ELSEVIER
Topology and its Applications 121 (2002) 535–542
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

The skein polynomial of freely periodic knots

Nafaa Chbili

Département de Mathématiques, Faculté des Sciences de Monastir, Boulevard de l'Environnement, 5000 Monastir, Tunisia

Received 19 October 2000; received in revised form 5 February 2001

Abstract

We use the first coefficient of the HOMFLY polynomial to find a necessary condition for a knot to be freely periodic. In particular for $p = 3$ we obtain a simple but powerful criterion. As an application we show that some knot cannot have a certain free period. © 2001 Elsevier Science B.V. All rights reserved.

AMS classification: 57M25

Keywords: Knots and links; HOMFLY polynomial; Freely periodic

1. Introduction

An old question asked by Fox is: which knots may be fixed by a p-periodic transformation φ of the three sphere S^3? According to the type of the fixed point set $\mathcal{C}$ of φ and its relationship to the knot, Fox distinguished eight cases. Each one corresponds to one kind of symmetry. By the positive solution of the Smith conjecture, we know that if $\mathcal{C}$ is an unknotted circle then φ is conjugate to a rotation by a $2\pi/p$ angle. A link which is fixed by φ and disjoint from $\mathcal{C}$ is said to be p-periodic. However, if the action defined by φ has no fixed point, then K is called a p-freely periodic knot. It has long been conjectured that if the group $\mathbb{Z}/p\mathbb{Z}$ acts freely on S^3 then the action is topologically conjugate to a free linear action [9]. Equivalently if φ^i (for $i = 1, 2, \ldots, p-1$) has no fixed point then there exists an integer s coprime with p such that φ is conjugate to the lens transformation $\varphi_{p,s}$ defined as follows:

$$\begin{aligned} \varphi_{p,s} : S^3 &\longrightarrow S^3 \\ (z_1, z_2) &\longmapsto \left(e^{2i\pi/p} z_1, e^{2is\pi/p} z_2\right). \end{aligned}$$

A link which is fixed by $\varphi_{p,s}$ is said to be a (p, s)-lens link.

E-mail address: nafaa.chbili@esstt.rnu.tn (N. Chbili).

PII: S0166-8641(01)00093-1

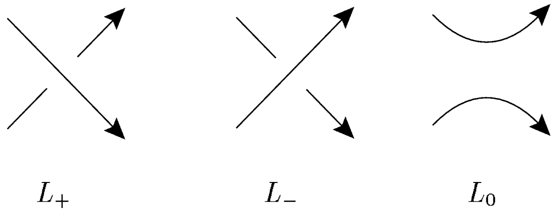


Fig. 1.

The new invariants of periodic knots have been subject of extensive literature. Several relationships between the Jones polynomials of a p-periodic link and its factor link were proved by Murasugi [6]. Traczyk [10,11] used the first coefficient of the HOMFLY polynomial to find simple but powerful criteria for testing a knot for possible periods. Tarczyk's criteria were extended by Yokota [12] to the $\frac{1}{2}(p-1)$ first coefficients of the HOMFLY polynomial. Przytycki [7] studied the Vassiliev–Gusarov invariants of periodic knots. However, the new invariants of freely periodic knots received no attention -to my knowledge- until now.

The HOMFLY polynomial is an invariant of oriented links which can be defined uniquely by the conditions

$$P_{\bigcirc}(v,z)=1,$$
$$v^{-1}P_{L_+}(v,z)-vP_{L_-}(v,z)=zP_{L_0}(v,z),$$

where L_+, L_- and L_0 are three oriented links which are identical except near one crossing where they look like in Fig. 1.

2. Statement of the main results

If K is a knot, then we have $P_K(v,z)=\sum_{i\geqslant 0}P_{2i,K}(v)z^{2i}$, where $P_{2i,K}(v)\in\mathbb{Z}[v^{\pm 2}]$ (see [5]). Let p be a prime. Throughout this paper we denote by $\mathbb{F}_p$ the field with p elements and by $P_K(v)_p$ the polynomial $P_{0,K}(v)$ with coefficients reduced modulo p. If n and m are two integers, then we denote by $\mathcal{T}(n,m)$ the torus link of type(n,m). Our main theorem is the following:

Theorem 1. *Let p be an odd prime and $s=\pm 1$. If K is a (p,s)-lens knot then $P_K(v)_p\in\Lambda_{p,s}$, where $\Lambda_{p,s}$ is the $\mathbb{F}_p[v^{\pm 2p}]$-module generated by the $P_{\mathcal{T}(\beta,\beta s\pm p)}(v)_p$, for $1\leqslant\beta\leqslant p-1$.*

The module $\Lambda_{p,s}$ is easy to handle for small values of p. In fact the Jones formula for the HOMFLY polynomial of torus knots can be used here to compute the generators of $\Lambda_{p,s}$. For $p=3$ and $p=5$ the result takes a nice and simple form.

Corollary 2. *If K is a freely periodic knot with period 3, then we have*:

$$P_K(v)_3\in\mathbb{F}_3\left[v^{\pm 6}\right].$$

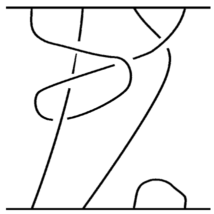
Corollary 3. *If K is a $(5, \pm 1)$-lens knot, then $P_K(v)_5 = \sum a_{2i} v^{2i}$ with: $a_{10k+4} = 2a_{10k+2}$ and $a_{10k+6} = 2a_{10k+8}$, for all $k \in \mathbb{Z}$.*

Applications. The following theorem [3] provides a combinatorial description of diagrams of lens links. Its proof uses a much more delicate analysis than that of the obvious version of periodic knots used previously by many authors.

Theorem 4. *A link K of S^3 is a (p, s)-lens link if and only if there exists an integer $n \neq 0$ and an n-tangle T such that:*

$$K = T^p \widehat{\left(\sigma_1 \sigma_2 \cdots \sigma_{n-1}\right)^{ns}}.$$

(1) Let K be the closure of $T^5(\sigma_1\sigma_2\sigma_3)^4$ where T is the 4-tangle as in the following picture:



By the previous theorem K is a $(5, 1)$-lens knot. A simple computation of the HOMFLY polynomial shows that $P_K(v)_5 = 4v^{16} + 2v^{18}$.

(2) Hartley [4] used the Alexander polynomial to find a criterion for free periodicity. This criterion does not decide whether the knot 9_{27} is freely 3-periodic or not. We have: $P_{9_{27}}(v)_p = v^{-2} - 2 + v^4 \notin \mathbb{F}_3[v^{\pm 6}]$ and by Corollary 2, 9_{27} is not freely 3-periodic. A simple application of this corollary excludes the possibility of free period 3 for all but 12 among the 84 prime knots with less than 9 crossings. The remaining knots are given by the following list:

$$5_1, 7_1, 8_2, 8_{10}, 8_{21}, 9_3, 9_6, 9_{26}, 9_{38}, 9_{41}, 9_{48}, 9_{49}.$$

The range of the given criterion can be expanded by applying it to appropriate cables of the 12 remaining knots. Indeed, the application given by Traczyk in the case of periodic knots (see [11]) indicates that prospects are good (in our case) for getting stronger results using cablings. In the case of $(5, \pm 1)$-lens knots, Corollary 3 rules out 60 among the 84 knots. So, it remains undecided for 24 knots (this means that the methods of this paper do not apply to 24 knots).

3. Proof of Theorem 1

The proof of Theorem 1 will be done in two steps. In the first one we show that the techniques used by Traczyk for periodic knots can be adapted to our case to reduce the problem to torus knots. In the second step we study the $\mathbb{F}_p[v^{\pm 2p}]$-module generated by the polynomial of torus knots of type $\mathcal{T}(n, ns+p)$. We show that this module is of finite type and we obtain a finite set of generators.

3.1. Reduction to torus knots

In this section we shall prove the following proposition.

Proposition I. *Let p be a prime, s an integer such that $\gcd(p,s)=1$ and K a (p,s)-lens knot. Then we have: $P_K(v)_p \in \Lambda'_{p,s}$ where $\Lambda'_{p,s}$ is the $\mathbb{F}_p[v^{\pm 2p}]$-module generated by $P_{\mathcal{T}(n,ns+p)}(v)_p$ for all integers n coprime with p.*

Proof. Let T be an m-tangle. We denote by T_+, T_- and T_0 three tangles which are the same except near one crossing where they are as in Fig. 1. We denote by Ω_m the central braid $(\sigma_1\sigma_2\cdots\sigma_{m-1})^m$. Let D be a (p,s)-lens diagram. By D_+, D_- and D_0 we denote the diagrams of the links $\widehat{T_+^p\Omega_m^s}$, $\widehat{T_-^p\Omega_m^s}$ and $\widehat{T_0^p\Omega_m^s}$. We shall prove Proposition I by induction on the number of crossings of the diagram D. If D has no crossings, then it is trivial and $P_D(v)_p = 1 \in \Lambda'_{p,s}$. Now, we suppose that $P_{D'}(v)_p \in \Lambda'_{p,s}$ for all (p,s)-lens diagram with less crossings than D. Using the fact that the braid Ω_m^s is central we can prove the following lemma (see also [7]). □

Lemma 1. *Let p be a prime, then we have*:

$$v^{-p}P_{D_+}(v,z) - v^p P_{D_-}(v,z) = z^p P_{D_0}(v,z) \mod p.$$

Note that if D_+ is a knot, then D_- is also a knot. However D_0 is a link with 2 or $p+1$ components. In the case of $p+1$ components $D_1, D_2, \ldots, D_{p+1}$, it is obvious that one component D_1 is a (p,s)-lens knot, while the other p components $D_2, \ldots, D_{p+1}$ are permuted cyclically by the lens action. In the same way as in [11] we can prove the following lemma:

Lemma 2. *If D_0 has two components, then*:

$$v^{-p}P_{D_+}(v)_p - v^p P_{D_-}(v)_p = 0.$$

If D_0 has $p+1$ components, then:

$$v^{-p}P_{D_+}(v)_p - v^p P_{D_-}(v)_p - v^{2\lambda}\left(v - v^{-1}\right)^p P_{D_1}(v)_p \left(P_{D_2}(v)_p\right)^p = 0.$$

In the previous lemma λ denotes the total linking number of the link D_0. The key observation in the proof of Proposition I is given by the following lemma.

Lemma 3. $P_{D_+}(v)_p \in \Lambda'_{p,s}$ *if and only if* $P_{D_-}(v)_p \in \Lambda'_{p,s}$.

Proof. The proof is obvious if D_0 has two components. If D_0 has $p+1$ components, then

$$\begin{aligned} P_{D_+}(v)_p &= v^{2p} P_{D_-}(v)_p + v^p v^{2\lambda}\big(v - v^{-1}\big)^p P_{D_1}(v)_p \big(P_{D_2}(v)_p\big)^p \\ &= v^{2p} P_{D_-}(v)_p + \big(v^2\big)^{\lambda}\big(v^2 - 1\big)^p P_{D_1}(v)_p \big(P_{D_2}(v)_p\big)^p. \end{aligned}$$

It is easy to see that $(1 - v^2)^p (P_{D_2}(v)_p)^p$ belongs to $\Lambda'_{p,s}$. The component D_1 has less crossings than D, and by the induction hypothesis $P_{D_1}(v)_p \in \Lambda'_{p,s}$. Now, we shall prove that p divides the linking number λ. We have

$$\lambda = \sum_{1 \leqslant i < j \leqslant p+1} \lambda_{i,j} = \sum_{1 < j \leqslant p+1} \lambda_{1,j} + \sum_{2 \leqslant i < j \leqslant p+1} \lambda_{i,j}.$$

We know that all the integers $\lambda_{1,j}$ are equal, so $\sum_{1<j\leqslant p+1} \lambda_{1,j} = p\lambda_{1,2}$. In the other hand the components D_i are cyclically permuted by $\varphi_{p,s}$. Then the linking number of D_i and D_j is equal to the linking number of $\varphi^l_{p,s}(D_i)$ and $\varphi^l_{p,s}(D_j)$ for all $1 \leqslant l \leqslant p-1$. Finally p divides $\sum_{2\leqslant i<j\leqslant p+1} \lambda_{i,j}$ and the lemma is proved. □

In the rest of this paper we denote by $D_+ \leftrightarrow D_-$ the transformation that consists of modifying p crossings to go from the diagram D_+ (respectively D_-) to D_- (respectively D_+). For $n \in \mathbb{N}^*$, let $\mathcal{B}_n$ be the n-braid group.

Lemma 4. *Every (p,s)-lens diagram may be transformed into a (p,s)-lens closed braid by a series of operations $D_+ \leftrightarrow D_-$ without increasing the number of crossings.*

Proof. More details about the arguments used to prove this lemma can be found in [10]. Here we follow the proof briefly. First, assume that the tangle T contains an arc (XY) with its two ends lying in the same side of the tangle. Assume also that there is no other arc of the same type with the two ends lying both between X and Y. By a series of operations $D_+ \leftrightarrow D_-$, this arc may be transformed into an arc lying above the rest of the tangle (see Fig. 2). By reversing self crossings, (XY) may be isotoped into an arc without self crossings. Of course $D_+ \leftrightarrow D_-$ means that we do the same thing for each copy of T. Now, we can push our arc (XY) from T to the next copy of T. Obviously, we can omit all arcs of type (XY) by this procedure to transform the m-tangle T into an n-braid B. Note that when we push arcs of type (XY) throughout $(\sigma_1\sigma_2\cdots\sigma_{m-1})^{ms}$, this braid is transformed into a central braid with less strings to finally get $(\sigma_1\sigma_2\cdots\sigma_{n-1})^{ns}$ (see Fig. 3). □

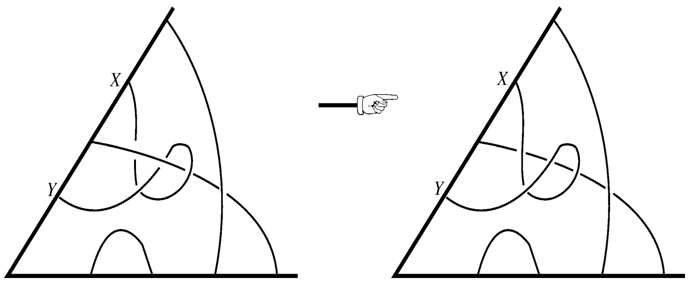


Fig. 2.

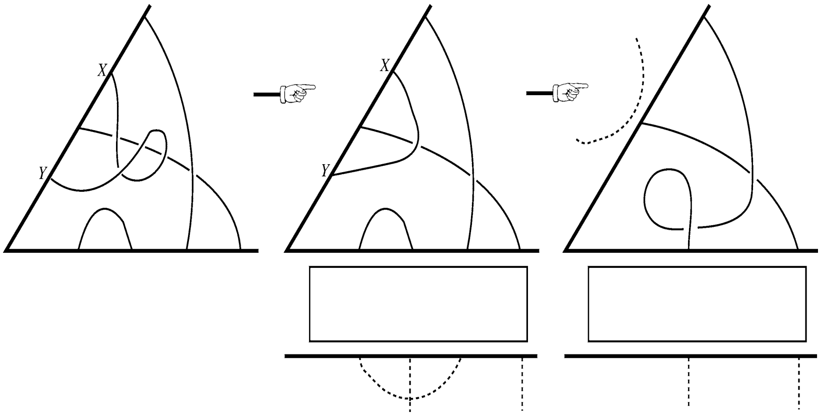


Fig. 3.

Lemma 5. *Let B be an n-braid. The (p,s)-lens knot $\widehat{B^p(\sigma_1\sigma_2\cdots\sigma_{n-1})^{ns}}$ may be transformed into the torus knot $\mathcal{T}(n, ns+p)$ by a series of operations $D_+ \leftrightarrow D_-$.*

Proof. As $\widehat{B^p(\sigma_1\sigma_2\cdots\sigma_{n-1})^n}$ is a knot, the permutation induced by B, $i(B)$ is an n-cycle. Hence, there exists an n-braid α such that $i(\alpha B\alpha^{-1}) = i(\sigma_1\sigma_2\cdots\sigma_{n-1}) = (n, 1, 2, \ldots, n-1)$. Let $B' = \alpha T\alpha^{-1}$, we have:

$$\begin{aligned} K &= \widehat{\left(\alpha^{-1}B'\alpha\right)^p(\sigma_1\sigma_2\cdots\sigma_{n-1})^{ns}} \\ &= \widehat{\alpha^{-1}B'^p\alpha(\sigma_1\sigma_2\cdots\sigma_{n-1})^{ns}} \\ &= \widehat{\alpha^{-1}B'^p(\sigma_1\sigma_2\cdots\sigma_{n-1})^{ns}\alpha} \quad \text{because } (\sigma_1\sigma_2\cdots\sigma_{n-1})^{ns} \in \text{ center of } \mathcal{B}_n \\ &= \widehat{B'^p(\sigma_1\sigma_2\cdots\sigma_{n-1})^{ns}}. \end{aligned}$$

Therefore we can assume without loss of generality that $i(B) = (n, 1, 2, \ldots, n-1)$. Now we can use operations $D_+ \leftrightarrow D_-$ to put the first string of B above the rest of the braid. Then to put the second string above strings $2, 3, \ldots, n$. Obviously, this procedure transforms B into $(\sigma_1\sigma_2\cdots\sigma_{n-1})$. Finally K is transformed into the torus knot $\mathcal{T}(n, ns+p)$. $\square$

3.2. The module $\Lambda'_{p,s}$

In this section we shall prove that in the case $s = \pm 1$, the module $\Lambda'_{p,s}$ is of finite type and we obtain a family of generators. More precisely we prove the following proposition:

Proposition II. *Let p be a prime, n an integer such that* $\gcd(p,n) = 1$ *and $s = \pm 1$. Then $P_{T(n,ns+p)}(v)_p \in \Lambda_{p,s}$ where $\Lambda_{p,s}$ is the $\mathbb{F}_p[v^{\pm 2p}]$-module generated by $P_{T(\beta,\beta s\pm p)}(v)_p$ for $1 \leqslant \beta \leqslant p-1$.*

Proof. Let us first consider $s = 1$ as a special case. We will deal with the case $s = -1$ later. The proof is done by induction on n, note here that $n = p\alpha + \beta$, where $\alpha \in \mathbb{N}$ and

$1 \leqslant \beta \leqslant p-1$. In fact we will prove by induction on α that $P_{T(p\alpha+\beta, p\alpha+\beta\pm p)}(v)_p \in \Lambda_{p,1}$. It is obvious that the induction is true for $\alpha = 0$. Now assume that for all $0 \leqslant m \leqslant \alpha - 1$ we have $P_{T(pm+\beta, pm+\beta\pm p)}(v)_p \in \Lambda_{p,1}$. Note that if we write Lemma 2 in the case where D is the torus knot $\mathcal{T}(n, n \pm p)$, the component D_1 is a torus knot of type $\mathcal{T}(r, r \pm p)$, where $r \leqslant \alpha - 1$. Then by induction we have $P_{D_1}(v)_p \in \Lambda_{p,1}$. Thus we conclude in the same way as in the proof of Proposition I that: $P_{D_+}(v)_p \in \Lambda_{p,1}$ if and only if $P_{D_-}(v)_p \in \Lambda_{p,1}$.

Therefore

$$
\begin{aligned}
&P_{\widehat{(\sigma_1\sigma_2\cdots\sigma_{n-1})^p(\sigma_1\sigma_2\cdots\sigma_{n-1})^n}}(v)_p \in \Lambda_{p,1}\\
&\quad\Longleftrightarrow P_{\widehat{(\sigma_1^{-1}\sigma_2\cdots\sigma_{n-1})^p(\sigma_1\sigma_2\cdots\sigma_{n-1})^n}}(v)_p \in \Lambda_{p,1}\\
&\quad\Longleftrightarrow P_{\widehat{(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-1})^p(\sigma_1\sigma_2\cdots\sigma_{n-1})^n}}(v)_p \in \Lambda_{p,1}\\
&\qquad\vdots\\
&\quad\Longleftrightarrow P_{\widehat{(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-1}^{-1})^p(\sigma_1\sigma_2\cdots\sigma_{n-1})^n}}(v)_p \in \Lambda_{p,1}.
\end{aligned}
$$

Recall that for all integers $n > 0$ we have

$$(\sigma_1\sigma_2\cdots\sigma_{n-1})^n = (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)^n.$$

Consequently

$$P_{\widehat{(\sigma_1\sigma_2\cdots\sigma_{n-1})^p(\sigma_1\sigma_2\cdots\sigma_{n-1})^n}}(v)_p \in \Lambda_{p,1} \Longleftrightarrow P_{\widehat{(\sigma_1\sigma_2\cdots\sigma_{n-1})^{-p}(\sigma_1\sigma_2\cdots\sigma_{n-1})^n}}(v)_p \in \Lambda_{p,1},$$

this means that $P_{\mathcal{T}(n,n-p)}(v)_p \in \Lambda_{p,1}$. Using the well known fact that $\mathcal{T}(n, n-p) = \mathcal{T}(n-p, n)$ and the induction hypothesis we conclude that $P_{\mathcal{T}(n,n+p)}(v)_p \in \Lambda_{p,1}$.

For the case $s = -1$, let us consider the involution:

$$
\begin{aligned}
f : \mathbb{F}_p\big[v^{\pm 1}\big] &\longrightarrow \mathbb{F}_p\big[v^{\pm 1}\big]\\
v &\longmapsto v^{-1},
\end{aligned}
$$

and denote by $\bar{\Lambda}_{p,1}$ the module $f(\Lambda_{p,1})$. From the case $s = 1$ we have $P_{\mathcal{T}(n,n-p)}(v)_p = P_{\mathcal{T}(n,-n+p)}(v^{-1})_p \in \Lambda_{p,1}$. Thus $P_{\mathcal{T}(n,n-p)}(v)_p \in \bar{\Lambda}_{p,1}$. It is obvious that $\bar{\Lambda}_{p,1}$ is the module generated by elements of type

$$
\begin{aligned}
&P_{\mathcal{T}(\beta,\beta-p)}\big(v^{-1}\big)_p = P_{\mathcal{T}(\beta,-\beta+p)}(v)_p, \qquad \text{for all } 1 \leqslant \beta \leqslant p-1.\\
&P_{\mathcal{T}(\beta,\beta+p)}\big(v^{-1}\big)_p = P_{\mathcal{T}(\beta,-\beta-p)}(v)_p, \qquad \text{for all } 1 \leqslant \beta \leqslant p-1.
\end{aligned}
$$

This proves that $\bar{\Lambda}_{p,1} = \Lambda_{p,-1}$ and completes the proof of Proposition II. $\square$

Acknowledgements

I would like to express my gratitude to Professor P. Vogel for his valuable suggestions.

References

[1] G. Burde, H. Zieshang, Knots, De Gruyter, Berlin, 1985.
[2] N. Chbili, On the invariants of lens knots, in: Proceedings of Knots ’96, World Scientific, Singapore, 1997, pp. 365–375.
[3] N. Chbili, The Jones polynomials of freely periodic knots, J. Knot Theory Ramifications 9 (7) (2000) 885–891.
[4] R. Hartely, Knots with free period, Canad. J. Math. 33 (1) (1981) 91–102.
[5] W.B.R. Lickorish, K.C. Millet, A polynomial invariant of oriented links, Topology 26 (1987) 107–141.
[6] K. Murasugi, Jones polynomials of periodic links, Pacific J. Math. 131 (1988) 319–329.
[7] J.H. Przytycki, Vassiliev–Gusarov skein modules of 3 manifolds and criteria for knot’s periodicity, Preprint, December 1992.
[8] D. Rolfsen, Knots and Links, Mathematics Lecture Series, Publish or Perish, Houston, TX, 1990.
[9] J.H. Rubinstein, An algorithm to recognize the 3-sphere, in: Proc. Internat. Congress Math. Zürich, 1994, pp. 601–611.
[10] P. Traczyk, A criterion for knots of period 3, Topology Appl. 36 (1990) 275–281.
[11] P. Traczyk, Periodic knots and the skein polynomial, Invent. Math. 106 (1) (1991) 73–84.
[12] Y. Yokota, The skein polynomial of periodic knots, Math. Ann. 291 (1991) 281–291.