sets); we will call the new sequence A_m ($m \in \mathcal{M} \subset \mathbb{N}$).

It remains to show that $V_{1/2}(v_f, A_m) < \infty$, where $m \in \mathcal{M}$. Let $m \in \mathcal{M}$, and fix $j, k \in \mathbb{N}$ such that $A_m = D_{jk} \cap K_f$. Let $x < y$ be such that $x, y \in A_m$. If $\{x, y\} \neq \{a, b\}$, then note that (since we can assume that for example $x \neq a$ and thus $f'(x) = 0$ by (3.1)) using (2.1), we obtain

$$\big|v_f(y) - v_f(x)\big| = \int_x^y \|f'(s)\|\, ds \leqslant j(y - x)^2. \tag{3.2}$$

If $\{x, y\} = \{a, b\}$, then $|v_f(y) - v_f(x)| = V(f, [a,b])$. Applying this observation together with (3.2) to $[x, y] = [a_i, b_i]$, $i \in \{1, \ldots, N\}$, where $[a_i, b_i]$ are non-overlapping intervals with $a_i, b_i \in A_m$, we obtain

$$\sum_{i=1}^N \big|v_f(b_i) - v_f(a_i)\big|^{\frac{1}{2}} \leqslant \sqrt{j} \sum_{i=1}^N (b_i - a_i) + V\big(f, [a,b]\big)^{\frac{1}{2}} \leqslant \sqrt{j}(b - a) + V\big(f, [a,b]\big)^{\frac{1}{2}}. \tag{3.3}$$

By taking a supremum in (3.3) over all sequences $\{[a_i, b_i]\}_{i=1}^N$ as above, we obtain that $V_{1/2}(v_f, A_m) < \infty$. □

Lemma 9. *Let $\zeta : [\sigma, \tau] \to \mathbb{R}$ be a continuous strictly increasing Lipschitz function with $\zeta(\sigma) = 0$, and $\lambda(F) = 0$ for some closed $F \subset [\sigma, \tau]$ with $\sigma, \tau \in F$. Then $\lambda(\sqrt{\zeta}(F)) = 0$, where $\sqrt{\zeta}(x) := \sqrt{\zeta(x)}$ for $x \in [\sigma, \tau]$.*

Proof. Since the function $g(x) = \sqrt{x}$ on $[0, \infty)$ has Luzin's property (N) (i.e. it maps zero sets onto zero sets), the conclusion easily follows. □

We will need the following simple lemma.

Lemma 10. *Let $h_m : [a,b] \to [c_m, d_m]$ ($m \in \mathcal{M} \subset \mathbb{N}$) be continuous increasing functions such that*

$$\sum_{m \in \mathcal{M}} h_m(x) < \infty \quad \textit{for all } x \in [a,b].$$

Let $K \subset [a,b]$ be closed and such that $\lambda(h_m(K)) = 0$ for all $m \in \mathcal{M}$. Then $h : [a,b] \to [c,d]$, defined as $h(x) := \sum_{m\in\mathcal{M}} h_m(x)$, is a continuous and increasing function (for some $c, d \in \mathbb{R}$) such that $\lambda(h(K)) = 0$.

Proof. The continuity and monotonicity of h follows easily by the assumptions. Let $K \subset [a,b]$ be closed with $\lambda(h_m(K)) = 0$ for all $m \in \mathcal{M}$. Without any loss of generality, we can assume that $\{a,b\} \subset K$. Let (c_p, d_p) ($p \in \mathcal{P} \subset \mathbb{N}$) be all the intervals contiguous to K in $[a,b]$. Let $\varepsilon > 0$ and find $M \in \mathbb{N}$ such that $\sum_{m\in\mathcal{M}\cap(M,\infty)}(h_m(b) - h_m(a)) < \varepsilon$. Then

$$\lambda\big(h\big([a,b]\big)\big) = \sum_{m\in\mathcal{M}}\big(h_m(b) - h_m(a)\big) \leqslant \varepsilon + \sum_{m\in\mathcal{M}\cap[1,M]}\big(h_m(b) - h_m(a)\big)$$
$$= \varepsilon + \sum_{m\in\mathcal{M}\cap[1,M]}\sum_{p\in\mathcal{P}}\big(h_m(d_p) - h_m(c_p)\big) \leqslant \varepsilon + \sum_{p\in\mathcal{P}}\lambda\big(h(c_p,d_p)\big),$$

where we used Lemma 4 to obtain the second equality. Since $\operatorname{card}(h((c_p,d_p)) \cap h((c_q,d_q))) \leqslant 1$ for $p, q \in \mathcal{P}$, $p \neq q$, we obtain the equality $\lambda(h([a,b])) = \lambda(h(\bigcup_{p\in\mathcal{P}}(c_p,d_p)))$. Since the set $h(K) \cap h(\bigcup_{p\in\mathcal{P}}(c_p,d_p))$ is countable, we get $\lambda(h(K)) = 0$. □

Lemma 11. *Suppose that X is a normed linear space with a Gâteaux smooth norm. Let $f : [a,b] \to X$ be a continuous $VBG_{1/2}$ function which is not constant on any interval. Then there exists a continuous strictly increasing $v : [a,b] \to [\alpha,\beta]$ such that $\lambda(v(K_f)) = 0$, $f \circ v^{-1}$ is twice differentiable on $[\alpha,\beta] \setminus v(K_f)$ with $(f \circ v^{-1})'(x) \neq 0$ for $x \in [\alpha,\beta] \setminus v(K_f)$, and for each $x \in K_f$ there exists $0 < C_x < \infty$ such that*

$$\big\|f(y) - f(z)\big\| \leqslant C_x\big|v(z) - v(y)\big|\big(\big|v(z) - v(x)\big| + \big|v(y) - v(x)\big|\big), \tag{3.4}$$

whenever $y, z \in [a,b]$, and $\operatorname{sgn}(y - x) = \operatorname{sgn}(z - x)$.

Proof. Let A_m ($m \in \mathcal{M} \subset \mathbb{N}$) be as in the definition of $VBG_{1/2}$ for $g = f \circ v_f^{-1}$. Note that g is 1-Lipschitz, and

$$K_g = v_f(K_f) \tag{3.5}$$

(note that v_f is a homeomorphism because f is not constant on any interval; see (2.2)). Since f is $VBG_{1/2}$, by Lemma 5 we have $\lambda(v_f(K_f)) = \lambda(v_g(K_g)) = 0$. Let $\ell = v_f(b)$. Note that because g is an arc-length parametrization of f, we have $V(g,[c,d]) = d - c$ for all $0 \leqslant c < d \leqslant \ell$ (we will use this fact frequently without necessarily repeating it). Let (c_p, d_p) ($p \in \mathcal{P} \subset \mathbb{N}$) be all the intervals contiguous to K_g in $[0,\ell]$. Since $\lambda(v_g(K_g)) = 0$, by Lemma 4 (applied to $f = v_g$) we have $V(g,[0,\ell]) = \ell = \sum_{p\in\mathcal{P}} V(g,[c_p,d_p]) = \sum_{p\in\mathcal{P}}(d_p - c_p)$, and thus $\lambda(K_g) = \ell - \lambda(\bigcup_{p\in\mathcal{P}}(c_p,d_p)) = 0$. For $m \in \mathcal{M}$ and $x \in [0,\ell]$, we define $v_m(x)$ as a supremum of the sums

$$\sum_{i=1}^{N}(b_i - a_i)^{\frac{1}{2}}, \tag{3.6}$$

where the supremum is taken over all finite sequences $\{[a_i,b_i]\}_{i=1}^N$ of non-overlapping intervals in $[0,\ell]$ such that $a_i, b_i \in (A_m \cup \{0,x\}) \cap [0,x]$ for $i = 1,\ldots,N$. Similarly, we define $\tilde{v}_m(x)$ for $x \in [0,\ell]$ as a supremum of the sums in (3.6), where the supremum is taken over all finite sequences $\{[a_i,b_i]\}_{i=1}^N$ of non-overlapping intervals in $[0,\ell]$ such that $a_i, b_i \in (A_m \cup \{x,\ell\}) \cap [x,\ell]$ for $i = 1,\ldots,N$. Note that v_m is increasing and $\tilde{v}_m$ is decreasing on $[0,\ell]$. Note that v_g is affine on each $[c_p,d_p]$, and

$$v_m(x) = v_m(z) + (x - z)^{\frac{1}{2}} \quad \text{for } x \in [c_p,d_p], \tag{3.7}$$

where $z = \max((A_m \cup \{0\}) \cap [0,c_p])$, and similarly for $\tilde{v}_m$. Thus v_m (and similarly $\tilde{v}_m$) is twice (or even infinitely many times) differentiable on $[0,\ell] \setminus v_f(K_f)$ with $v_m'(x) > 0$ for all $x \in [0,\ell] \setminus v_f(K_f)$. Find $\varepsilon_m > 0$ such that

(a) if we define $w(x) := \sum_m \varepsilon_m \cdot (v_m(x) - \tilde{v}_m(x))$, then $w(0)$, and $w(\ell)$ are finite (thus, $w(x)$ is finite for all $x \in [0,\ell]$), and w is continuous on $[0,\ell]$ (provided all v_m, $\tilde{v}_m$ were continuous),
(b) for all $m \in \mathcal{M}$ and $p \in \mathcal{P}$ with $c_p + 1/m < d_p - 1/m$ and all $x \in (c_p + 1/m, d_p - 1/m)$, we have $\varepsilon_m \cdot \max(v_m'(x), |v_m''(x)|, -\tilde{v}_m'(x), |\tilde{v}_m''(x)|) < 2^{-m}$.

By (b), it is easy to see that $w'(x)$ exists, is positive, and $w''(x)$ exists for each $x \in [0,\ell] \setminus v_f(K_f)$. Put $v := w \circ v_f$, $\alpha = v(a)$, and $\beta = v(b)$.

To show that v is strictly increasing, it is enough to show that w is strictly increasing (as v_f is strictly increasing by the fact that f is not constant on any interval). On the other hand, to show that w is strictly increasing, it is enough to show that v_m is strictly increasing for each $m \in \mathcal{M}$. Fix $m \in \mathcal{M}$. Let $x, y \in [0,\ell]$ with $x < y$. If $x, y \in [c_p, d_p]$ for some $p \in \mathcal{P}$, then (3.7) implies that $v_m(x) < v_m(y)$, and similarly if $x \in (c_p, d_p)$ or $y \in (c_{p'}, d_{p'})$ for some $p \in \mathcal{P}$ (respectively $p' \in \mathcal{P}$). If $x, y \in K_f$, and $(x, y) \cap A_m = \emptyset$, then

$$v_m(t) = v_m(z) + \sqrt{t-z} \quad \text{for all } t \in [x, y], \tag{3.8}$$

where $z = \max((A_m \cup \{0\}) \cap [0, x])$, and thus $v_m(x) < v_m(y)$. Finally, if there exists $q \in A_m \cap (x, y)$, then $v_m(x) \leqslant v_m(q) < v_m(q) + \sqrt{y-q} \leqslant v_m(y)$, and thus $v_m(x) < v_m(y)$ also in this case. By a similar argument, $\tilde{v}_m$ is strictly decreasing.

For a fixed $m \in \mathcal{M}$, we will prove that whenever $r, s \in A_m \cup \{0, \ell\}$ with $r < s$, then

$$v_m(s) - v_m(r) \leqslant \sum_{\substack{p\in\mathcal{P}:\\ (c_p,d_p)\cap[r,s]\neq\emptyset}} \big(v_m(d_p) - v_m(c_p)\big). \tag{3.9}$$

A symmetrical argument then shows that

$$\tilde{v}_m(r) - \tilde{v}_m(s) \leqslant \sum_{\substack{p\in\mathcal{P}:\\ (c_p,d_p)\cap[r,s]\neq\emptyset}} \big(\tilde{v}_m(c_p) - \tilde{v}_m(d_p)\big). \tag{3.10}$$

To prove (3.9), fix $\varepsilon_0 > 0$, and let $\{[a_i, b_i]\}_{i=1}^N$ be non-overlapping intervals in $[r, s]$ such that $a_i, b_i \in (A_m \cup \{r, s\}) \cap [r, s]$ for $i = 1, \dots, N$ such that $v_m(s) = v_m(r) + \sum_{i=0}^{N-1} (b_i - a_i)^{1/2} + \varepsilon$, for some $0 \leqslant \varepsilon < \varepsilon_0/2$. For $i \in \{1, \dots, N\}$ by Lemma 4 applied to $f = v_g$ on $[a, b] = [a_i, b_i]$ and $B = (A_m \cup \{r, s\}) \cap [a_i, b_i]$ (note that $\lambda(v_g(A_m)) = 0$ since $\lambda(v_g(K_g)) = 0$, and thus $\lambda(v_g(B)) = 0$), let (γ_j^i, δ_j^i) $(j \in \{1, \dots, J^i\})$ be a finite collection of intervals contiguous to $A_m \cup \{r, s\}$ in $[a_i, b_i]$ such that $(b_i - a_i) \leqslant \sum_{j=1}^{J^i} (\delta_j^i - \gamma_j^i) + (\frac{\varepsilon_0}{2N})^2$. Then

$$v_m(s) - v_m(r) \leqslant \sum_{i=1}^{N} \sum_{j=1}^{J^i} \big(\delta_j^i - \gamma_j^i\big)^{\frac{1}{2}} + \frac{\varepsilon_0}{2} + \varepsilon. \tag{3.11}$$

By Lemma 9 applied to $\zeta(x) = x - \gamma_j^i$ on $[\sigma, \tau] = [\gamma_j^i, \delta_j^i]$, $F = K_g \cap [\gamma_j^i, \delta_j^i]$, and because $v_m(x) = v_m(\gamma_j^i) + (x - \gamma_j^i)^{1/2}$ for $x \in [\gamma_j^i, \delta_j^i]$, we have that $\lambda(v_m(K_g \cap [\gamma_j^i, \delta_j^i])) = 0$, and by Lemma 4 applied to $f = v_m$ on $[a, b] = [\gamma_j^i, \delta_j^i]$, and $B = K_g \cap [\gamma_j^i, \delta_j^i]$, we obtain that

$$\big(\delta_j^i - \gamma_j^i\big)^{\frac{1}{2}} \leqslant \sum_{\substack{p\in\mathcal{P}:\\ (c_p,d_p)\subset[\gamma_j^i,\delta_j^i]}} \big(v_m(d_p) - v_m(c_p)\big)$$

for each $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, J^i\}$. Combining this inequality with (3.11), we get

$$v_m(s) - v_m(r) \leqslant \sum_{\substack{p\in\mathcal{P}:\\ (c_p,d_p)\cap[r,s]\neq\emptyset}} \big(v_m(d_p) - v_m(c_p)\big) + \varepsilon_0,$$

and by sending $\varepsilon_0 \to 0$ it follows that (3.9) holds.

To show that v is continuous, it is enough to show that each v_m is continuous (as this implies that w is continuous by the choice of ε_m's, and the continuity of v_f follows from e.g. [6, §2.5.16]). Fix $m \in \mathcal{M}$. From (3.7), it follows that

(∗) v_m is continuous from the right at all points $x \in \bigcup_{p\in\mathcal{P}} [c_p, d_p)$, and continuous from the left at all points $x \in \bigcup_{p\in\mathcal{P}} (c_p, d_p]$.

If $(x, y) \cap A_m = \emptyset$ for some $y > x$ with $y \in (0, \ell] \cap K_g$, then (3.8) implies that v_m is continuous from the right at x. If $x \in A_m$ is a right-hand side accumulation point of A_m (i.e. $A_m \cap (x, x+\delta) \neq \emptyset$ for all $\delta > 0$), then (3.9) implies that $\lim_{\substack{y\to x+\\ y\in A_m}} v_m(y) = v_m(x)$, since

$$\sum_{\substack{p\in\mathcal{P}:\\ (c_p,d_p)\cap[x,y]\neq\emptyset}} \big(v_m(d_p) - v_m(c_p)\big) \to 0 \tag{3.12}$$

as $y \to x+$. Now the monotonicity of v_m implies that it is continuous from the right at x. Concerning the continuity from the left, by $(*)$ it is enough to prove that v_m is continuous from the left at all points $y \in (K_g \cap (0, \ell]) \setminus \bigcup_{p\in\mathcal{P}}\{d_p\}$. Fix such a point y. If there is an $x \in [0, y)$ such that $(x, y) \cap A_m = \emptyset$, then (3.8) implies that v_m is continuous from the left at y. If y is a left-hand side accumulation points of A_m, then (3.9) together with (3.12) imply that v_m is continuous from the left at y. A similar argument as above yields the continuity of $\tilde{v}_m$.

Now we will prove that $\lambda(v(K_f)) = 0$. Note that we already established that $\lambda(K_g) = 0$. Because $K_g = v_f(K_f)$ by (3.5), it is enough to prove that $\lambda(w(K_g)) = 0$. To apply Lemma 10 to h_k, where $h_{2k} := \varepsilon_k \cdot v_k$, and $h_{2k+1} := -\varepsilon_k \cdot \tilde{v}_k$, we have to check that $\lambda(v_m(K_g)) = 0$ and $\lambda(\tilde{v}_m(K_g)) = 0$ for all $m \in \mathcal{M}$. Let $m \in \mathcal{M}$. Then (3.9) applied to $r = 0$, and $s = \ell$ shows that $v_m(\ell) - v_m(0) \leqslant \sum_{p\in\mathcal{P}}(v_m(d_p) - v_m(c_p))$, and since $v_m(K_g) \cap v_m(\bigcup_{p\in\mathcal{P}}(c_p, d_p)) = \emptyset$, we get $\lambda(v_m(K_g)) = 0$. Similarly, we obtain $\lambda(\tilde{v}_m(K_g)) = 0$. Thus, Lemma 10 shows that $\lambda(w(K_g)) = 0$.

To prove that the second derivative of $f \circ v^{-1}$ exists and the first derivative is non-zero on $[\alpha, \beta] \setminus v(K_f)$, let $x \in [\alpha, \beta] \setminus v(K_f)$. Put $y = w^{-1}(x)$. There exists $p \in \mathcal{P}$ and $q \in \mathbb{N}$ such that $y \in (c_p + 1/q, d_p - 1/q)$. Since (by the chain rule and the smoothness of the norm on X) g is twice differentiable on (c_p, d_p) and $\|g'(x)\| = 1$ for all $x \in (c_p, d_p)$ (because g is an arc-length parametrization of f, because $t - s = \int_s^t \|g'\|$ for $c_p < s < t < d_p$, and since g' is continuous with $\|g'\| \leqslant 1$ on (c_p, d_p), it follows that $\|g'(x)\| = 1$ for all $x \in (c_p, d_p)$) it is enough to prove that $w'(y)$ exists, is non-zero, and $w''(y)$ exists (since then $(f \circ v^{-1})'(x) = g'(y) \cdot (w^{-1})'(x)$, and $(f \circ v^{-1})''(x) = g''(y) \cdot ((w^{-1})'(x))^2 + g'(y) \cdot (w^{-1})''(x)$). But by the choice of ε_m (for $m > q$), and by the properties of v_m, $\tilde{v}_m$ for all m, we see that $w'(y)$ exists, $w'(y) > 0$, and $w''(y)$ exists; the rest is a straightforward application of the "derivative of the inverse function" rule.

To prove (3.4) for f and v, by a substitution using v_f, it follows that it is enough to establish a version of (3.4), where f is replaced by g, and v by w. To that end, take $m \in \mathcal{M}$ such that $x \in A_m$, and let $C_m = (\varepsilon_m)^{-2}$. Take $y, z \in [0, \ell]$. Without any loss of generality, we can assume that $x < y < z$ (if $y < x$, then a symmetric estimate using $\tilde{v}_m$ yields the conclusion). Let $0 < \varepsilon_0 < v_m(z) - v_m(x)$. Find a sequence $\{[a_i, b_i]\}_{i=1}^N$ of non-overlapping intervals with endpoints in $(A_m \cup \{x, y\}) \cap [x, y]$ with $b_i < a_{i+1}$ for $i = 1, \dots, N-1$, and such that $v_m(y) = v_m(x) + \sum_{i=1}^N (b_i - a_i)^{1/2} + \varepsilon$, for some $0 \leqslant \varepsilon \leqslant \varepsilon_0$ (such a choice of $\{[a_i, b_i]\}_{i=1}^N$ is possible because $x \in A_m$). Then

$$\begin{aligned} v_m(z) &\geqslant v_m(x) + \sum_{i=1}^{N-1}(b_i - a_i)^{\frac{1}{2}} + (z - a_N)^{\frac{1}{2}} \\ &\geqslant v_m(x) + \sum_{i=1}^{N-1}(b_i - a_i)^{\frac{1}{2}} + (z - y + b_N - a_N)^{\frac{1}{2}}. \end{aligned}$$

Thus (since $z - y \geqslant \|g(z) - g(y)\|$), we get

$$\begin{aligned} v_m(z) - v_m(y) &\geqslant (b_N - a_N + z - y)^{\frac{1}{2}} - (b_N - a_N)^{\frac{1}{2}} - \varepsilon \\ &= \frac{\|g(y) - g(z)\|}{(b_N - a_N + z - y)^{\frac{1}{2}} + (b_N - a_N)^{\frac{1}{2}}} - \varepsilon \\ &\geqslant \frac{\|g(y) - g(z)\|}{v_m(z) - v_m(x) + v_m(y) - v_m(x)} - \varepsilon. \end{aligned}$$

By sending $\varepsilon_0 \to 0$, we obtain

$$\big\|g(y) - g(z)\big\| \leqslant \big(v_m(z) - v_m(y)\big)\big(v_m(z) - v_m(x) + v_m(y) - v_m(x)\big). \tag{3.13}$$

To finish the proof of (3.4) for g and w, note that $v_m(\tau) - v_m(\sigma) \leqslant \frac{1}{\varepsilon_m}(w(\tau) - w(\sigma))$ for any $0 \leqslant \sigma < \tau \leqslant \ell$; thus (3.4) follows from (3.13). □

We will need the following version of Zahorski's lemma. See e.g. [8] for a proof of a slightly weaker statement.

Lemma 12. *Let $F \subset [\alpha, \beta]$ be closed, $\{\alpha, \beta\} \subset F$, and $\lambda(F) = 0$. Then there exists an (increasing) continuously differentiable homeomorphism h of $[\alpha, \beta]$ onto itself such that $h'(x) = 0$ if and only if $x \in h^{-1}(F)$, h is twice differentiable on $[\alpha, \beta] \setminus h^{-1}(F)$, and h^{-1} is absolutely continuous.*

Proof. Since we were not able to locate a reference in the literature for this exact statement, we will sketch the proof. Let (a_i, b_i) (where $i \in \mathcal{I} \subset \mathbb{N}$) be all the intervals contiguous to F in $[\alpha, \beta]$. For each $i \in \mathcal{I}$ find a C^1 function $\psi_i : (a_i, b_i) \to \mathbb{R}$ such that

- $\psi_i(x) \geqslant 0$ for all $x \in (a_i, b_i)$, and $\lim_{x \to a_i+} \psi_i(x) = \lim_{x \to b_i-} \psi_i(x) = \infty$,
- $m_i := \min_{x \in (a_i, b_i)} \psi_i(x) > 0$, and if $|\mathcal{I}| = \aleph_0$, then $\lim_{i \to \infty, i \in \mathcal{I}} m_i = \infty$,
- $\sum_{i \in \mathcal{I}} \int_{a_i}^{b_i} \psi_i(t)\, dt < \infty$.

Such functions ψ_i clearly exist. Define $\psi : [\alpha, \beta] \to \mathbb{R}$ as $\psi(x) := \psi_i(x)$ for $x \in (a_i, b_i)$, and $\psi(x) = 0$ for $x \in F$. By the choice of ψ_i, it follows that ψ is integrable. Define $k(x) := \int_\alpha^x \psi(t)\, dt$; then k is continuous and (strictly) increasing. By integrability of ψ, it follows that k has Luzin's property (N), and thus k is absolutely continuous by the Banach–Zarecki theorem (see e.g. [14, Theorem 3]). Since k is an integral of a positive C^1 function on $[\alpha, \beta] \setminus F$, it follows that $k''(x)$ exists for all $x \in [\alpha, \beta] \setminus F$ and also $k'(x) > 0$. We also have that $k'(x) = \infty$ for $x \in F \setminus (\bigcup_i \{a_i\})$, as for $x \in F$ and $t > 0$ small enough, we have

$$k(x+t) - k(x) \geqslant m_j(x + t - a_j) + \sum_{(a_i, b_i) \subset [x, x+t]} m_i(b_i - a_i) \geqslant m_t \cdot t,$$

where $j \in \mathcal{I}$ is such that $x + t \in (a_j, b_j)$ and for $m_t := \min\{m_k\colon (a_k, b_k) \cap [x, x+t] \neq \emptyset\}$ we have $\lim_{t \to 0+} m_t = \infty$ by the choice of ψ_i. If $x = a_i$ for some $i \in \mathcal{I}$, then we have $k(x+t) - k(x) \geqslant t \cdot \min_{y \in [x, x+t]} \psi_i(y)$, and the minimum goes to infinity with $t \to 0+$ by the choice of ψ_i. By continuity and symmetry, the rest follows. Now define $\varphi(x) := \alpha + \frac{\beta - \alpha}{k(\beta)} k(x)$, $h := \varphi^{-1}$, and the conclusion of the lemma follows. □

4. Proofs of the main results

Proof of Theorems 1 and 2. We will prove the theorems simultaneously. The implication (i) ⇒ (ii) in both theorems is trivial. To prove that (ii) ⇒ (iii) in Theorem 1, let h be a homeomorphism such that $g = f \circ h$ has pointwise Lipschitz derivative and such that $g''(x)$ exists whenever $g'(x) \neq 0$. Then Lemma 8 implies that g is $VBG_{1/2}$. By a remark following Definition 6, it follows that f is $VBG_{1/2}$.

To prove that (ii) ⇒ (iii) in Theorem 2, conclude that f is $VBG_{1/2}$ as in the corresponding implication of Theorem 1. Further, note that if $g'(x) \neq 0$ for a.e. $x \in [a, b]$, then g is not constant in any interval. This notion is clearly stable with respect to Lebesgue equivalence.

To prove that (iii) ⇒ (i) in Theorem 2, without any loss of generality, we can assume that the norm on X is Gâteaux differentiable (see Preliminaries). Lemma 11 implies that there exists an increasing homeomorphism $v : [a, b] \to [\alpha, \beta]$ such that $f \circ v^{-1}$ is differentiable on $[\alpha, \beta]$, twice differentiable on $[\alpha, \beta] \setminus v(K_f)$, and $\lambda(v(K_f)) = 0$. Apply Lemma 12 to $F = v(K_f)$ to obtain an (increasing) continuously differentiable homeomorphism $h : [\alpha, \beta] \to [\alpha, \beta]$ such that $h'(x) = 0$ if $x \in h^{-1}(v(K_f))$, and such that h is twice differentiable on $[\alpha, \beta] \setminus h^{-1}(v(K_f))$. Let $g = f \circ v^{-1} \circ h$. By the chain rule for derivatives, we have that g is twice differentiable on $[\alpha, \beta] \setminus h^{-1}(v(K_f))$. Let $x \in h^{-1}(v(K_f))$. Then by (3.4) there exists a $C_x > 0$ such that

$$\frac{\|f \circ v^{-1}(y) - f \circ v^{-1}(z)\|}{|y - z|} \leqslant 2C_x |z - h(x)| \tag{4.1}$$

for $z < y < h(x)$ or $h(x) < y < z$ (and by continuity this holds also for $y = h(x)$), and $y, z \in [\alpha, \beta]$. It follows that $(f \circ v^{-1})'(h(x)) = 0$. Thus, $g'(x) = 0$ by the chain rule. It also follows from (4.1) that $(f \circ v^{-1})'(\cdot)$ is pointwise-Lipschitz at $h(x)$ with constant $2C_x$. This implies that

$$\begin{aligned} \left\| \frac{g'(x+t) - g'(x)}{t} \right\| &= \left\| \frac{(f \circ v^{-1})'(h(x+t))h'(x+t)}{t} \right\| \\ &= \left\| \frac{(f \circ v^{-1})'(h(x+t)) - (f \circ v^{-1})'(h(x))}{t} \right\| \cdot h'(x+t) \\ &\leqslant 2C_x \cdot \left| \frac{h(x+t) - h(x)}{t} \right| \cdot h'(x+t), \end{aligned}$$

for all $x + t \in [\alpha, \beta]$. The continuity of h' at x shows that $g''(x) = 0$. We see that f is Lebesgue equivalent to g (by composing $v^{-1} \circ h$ with an affine change of parameter). To see that the function g has non-zero derivative almost everywhere, we note that the homeomorphism h obtained by applying the Lemma 12 has an absolutely continuous inverse and $(f \circ v^{-1})'(x) \neq 0$ for all $x \in v(K_f)$ by Lemma 11, where $\lambda(v(K_f)) = 0$.

To show that (iii) $\Rightarrow$ (i) in Theorem 1, suppose that f is constant on some interval, and let (c_i, d_i) $(i \in \mathcal{I} \subset \mathbb{N})$ be the collection of all maximal open intervals such that f is constant on each (c_i, d_i). It is easy to see that we can find a continuous function $\tilde{f} : [a, b] \to X$ such that $f = \tilde{f}$ on $[a, b] \setminus \bigcup_i (c_i, d_i)$, $\tilde{f}$ is affine and non-constant on $(c_i, 2c_i/3 + d_i/3)$, $(2c_i/3 + d_i/3, c_i/3 + 2d_i/3)$, $(c_i/3 + 2d_i/3, d_i)$, such that $c_i, d_i \in K_{\tilde{f}}$ for $i \in \mathcal{I}$, such that $\tilde{f}$ is $VBG_{1/2}$, and such that if $u = f \circ \xi$ (respectively $v = f \circ \eta$) is an arc-length parametrization of $f|_{[c_i - \delta, c_i]}$ (respectively $f|_{[d_i, d_i + \delta]}$) for some $\delta > 0$, and $u'_-(\xi^{-1}(c_i))$ (respectively $v'_+(\eta^{-1}(d_i))$) exists, then $\tilde{f}'_+(c_i)/\|\tilde{f}'_+(c_i)\| \neq u'_-(\xi^{-1}(c_i))$ (respectively $\tilde{f}'_-(d_i)/\|\tilde{f}'_-(d_i)\| \neq v'_+(\eta^{-1}(d_i))$). Then, $K_{\tilde{f}} = K_f \cup \bigcup_{i \in \mathcal{I}} \{c_i, 2c_i/3 + d_i/3, c_i/3 + 2d_i/3, d_i\}$. By the previous paragraph, there exists a homeomorphism h of $[a, b]$ onto itself such that $\tilde{f} \circ h$ is twice differentiable. It follows that $f \circ h$ is twice differentiable (since $(\tilde{f} \circ h)'(x) = (\tilde{f} \circ h)''(x) = 0$ for all $x \in h^{-1}(\bigcup_i \{c_i, d_i\})$ by the construction). $\square$

The following example shows that even in the case of $X = \mathbb{R}$, $VBG_{1/2}$ functions do not coincide with continuous functions satisfying $V_{1/2}(f, K_f) < \infty$.

Example 13. There exists a continuous $VBG_{1/2}$ function $f : [0, 1] \to \mathbb{R}$ such that $V_{1/2}(f, K_f) = \infty$ (and thus f is not Lebesgue equivalent to a C^2 function by [10, Remark 3.6]).

Proof. Let $a_n \in (0, 1)$ be such that $a_n \downarrow 0$. Define $f(a_{2k}) = 0$, $f(a_{2k+1}) = 1/k^2$ for $k = 1, \ldots,$ and $f(0) = f(1) = 0$. Extend f to be continuous and affine on the intervals $[a_{2k+1}, a_{2k}]$ and $[a_{2k+2}, a_{2k+1}]$. Then $K_f = \{0, 1\} \cup \{a_n\colon n \geqslant 2\}$ and it follows that f is $VBG_{1/2}$ (with the obvious decomposition $A_0 = \{0, 1\}$, $A_n = \{a_n\}$ for $n \geqslant 2$), but $V_{1/2}(f, K_f) = \infty$ since

$$\sum_{k \geqslant 2} \sqrt{\left| f(a_{2k}) - f(a_{2k+1}) \right|} = \sum_{k \geqslant 2} \frac{1}{k} = \infty. \qquad \square$$

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