The Stable Discrete Numerical Solution of Strongly Coupled Mixed Partial Differential Systems

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Abstract—This paper is concerned with the discrete numerical solution of coupled partial differential mixed problems with non-Dirichlet coupled boundary value conditions. By application of a discrete separation of variables method, the proposed numerical solution of the problem turns out to be the exact solution of certain coupled partial difference system, appearing from the discretization of the continuous partial differential system. Our approach avoids the iterative solution of algebraic systems which appears when one uses simple discretization methods. Existence, stability and the construction of solutions are considered. © 1999 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Coupled partial differential systems with coupled boundary value conditions are frequent in quantum mechanical scattering problems [1,2], chemical physics [3], modelling of coupled thermoelasto-plastic response of clays subjected to nuclear waste heat [4], and coupled diffusion problems [4,5]. The solution of these problems has motivated the study of vector and matrix Sturm-Liouville problems [4,6].

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This paper deals with coupled partial differential systems of the type

\begin{align}
\frac{u_t}{u_t}(x, t) - A \frac{u_{xx}}{u_{xx}}(x, t) &= 0, & 0 < x < 1, & t > 0, \\
\frac{u}{u_{x}}(0, t) + u_x(0, t) &= 0, & t > 0, \\
Bu(1, t) + Cu_x(1, t) &= 0, & t > 0, \\
\frac{u}{u}(x, 0) &= f(x), & 0 \leq x \leq 1,
\end{align}

where the unknown \( u = (u_1, u_2, \ldots, u_m)^T \) and \( f(x) = (f_1(x), \ldots, f_m(x))^T \) are \( m \)-dimensional vectors and \( A, B, C \) are \( m \times m \) complex matrices, elements of \( \mathbb{C}^{mxm} \). An analytic method for solving problems of type (1.1)-(1.4) has been recently proposed in [7-9] under rather restrictive hypotheses.

Due to physical conditions, the existence of solutions for the proposed problem is sometimes known, and in this these cases the construction of stable discrete numerical solutions is required. Matrix difference schemes have been used in [10], for solving coupled diffusion problems of type (1.1) with Dirichlet boundary conditions. Standard difference methods for problems without coupling in the boundary conditions, or scalar problems with non-Dirichlet boundary conditions have been considered in [5,11].

The organization of the paper is as follows. Section 2 deals with the discretization of the problem and the study of the existence of nontrivial stable solutions of the discretized partial boundary value difference system. In Section 3, the construction of numerical solutions of the mixed partial differential, stability and examples are considered.

Throughout this paper, the set of all eigenvalues of a matrix in \( \mathbb{C}^{mxm} \) is denoted by \( \sigma(D) \), the spectral radius of \( D \), defined by the maximum of the set \( \{ \|z\|; z \in \sigma(D) \} \) is denoted by \( \rho(D) \). We recall that \( D \) is said to be convergent if the sequence \( \{D^n\}_{n \geq 0} \) tends to the zero matrix of \( \mathbb{C}^{mxm} \), and by [12, Theorem 1.3.9], a matrix \( D \) in \( \mathbb{C}^{mxm} \) is convergent if \( \rho(D) < 1 \). If \( D \) is an Hermitian matrix, then \( \sigma(D) \) is contained in the real line and we denote by \( \lambda_{\text{min}}(D) \) the minimum of \( \sigma(D) \). If \( S \) is a matrix in \( \mathbb{C}^{mxm} \), we denote by \( S^+ \) its Moore-Penrose pseudoinverse. An account of properties, examples and applications of this concept may be found in [13,14]. In particular, the kernel of \( S \), denoted by \( \text{ker } S \) coincides with \( \text{Im}(I - S^+S) \), the image of the matrix \( I - S^+S \). The Moore-Penrose pseudoinverse \( S^+ \) of a matrix \( S \) can be computed efficiently with the MATLAB package. Finally, the vector subspace generated by a vector \( \omega \) is denoted by \( \text{Lin}\{\omega\} \).

2. THE DISCRETIZED PARTIAL DIFFERENCE BOUNDARY VALUE PROBLEM

Let us divide the domain \( [0, 1] \times [0, \infty] \) into equal rectangles of sides \( \Delta x = h \) and \( \Delta t = k \), and introduce coordinates of a typical mesh point \( p = (ih, jk) \) and let us represent \( u(ih, jk) = U(i, j) \). Approximating the partial derivatives appearing in (1.1) by the forward difference approximations

\begin{align*}
\frac{u_t}{u_t}(ih, jk) &\approx \frac{U(i, j + 1) - U(i, j)}{k}, \\
\frac{u_{xx}}{u_{xx}}(ih, jk) &\approx \frac{U(i + 1, j) - 2U(i, j) + U(i - 1, j)}{h^2},
\end{align*}

equation (1.1) takes the form

\begin{equation}
\frac{1}{k} [U(i, j + 1) - U(i, j)] = \frac{A}{h^2} [U(i, j + 1) - 2U(i, j) + U(i - 1, j)],
\end{equation}

where \( h = 1/N, 1 \leq i \leq N - 1, j \geq 0 \). Let \( r = k/h^2 \) and write the last equation in the form

\begin{equation}
\begin{array}{l}
ra [U(i + 1, j) + U(i - 1, j)] + (I - 2ra)U(i, j) - U(i + 1, j) = 0, \\
1 \leq i \leq N - 1, \quad j \geq 0.
\end{array}
\end{equation}
Boundary conditions (1.2),(1.3) take the form

\[ U(0, j) + N [U(1, j) - U(0, j)] = 0, \quad j \geq 0, \]  
\[ U(N, j) + NC [U(N, j) - U(N - 1, j)] = 0, \quad j \geq 0. \]  

(2.2)

(2.3)

We seek solutions of problem (2.1)-(2.3) of the form

\[ U(i, j) = G(j)H(i), \quad G(j) \in C^{mxm}, \quad H(i) \in C^{m}, \]  

(2.4)

where

\[ Nh = 1, \quad r = \frac{k}{h^2}. \]

(2.5)

By imposing to \( U(i, j) \) given by (2.4) that satisfies (2.1) one gets

\[ rAG(j) [H(i + 1) - H(i - 1)] + (I - 2rA)G(j)H(i) - G(j + 1)H(i) = 0. \]  

(2.6)

If \( \rho \) is a real number, by adding and subtracting the term \( \rho AG(j)H(i) \) to the left-hand side of (2.5) one gets

\[ 0 = rAG(j) \left[ H(i + 1) - H(i - 1) \right] + [(I + \rho A)G(j) - G(j + 1)]H(i). \]  

(2.7)

Note that (2.6) holds true if \{H(i)\}, \{G(j)\} verify

\[ G(j + 1) - (I + \rho A)G(j) = 0, \quad j \geq 0, \]  

(2.8)

\[ H(i + 1) + \left( \frac{-2r - \rho}{r} \right) H(i) + H(i - 1) = 0, \quad 1 \leq i \leq N - 1. \]  

(2.9)

The solution of (2.7) satisfying \( G(0) = I \) is given by

\[ G(j) = (I + \rho A)^j, \quad j \geq 0. \]  

(2.10)

Since coefficients of the vector equation (2.8) are scalar and for those values of \( \rho \) such that

\[ -4r < \rho \leq 0, \]  

(2.11)

one gets \(|(2r + \rho)/2r| \leq 1\), and that for \(-4r < \rho < 0\), the two different roots of

\[ z^2 - \left( \frac{2r + \rho}{r} \right) z + 1 = 0, \]

are

\[ z_0 = \left\{ \frac{2r + \rho}{2r} + i \left[ 1 - \left( \frac{2r + \rho}{2r} \right)^2 \right]^{1/2} \right\}, \]  

(2.12)

\[ z_1 = \left\{ \frac{2r + \rho}{2r} - i \left[ 1 - \left( \frac{2r + \rho}{2r} \right)^2 \right]^{1/2} \right\}. \]

If \( \rho \) and \( r \) are given, there exists \( \theta \in [0, 2\pi] \) such that \( z_0 = e^{i\theta}, \ z_1 = e^{-i\theta}, \) and

\[ z_0^\rho = \cos(n\theta) + i \sin(n\theta), \quad z_1^\rho = \cos(n\theta) - i \sin(n\theta). \]  

(2.13)

Hence, the solution set of the vector equation (2.8) is given by

\[ H(i) = z_0^i c + z_1^i d, \quad c, d \in C^{m}, \quad 1 \leq i \leq N - 1. \]  

(2.14)
Equation (2.4) implies \( G(j)[H(0) + N(H(1) - H(0))] = 0 \), for \( j \geq 0 \). Since we are interested in nonzero solutions of (2.1)-(2.3), \( \{G(j)\} \) cannot be identically zero and thus one gets the condition \( (N - 1)H(0) - NH(1) = 0 \), \( N > 1 \). By (2.12), (2.13), for \(-4r \leq \rho < 0\) the last condition implies

$$ c = -\left[ \frac{Nz_1 + (1-N)}{Nz_0 + (1-N)} \right] d, \quad (2.14) $$

and the solution set of (2.8) satisfying (2.2) takes the form

$$ H(i) = \{\sin(i\theta) - N[\sin(i\theta) - \sin((i-1)\theta)]\} d, \quad d \in C^m. \quad (2.15) $$

By (2.12), one gets \( \cos \theta = (2r + \rho)/2r \),

$$ \rho = -4r \sin^2 \left( \frac{\theta}{2} \right), \quad 0 < \theta < \pi \quad \text{or} \quad \pi < \theta < 2\pi, \quad (2.16) $$

because \( \theta = 0 \) and \( \theta = \pi \) yield the trivial solution in (2.15). Substituting (2.15) in (2.4) and using the formula \( \sin a - \sin b = 2 \sin((a - b)/2) \cos((a + b)/2) \), condition (2.3) takes the form

$$ \left\{ 2N(C - B) \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{2N-1}{2} \theta \right) + B \sin(N\theta) \right. $$

$$ + 4N^2C \sin^2 \left( \frac{\theta}{2} \right) \sin((N-1)\theta) \right\} G(j)d = 0, \quad j \geq 0. \quad (2.17) $$

By adding and subtracting \( B \sin((N-1)\theta) \) to the bracket of the last expression and using (2.9), one gets

$$ \left\{ 2 \left[ B - N(B - C) \right] \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{2N-1}{2} \theta \right) \right. $$

$$ + \left[ B + 4N^2C \sin^2 \left( \frac{\theta}{2} \right) \right] \sin((N-1)\theta) \right\} (I + \rho A)^j d = 0, \quad j \geq 0, \quad d \in C^m. \quad (2.18) $$

By the Cayley-Hamilton theorem [9, p. 206], if \( p \) is the degree of the minimal polynomial of \( A \), then for \( j \geq p \), the powers \( A^j \) can be expressed in terms of matrices \( I, A, A^2, \ldots, A^{p-1} \). Since \( \rho \neq 0 \), condition (2.17) is equivalent to the conditions

$$ T(\theta)A^j d = 0, \quad 0 \leq j < p, \quad d \in C^m, \quad d \neq 0, \quad (2.18) $$

where \( T(\theta) \) is the matrix defined by

$$ T(\theta) = 2 \left[ B - N(B - C) \right] \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{2N-1}{2} \theta \right) $$

$$ + \left[ B + 4N^2C \sin^2 \left( \frac{\theta}{2} \right) \right] \sin((N-1)\theta). \quad (2.19) $$

Thus problem (2.1)-(2.3) has nontrivial solutions of form (2.4) if condition (2.18) holds. A necessary condition is that \( T(\theta) \) be singular. Let us assume the following hypothesis.

There exists \( \lambda_0 \geq 0 \) such that \( B + \lambda_0 C \) is invertible. \( \quad (2.20) \)

**Remark 1.** Condition (2.20) holds if some of the matrices \( B \) or \( C \) is invertible. In fact, if \( B \) is invertible then taking \( \lambda_0 = 0 \) one gets (2.20). Let us suppose now that \( C \) is invertible and note that for \( \lambda > 0 \), the matrix \( B + \lambda C \) is invertible if and only if \( ((1/\lambda)B + C)\lambda \) is invertible, or \( (1/\lambda)B + C \) is invertible. Since \( C \) is invertible, the perturbation lemma [10, p. 584] implies that
Stable Discrete Numerical Solution 173

(1/\lambda)B + C is invertible if (1/\lambda)\|B\| < \|C^{-1}\|^{-1}$, or $\lambda > \|B\| \|C^{-1}\|$, thus condition (2.20) holds true for $\lambda > \|B\| \|C^{-1}\|$, if $C$ is invertible.

Let $\lambda_0$ be given by (2.20) and let us write $T(\theta)$ in the form

$$T(\theta) = \left[ (B + \lambda_0 C) + C \left( 4N^2 \sin^2 \left( \frac{\theta}{2} \right) - \lambda_0 I \right) \right] \sin((N - 1)\theta)$$

$$+ 2 \left[ (1 - N) \left( B + \lambda_0 C \right) + (N + \lambda_0 (N - 1)) C \right] \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{2N - 1}{2} \theta \right),$$

and premultiplying the last expression by $(B + \lambda_0 C)^{-1}$, one gets that condition $T(\theta)$ is singular, is equivalent to the singularity of the matrix

$$(B + \lambda_0 C)^{-1} T(\theta) = \left[ I + (B + \lambda_0 C)^{-1} C \left( 4N^2 \sin^2 \left( \frac{\theta}{2} \right) - \lambda_0 I \right) \right] \sin((N - 1)\theta)$$

$$+ 2 \left[ (1 - N)I + (N + \lambda_0 (N - 1)) (B + \lambda_0 C)^{-1} C \right] \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{2N - 1}{2} \theta \right).$$

Note that the last condition can be expressed in the form

$$\left[ I + 4N^2 \sin^2 \left( \frac{\theta}{2} \right) \right] \sin((N - 1)\theta)$$

$$+ 2 \left[ (1 - N)I + (N + \lambda_0 (N - 1)) \Omega \right] \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{2N - 1}{2} \theta \right),$$

where $\Omega = (B + \lambda_0 C)^{-1} C$. By the spectral mapping theorem [10, p. 569] condition (2.22) holds if one satisfies the following condition

There exists a real number $\alpha_0 \in \sigma \left( (B + \lambda_0 C)^{-1} C \right)$ and $\theta \in ]0, \pi[ \cup ]\pi, 2\pi[ \text{ such that } g(\theta) = 2 \left[ N \left[ 1 + \lambda_0 \right] \alpha_0 - 1 \right] + 1 - \lambda_0 \alpha_0$

$$\cos \left( \frac{2N - 1}{2} \theta \right) + \left[ 1 - \lambda_0 \alpha_0 + 4N^2 \alpha_0 \sin^2 \left( \frac{\theta}{2} \right) \right] \sin((N - 1)\theta) = 0.$$ (2.33)

Consider the following subcases of case (2.20):

$$(1 + \lambda_0) \alpha_0 - 1 = 0,$$ (2.24)

$$(1 + \lambda_0) \alpha_0 - 1 \neq 0.$$ (2.25)

Under condition (2.24), it follows that $1 - \lambda_0 \alpha_0 = \alpha_0$, with $\alpha_0 \neq 0$. Furthermore, in this case one gets $\sin((N - 1)\theta) \neq 0$ for $\theta \in ]0, \pi[ \cup ]\pi, 2\pi[ \text{ satisfying (2.33).}$ In fact, by (2.23) if $\sin((N - 1)\theta) = 0$, one gets $N\alpha_0 \sin(N\theta) = 0$. Hence, we also have $\sin(N\theta) = 0$, but this only occurs for $\theta = \pi$ or $\theta = 0$. Since $\sin((N - 1)\theta) \neq 0$ and $\alpha_0 \neq 0$ by (2.23), condition $g(\theta) = 0$ in (2.23) implies

$$2 \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{2N - 1}{2} \theta \right) + \left[ 1 + 4N^2 \sin^2 \left( \frac{\theta}{2} \right) \right] \sin((N - 1)\theta) = 0,$$

or

$$\sin(N\theta) + 4N^2 \sin^2 \left( \frac{\theta}{2} \right) \sin((N - 1)\theta) = 0.$$

Since $\sin((N - 1)\theta) \neq 0$, (2.33) and (2.24) imply

$$\frac{\sin(N\theta)}{\sin((N - 1)\theta)} = -4N^2 \sin^2 \left( \frac{\theta}{2} \right).$$ (2.26)
Equation (2.26) can be written in the form

\[
\cot ((N - 1)\theta) = (2N^2 - 1) \left\{ \frac{\cos \theta - 2N^2/(2N^2 - 1)}{\sin \theta} \right\},
\]

and since \( \cos \theta - 2N^2/(2N^2 - 1) < 0 \), one gets the equation

\[
\tan ((N - 1)\theta) = \frac{\sin \theta}{(2N^2 - 1) [\cos \theta - 2N^2/(2N^2 - 1)]}.
\]  

(2.27)

Note that the right-hand side of (2.27) is continuous and bounded in \([0, \pi] \cup [\pi, 2\pi] \). Furthermore, \( \tan((N - 1)\theta) \) is a transformation of each interval \( I_k = [(k - 1)/(N - 1))\pi, (k/(N - 1))\pi \) onto the all real line for \( k = 1, 2, \ldots, 2N - 2 \). Thus, for each \( k = 1, 2, \ldots, 2N - 2 \), there exists a root \( \theta_k \in I_k \) of (2.27) satisfying

\[
\tan ((N - 1)\theta_k) = \frac{\sin \theta_k}{(2N^2 - 1) [\cos \theta_k - 2N^2/(2N^2 - 1)]}, \quad 1 \leq k \leq N - 2, \quad \theta_k \in I_k. \tag{2.28}
\]

By (2.21),(2.26) it follows that

\[
\begin{align*}
\frac{T(\theta)}{\sin ((N - 1)\theta)} &= \left[ \frac{\sin (N\theta)}{\sin ((N - 1)\theta)} - 1 \right] (B - N(B - C)) + \left( B + 4N^2C \sin^2 \left( \frac{\theta}{2} \right) \right) \\
&= -\left( 1 + 4N^2 \sin^2 \left( \frac{\theta}{2} \right) \right) (B - N(B - C)) + \left( B + 4N^2C \sin^2 \left( \frac{\theta}{2} \right) \right) \\
&= 4N^2(C - B) \sin^2 \left( \frac{\theta}{2} \right) - N(C - B) \left( 1 + 4N^2 \sin^2 \left( \frac{\theta}{2} \right) \right) \\
&= -\left( N + 4N^2(N - 1) \sin^2 \left( \frac{\theta}{2} \right) \right) (C - B) \\
&= -(N - 1) \left[ \frac{N}{N - 1} + 4N^2 \sin^2 \left( \frac{\theta}{2} \right) \right] (C - B).
\end{align*}
\]

Since \( N/(N - 1) + 4N^2 \sin^2(\theta/2) > 0 \), from the last expression and previous comments it follows that

\[
\frac{-T(\theta)}{(N + 4(N - 1)N^2 \sin^2(\theta/2)) \sin ((N - 1)\theta)} = C - B,
\]

and under conditions (2.20) and (2.24) it follows that

\[
T(\theta) \text{ is singular if and only if } C - B \text{ is singular.} \tag{2.29}
\]

Consider the matrix \( \tilde{G} \) in \( \mathbb{C}^{mp \times m} \) defined by

\[
\tilde{G} = \begin{bmatrix}
C - B \\
(C - B)A \\
\vdots \\
(C - B)A^{p-1}
\end{bmatrix}.
\]  

(2.30)

By (2.18)-(2.20) and (2.30), the existence of nontrivial solutions of problem (2.1)-(2.3) is guaranteed if

\[
\text{rank } \tilde{G} < m. \tag{2.31}
\]

Under hypothesis (2.31), by [14, Theorem 2.3.1], the solution set of equation

\[
\tilde{G}d = 0, \quad d \in \mathbb{C}^m,
\]  

(2.32)

is given by

\[
d = (I - \tilde{G}^+\tilde{G}) S, \quad S \in \mathbb{C}^m - \{0\}.
\]

Summarizing from the previous comments, the following result has been established.
THEOREM 1. Assume the hypotheses (2.20) and (2.24). Let $\mathcal{G}$ be the matrix defined by (2.30) and assume that $\text{rank } \mathcal{G} < m$. Let $\theta_k$ be a solution of equation (2.28) in $I_k = [(k - 1)/(N - 1)]\pi, (k/(N - 1))\pi]$, and $\theta_k = -4r\sin^2(\theta_k/2)$, for $1 \leq k \leq 2N - 2$. Then a set of nontrivial solutions of the boundary value problem (2.1)–(2.3), is given by

$$U_k(i, j) = (I + \rho_k A)^j \{(1 - N) \sin (i\theta_k) + N \sin ((i - 1)\theta_k)\} d_k,$$

$$d_k = \left( I - \mathcal{G}^+ \mathcal{G} \right) S_k, \quad S_k \in C^m - \{0\}, \quad 1 \leq i \leq N - 1, \quad 0 \leq j < p.$$

Assume now hypothesis (2.25). The following lemma will play an important role in the following.

LEMMA 2. Under hypotheses and notation (2.20), (2.23), and (2.25), if $\alpha_0$ does not belong to the interval $[0, (1 + \lambda_0)^{-1}]$, then:

(i) $\sin((N - 1)\theta) \neq 0$;
(ii) $\alpha_0 + (N - 1)((1 + \lambda_0)\alpha_0 - 1) > 0$, if $\alpha_0 \geq (1 + \lambda_0)^{-1}$;
(iii) $\alpha_0 + (N - 1)((1 + \lambda_0)\alpha_0 - 1) < 0$, if $\alpha_0 \leq 0$ and $N > 1 + \alpha_0/(1 - \alpha_0(1 + \lambda_0))$.

PROOF.

(i) Note that equation $g(\theta) = 0$ in (2.23) can be written in the form

$$\left[(1 + \lambda_0)\alpha_0 - 1\right] \left[N \sin ((N - 1)\theta) - (N - 1) \sin (N\theta)\right]$$

$$= \alpha_0 \left[\sin(N\theta) + 4N^2 \sin^2 \left(\frac{\theta}{2}\right) \sin ((N - 1)\theta)\right],$$

or

$$[N \sin ((N - 1)\theta) - (N - 1) \sin (N\theta)] \left[(1 + \lambda_0)\alpha_0 - 1\right]$$

$$= \alpha_0 \left[\sin(N\theta) + 4N^2 \sin^2 \left(\frac{\theta}{2}\right) \sin ((N - 1)\theta)\right]. \quad (2.33)$$

If $(2.33)$ holds, then $\sin((N - 1)\theta) \neq 0$, because if $\sin((N - 1)\theta) = 0$, by $(2.33)$ one gets $\alpha_0 \sin(N\theta) = -(N - 1) \sin (N\theta)((1 + \lambda_0)\alpha_0 - 1)$. Since $\sin(N\theta) \neq 0$, this last condition implies

$$(1 + \lambda_0)\alpha_0 - 1 = -\frac{\alpha_0}{N - 1}. \quad (2.34)$$

Note that if $\alpha_0 \leq 0$ equation $(2.34)$ does not hold because the sign of both sides are different. If $\alpha_0 \geq (1 + \lambda_0)^{-1} > 0$, then $(1 + \lambda_0)\alpha_0 - 1 \geq 0$ which contradicts $(2.34)$. This prove that $\sin((N - 1)\theta) \neq 0$.

(ii) If $\alpha_0 \geq (1 + \lambda_0)^{-1} > 0$, then $(1 + \lambda_0)\alpha_0 \geq 1$ and $(N - 1)\alpha_0(1 + \lambda_0) \geq N - 1$ or

$$(1 + \lambda_0)\alpha_0 - 1 \geq 0.$$ 

Thus, $\alpha_0 + (N - 1)((1 + \lambda_0)\alpha_0 - 1) > 0$.

(iii) If $\alpha_0 \leq 0$, then $\alpha_0(1 + \lambda_0) - 1 < 0$ and for $N > 1 + \alpha_0/(1 - \alpha_0(1 + \lambda_0))$, $\alpha_0 + (N - 1)((1 + \lambda_0)\alpha_0 - 1) < 0$.

Under hypotheses and notation of Lemma 2, $\sin((N - 1)\theta) \neq 0$ and equation $(2.33)$ can be written in the form

$$\frac{\sin(N\theta)}{\sin((N - 1)\theta)} \{\alpha_0 + (N - 1)((1 + \lambda_0)\alpha_0 - 1)\} = N \left[(1 + \lambda_0)\alpha_0 - 1\right] - 4N^2 \sin^2 \left(\frac{\theta}{2}\right). \quad (2.35)$$

If $\alpha_0 \leq 0$ or $\alpha_0 \geq (1 + \lambda_0)^{-1}$, then $\alpha_0 + (N - 1)((1 + \lambda_0)\alpha_0 - 1) \neq 0$ for $N > 1 + \alpha_0/(1 - \alpha_0(1 + \lambda_0))$ and $(2.35)$ takes the form

$$\Gamma_1(\theta) = \Gamma_2(\theta), \quad (2.36)$$

where

$$\Gamma_1(\theta) = \frac{\sin(N\theta)}{\sin((N - 1)\theta)}, \quad (2.37)$$

and

$$\Gamma_2(\theta) = \frac{\sin(N\theta)}{\sin((N - 1)\theta)}. \quad (2.38)$$

If $\alpha_0 > 0$ and $\alpha_0 \leq (1 + \lambda_0)^{-1}$, then

$$\alpha_0 + (N - 1)((1 + \lambda_0)\alpha_0 - 1) = 0$$

for $N > 1 + \alpha_0/(1 - \alpha_0(1 + \lambda_0))$ and $(2.35)$ becomes

$$\Gamma_1(\theta) = \Gamma_2(\theta), \quad (2.36)$$

where

$$\Gamma_1(\theta) = \frac{\sin(N\theta)}{\sin((N - 1)\theta)}, \quad (2.37)$$

and

$$\Gamma_2(\theta) = \frac{\sin(N\theta)}{\sin((N - 1)\theta)}. \quad (2.38)$$
and
\[ \Gamma_2(\theta) = \frac{-4\alpha_0N^2\sin^2(\theta/2)}{\alpha_0 + (N - 1)(1 + \lambda_0)\alpha_0 - 1} + \frac{N[(1 + \lambda_0)\alpha_0 - 1]}{\alpha_0 + (N - 1)(1 + \lambda_0)\alpha_0 - 1} \]
\[ = \frac{N \left\{ 1 - 4\alpha_0N^2\sin^2(\theta/2) \alpha_0 / ((1 + \lambda_0)\alpha_0 - 1) \right\}}{(N - (\alpha_0\lambda_0 - 1))/((1 + \lambda_0)\alpha_0 - 1))} \] (2.38)

Note that for a fixed value of \( N > 1 \), the function \( \Gamma_2(\theta) \) defined by (2.38) is bounded in each interval \( ((k - 2)/(N - 1))\pi < \theta < ((k - 1)/(N - 1))\pi \) while the image of this interval by the function \( \Gamma_1(\theta) \) is all the real line. Hence,

there exists \( \theta_k \in \left\{ \left( \frac{k - 2}{N - 1} \right)\pi, \left( \frac{k - 1}{N - 1} \right)\pi \right\} \), such that \( \Gamma_1(\theta_k) = \Gamma_2(\theta_k), \quad 2 \leq k \leq 2N - 1 \) (2.39)

Let us assume now that \( \alpha_0 \in [0, (1 + \lambda_0)^{-1}] \) and \( \theta \) appearing in (2.23) satisfies \( \sin((N - 1)\theta) \neq 0 \); then \( \alpha_0 / ((1 + \lambda_0)\alpha_0 - 1) < 0 \) and \( (\alpha_0\lambda_0 - 1)/(1 + \lambda_0)\alpha_0 - 1) > 1 \). Let \( \epsilon > 0 \) be such that \( (\alpha_0\lambda_0 - 1)/(1 + \lambda_0)\alpha_0 - 1) = 1 + \epsilon \); then functions \( \Gamma_i(\theta) \) defined by (2.37),(2.38), for \( i = 1, 2 \) verify \( \lim_{\theta \to 0^+} \Gamma_1(\theta) = N/(N - 1) < N/(N - (1 + \epsilon)) = \lim_{\theta \to 0^+} \Gamma_2(\theta) \). Furthermore, \( \Gamma_2(\theta) > \Gamma_1(\theta) \) in \( [0, \pi/(N - 1)] \). Hence, equation (2.36) admits a solution \( \theta_k \in \left((k - 2)/(N - 1)\pi, (k - 1)/(N - 1)\pi\right) \) for \( k = 1, 2, \ldots, 2N - 1 \) such that

\[ \theta_k \in I_k = \left\{ \left( \frac{k - 2}{N - 1} \right)\pi, \left( \frac{k - 1}{N - 1} \right)\pi \right\}, \text{ if } \alpha_0 \geq (1 + \lambda_0)^{-1} \text{ or } \alpha_0 \leq 0, \]

\[ \theta_k \in J_k = \left\{ \left( \frac{k - 1}{N - 1} \right)\pi, \left( \frac{k}{N - 1} \right)\pi \right\}, \text{ if } \alpha_0 \in \left(0, (1 + \lambda_0)^{-1}\right]. \] (2.40)

Recall that from (2.21) and under the hypothesis \( \sin((N - 1)\theta) \neq 0 \) one gets

\[ \frac{T(\theta)}{\sin((N - 1)\theta)} = \left\{ \frac{(N + (N - 1)\lambda_0)}{\sin((N - 1)\theta)} - \frac{N \left( \lambda_0 + 1 - 4N\sin^2\left(\frac{\theta}{2}\right) \right)}{\sin((N - 1)\theta)} \right\} C \]
\[ - \frac{N}{(N - 1)} \left\{ \frac{\sin(N\theta)}{\sin((N - 1)\theta)} - \frac{N}{N - 1} \right\} (B + \lambda_0C). \] (2.41)

By (2.35), it is easy to show that

\[ \alpha_0 \left\{ \frac{(N + (N - 1)\lambda_0)}{\sin((N - 1)\theta)} - \frac{N \left( \lambda_0 + 1 - 4N\sin^2\left(\frac{\theta}{2}\right) \right)}{\sin((N - 1)\theta)} \right\} \]
\[ = (N - 1) \left\{ \frac{\sin(N\theta)}{\sin((N - 1)\theta)} - \frac{N}{N - 1} \right\}, \] (2.42)

and by (2.41),(2.42), it follows that

\[ \frac{T(\theta)}{\sin((N - 1)\theta)} = \left\{ \frac{(N + (N - 1)\lambda_0)}{\sin((N - 1)\theta)} - \frac{N \left( \lambda_0 + 1 - 4N\sin^2\left(\frac{\theta}{2}\right) \right)}{\sin((N - 1)\theta)} \right\} \]
\[ \times \left(C - \alpha_0 (B + \lambda_0C) \right). \] (2.43)

Let \( \theta \) satisfy (2.35) and let us write (2.35) in the form

\[ \alpha_0 \left\{ \frac{(N + (N - 1)\lambda_0)}{\sin((N - 1)\theta)} - \frac{N \left( \lambda_0 + 1 - 4N\sin^2\left(\frac{\theta}{2}\right) \right)}{\sin((N - 1)\theta)} \right\} \]
\[ = (N - 1) \left\{ \frac{\sin(N\theta)}{\sin((N - 1)\theta)} - \frac{N}{N - 1} \right\}. \] (2.44)
Note that if \( \sin((N - 1)\theta) \neq 0 \), then
\[
\sin(N\theta) + 4N^2 \sin^2 \left( \frac{\theta}{2} \right) \sin ((N - 1)\theta) = \sin ((N - 1)\theta) \left[ \frac{N}{N - 1} + 4N^2 \sin^2 \left( \frac{\theta}{2} \right) \right] \neq 0.
\]

Hence, and by (2.44), it follows that
\[
\alpha_0 = 0 \text{ if and only if } \frac{\sin(N\theta)}{\sin((N - 1)\theta)} = \frac{N}{N - 1}.
\] (2.45)

By (2.43),(2.45), if \( \theta_k \) with \( \sin((N - 1)\theta_k) \neq 0 \) satisfies (2.40) and \( \alpha_0 \neq 0 \), it follows that
\[
(C - \alpha_0 (B + \lambda_0 C)) T(\theta) = \frac{1}{(N + (N - 1)\lambda_0) \sin(N\theta) - N (\lambda_0 + 1 - 4N \sin^2(\theta/2)) \sin((N - 1)\theta)}.
\] (2.46)

Let \( \tilde{G}(\lambda_0, \alpha_0) \) be the matrix in \( C^{m \times m} \) defined by
\[
\tilde{G}(\lambda_0, \alpha_0) = \begin{bmatrix}
[C - \alpha_0 (B + \lambda_0 C)] & [C - \alpha_0 (B + \lambda_0 C)] A \\
& \vdots \\
& [C - \alpha_0 (B + \lambda_0 C)] A^{p-1}
\end{bmatrix},
\] (2.47)

and note that condition (2.18) is equivalent to
\[
\tilde{G}(\lambda_0, \alpha_0) d_k = 0, \quad d_k \in C^m.
\] (2.48)

System (2.48) admits nonzero solutions \( d_k \in C^m \) if and only if
\[
\text{rank } \tilde{G}(\lambda_0, \alpha_0) < m,
\] (2.49)

and under this condition by [14, Theorem 2.3.1], the solution set of (2.48) is given by
\[
d_k = \left( I - \tilde{G}(\lambda_0, \alpha_0)^+ \tilde{G}(\lambda_0, \alpha_0) \right) S_k, \quad S_k \in C^m - \{0\}.
\]

Summarizing, we have proved the following result.

**Theorem 3.** Assume hypotheses (2.20),(2.25) and that matrix \( \tilde{G}(\lambda_0, \alpha_0) \) defined by (2.47) satisfies (2.49). Let \( \theta_k \) be the roots of equation (2.36), given by (2.40), for \( 1 \leq k \leq 2N - 1 \). Then a set of nonzero solutions of the boundary value problem (2.1)-(2.3) is given by
\[
\begin{align*}
U_k(i, j) &= (I + \rho_k A)^j \{ (1 - N) \sin (i\theta_k) + N \sin ((i - 1)\theta_k) \} d_k, \\
\rho_k &= -4r \sin^2 \left( \frac{\theta_k}{2} \right), \quad d_k = \left( I - \tilde{G}(\lambda_0, \alpha_0)^+ \tilde{G}(\lambda_0, \alpha_0) \right) S_k, \\
S_k &\in C^m - \{0\}, \quad 1 \leq k \leq 2N - 1.
\end{align*}
\] (2.50)

**Remark 2.** The case \( \rho = 0 \), in provides the solution set for equation (2.8) of the form
\[
H(i) = c + id, \quad c, d \in C^m,
\]
and the solution of the corresponding equation (2.9) is \( G(i) = I, \ j \geq 0 \). Taking into account (2.4), condition (2.2) implies \( c = -Nd, \ H(i) = (N - i)d \). Boundary condition (2.3) means that
\[
Cd = 0, \quad d \in C^m.
\]
Thus, if $C$ is singular, a solution set of the boundary value problem (2.1)-(2.3) is given by

$$U(i, j) = (N - i)d, \quad d \in \ker C - \{0\}.$$

Recall that from Remark 1 if the matrix $B$ is invertible then condition (2.20) holds true. We now consider the boundary value problem (2.1)-(2.3) for the case where $B$ is singular. (2.51)

Suppose that $0 \in ]0, \pi[ \cup ]\pi, 2\pi[ $ satisfies the equation

$$4N\sin^2\left(\frac{\theta}{2}\right)\sin((N - 1)\theta) + \sin(N\theta) - \sin((N - 1)\theta)
= 4N\sin^2\left(\frac{\theta}{2}\right)\sin((N - 1)\theta) + 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{2N - 1}{2}\theta\right)$$

$$= \left[4N\sin^2\left(\frac{\theta}{2}\right) - 1\right]\sin((N - 1)\theta) + \sin(N\theta) = 0.$$ (2.52)

By (2.52), it follows that $\sin((N - 1)\theta) \neq 0$ and thus the equation can be written in the form

$$\frac{\sin(N\theta)}{\sin((N - 1)\theta)} = 1 - 4N\sin^2\left(\frac{\theta}{2}\right),$$ (2.53)

or

$$\cot((N - 1)\theta) = \frac{2\sin^2(\theta/2)(1 - 2N)}{\sin\theta} = (1 - 2N)\tan\left(\frac{\theta}{2}\right),$$ (2.54)

$$\tan((N - 1)\theta) = \cot(\theta/2) = \frac{1}{1 - 2N}. $$

Note that equation (2.54) has a solution $\theta_k$ in each interval $I_k = \left]\frac{(k - 1)}{(N - 1)}\pi, \frac{k}{(N - 1)}\pi\right[ $ for $1 \leq k \leq 2N - 2$. By (2.19) the matrix $T(\theta)$ can be written in the form

$$T(\theta) = \left\{2N\sin\left(\frac{\theta}{2}\right)\sin((N - 1)\theta) + \cos\left(\frac{2N - 1}{2}\theta\right)\right\}2NC\sin\left(\frac{\theta}{2}\right)$$

$$+ \left\{\sin(N\theta) - N[\sin(N\theta) - \sin((N - 1)\theta)]\right\}B.$$ By (2.52),(2.54), if $\theta_k$ is a root of (2.54), then by (2.53) it follows that

$$T(\theta_k) = \{\sin(N\theta_k) - N[\sin(N\theta_k) - \sin((N - 1)\theta_k)]\}B$$

$$= \frac{1}{\sin((N - 1)\theta_k)} \left\{N - (N - 1)\frac{\sin(N\theta_k)}{\sin((N - 1)\theta_k)}\right\}B$$

$$= \frac{(N - 1)}{\sin((N - 1)\theta_k)} \left\{\frac{1}{N - 1} + 4N\sin^2\left(\frac{\theta_k}{2}\right)\right\}B.$$ (2.55)

By (2.55), it follows that $((N - 1)/\sin((N - 1)\theta_k))\{1/(N - 1) + 4N\sin^2(\theta_k/2)\} \neq 0$ and

$$T(\theta_k)$$ is singular if and only if $B$ is singular. (2.56)

Let $\tilde{G}$ be a matrix in $C^{mp \times nm}$ defined by

$$\tilde{G} = \begin{bmatrix} B \\ BA \\ \vdots \\ BAP^{-1} \end{bmatrix},$$ (2.57)

and suppose that rank $\tilde{G} < m$; then a solution set of problem (2.1)-(2.3) is given by

$$U_k(i, j) = (I + \rho_k A)^{j}\{(1 - N)\sin(i\theta_k) + N\sin((i - 1)\theta_k)\}d_k,$$

$$\rho_k = -4\pi\sin^2\left(\frac{\theta_k}{2}\right), \quad d_k = (I - \tilde{G}^+\tilde{G})S_k, \quad S_k \in C^m - \{0\},$$ (2.58)

$$1 \leq i \leq N - 1, \quad j \geq 0, \quad 1 \leq k \leq 2N - 2.$$ The following result has been proved.
THEOREM 4. Let $B$ be a singular matrix in $C^{n \times m}$, $N > 1$ and let $\theta_k$ be a solution of (2.54) in $I_k = \{(k - 1)/(N - 1)\pi, (k/(N - 1))\pi\}$ for $1 \leq k \leq 2N - 2$. Let $\widetilde{G}$ be the matrix defined by (2.57) and suppose that rank $\widetilde{G} < m$. Then a set of nonzero solutions of the boundary value problems (2.1)-(2.3) is given by (2.58).

3. NUMERICAL SOLUTION OF THE MIXED PROBLEM

Theorems 1, 3, and 4 of Section 2 provide nonzero solutions of the boundary value problem (2.1)-(2.3) of the form

$$U_k(i, j) = \left( I - 4rA \sin\left( \frac{\theta_k}{2} \right) \right)^j \{(1 - N) \sin(i\theta_k) + N \sin((i - 1)\theta_k)\} d_k,$$

where $\theta_k$ are real numbers in $[0, \pi) \cup \{2\pi, \pi\}$, $d_k = (I - \tilde{G}^+\tilde{G})S_k$, $S_k \in C^m - \{0\}$, $1 \leq k \leq 2N - 2$, and $\tilde{G}$ is the matrix in $C^{mp \times m}$, defined by (2.30), (2.47), or (2.57), respectively, according to hypotheses of Theorems 1, 3, or 4, respectively. Superposition suggests to seek a solution of problem (2.1)-(2.3) of the form

$$U(i, j) = \sum_{k=1}^{2N-1} \left( I - 4rA \sin\left( \frac{\theta_k}{2} \right) \right)^j \{(1 - N) \sin(i\theta_k) + N \sin((i - 1)\theta_k)\} d_k,$$

where $d_{2N-1} = 0$ under by hypotheses of Theorems 1 and 4. By imposing to the vector function defined by (3.1) the initial condition

$$U(i, 0) = F(i), \quad F(i) = F\left( \frac{i}{N} \right), \quad 1 \leq i \leq N,$$

one gets

$$F(i) = \sum_{k=1}^{2N-1} \{(1 - N) \sin(i\theta_k) + N \sin((i - 1)\theta_k)\} d_k.$$

By the discrete series Fourier theory, see [15,16], one gets the following condition for the sequence \( \{F(i)\}_{i=1}^{N-1} \):

$$d_k = \frac{2}{N} \sum_{i=1}^{2N-1} \{(1 - N) \sin(i\theta_k) + N \sin((i - 1)\theta_k)\} F(i).$$

Since vectors $d_k$ must belong to ker $\tilde{G} = \text{Im}(I - \tilde{G}^+\tilde{G})$, condition (3.4) is satisfied if

$$F(i) \in \text{ker} \tilde{G}, \quad 1 \leq i \leq N.$$

Note that matrix $\tilde{G}$ has the form

$$\tilde{G} = \begin{bmatrix} G \\ GA \\ \vdots \\ GA^{p-1} \end{bmatrix},$$

where

$$G = C - B, \quad \text{in (2.30)},$$

$$G = C - \alpha_0 (B + \lambda_0 C), \quad \text{in (2.47)},$$

$$G = B, \quad \text{in (2.57)}.$$

Since ker $G = \text{Im}(I - G^+G)$, condition (3.5) is satisfied if

$$F(i) \in \text{ker} G \text{ for } 1 \leq i \leq N \text{ and ker } G \text{ is an invariant subspace of } A.$$

Summarizing, the following result has been proved.
THEOREM 5. Let $p$ be the degree of the minimal polynomial of the matrix $A \in \mathbb{C}^{m \times m}$, and let $G$ be the matrix in $\mathbb{C}^{mp \times m}$ defined by (3.6).

(i) Under hypotheses (2.20) and (2.24), let $G = C - B$, $\theta_k$, $\rho_k$ be given by Theorem 1 for $1 \leq k \leq 2N - 2$, and assume that rank $G < m$. Under condition (3.8), a solution of (2.1)–(2.3) satisfying $U(i, 0) = F(i)$, $1 \leq i \leq N - 1$, is given by (3.1), (3.4), where $1 \leq k \leq 2N - 2$, $j \geq 0$, $1 \leq i \leq N - 1$ and $d_{2N-1} = 0$.

(ii) Under hypotheses (2.20) and (2.25), let $G = C - \alpha_0(B + \lambda_0C)$, let $\theta_k$, $\rho_k$ be given by Theorem 3 for $1 \leq k \leq 2N - 1$. Assume that rank $G < m$. Under condition (3.8) a solution of (2.1)–(2.3) satisfying $U(i, 0) = F(i)$, $1 \leq i \leq N - 1$, is given by (3.1), (3.4), where $1 \leq k \leq 2N - 1$, $j \geq 0$, $1 \leq i \leq N - 1$ and $d_{2N-1} = 0$.

(iii) Let $B$ be a singular matrix in $\mathbb{C}^{m \times m}$, let $\theta_k$, $\rho_k$ be given by Theorem 4, and suppose that rank $G < m$. Under condition (3.8) a solution of (2.1)–(2.3) satisfying $U(i, 0) = F(i)$, $1 \leq i \leq N - 1$, is given by (3.1), (3.4), where $1 \leq k \leq 2N - 2$, $j \geq 0$, $1 \leq i \leq N - 1$, and $d_{2N-1} = 0$.

Note that $\{F(i)\}_{i \geq 1}$ is bounded; then the sine Fourier coefficients $d_k$ appearing in (3.1) and defined by (3.4) are bounded. Assume the following hypothesis:

\[
every\text{ eigenvalue } z \text{ of } \frac{A + A^H}{2} \text{ is positive.} \quad (3.9)
\]

By the Bromwich theorem [9, p. 389], one gets

\[
\lambda_{\min} \left( \frac{A + A^H}{2} \right) \leq \Re(a) \leq \lambda_{\max} \left( \frac{A + A^H}{2} \right), \quad a \in \sigma(A),
\]

(3.10)

and by the spectral mapping theorem [10, p. 569] it follows that

\[
\sigma \left( I - 4rA \sin^2 \left( \frac{\theta_k}{2} \right) \right) = \left\{ 1 - 4ra \sin^2 \left( \frac{\theta_k}{2} \right) : a \in \sigma(A) \right\}.
\]

For $a \in \sigma(A)$, one gets

\[
\left| 1 - 4r \Re(a) \sin^2 \left( \frac{\theta_k}{2} \right) \right|^2 = \left| \Re \left( 1 - 4ra \sin^2 \left( \frac{\theta_k}{2} \right) \right) \right|^2
\]
\[
= 1 - 8r \Re(a) \sin^2 \left( \frac{\theta_k}{2} \right) + \left| 4r \Re(a) \right|^2 \sin^4 \left( \frac{\theta_k}{2} \right),
\]
\[
\left| -4r \Im(a) \sin^2 \left( \frac{\theta_k}{2} \right) \right|^2 = \left| \Im \left( -4ra \sin^2 \left( \frac{\theta_k}{2} \right) \right) \right|^2 = \left| 4r \Im(a) \right|^2 \sin^4 \left( \frac{\theta_k}{2} \right).
\]

From (3.10), follows

\[
\left| 1 - 4ra \sin^2 \left( \frac{\theta_k}{2} \right) \right|^2 = 1 + \left( 4r \right)^2 \sin^4 \left( \frac{\theta_k}{2} \right) \left\{ \left( \Re(a) \right)^2 + \left( \Im(a) \right)^2 \right\} - 8r \Re(a) \sin^2 \left( \frac{\theta_k}{2} \right)
\]
\[
= 1 + 16r^2 \sin^4 \left( \frac{\theta_k}{2} \right) |a|^2 - 8r \Re(a) \sin^2 \left( \frac{\theta_k}{2} \right)
\]
\[
\leq 1 + 16r^2 \sin^4 \left( \frac{\theta_k}{2} \right) \left( \rho(A) \right)^2 - 8r \sin^2 \left( \frac{\theta_k}{2} \right) \lambda_{\min} \left( \frac{A + A^H}{2} \right).
\]

Assume that

\[
r < \frac{\lambda_{\min} \left( \frac{(A + A^H)/2}{2} \right)}{2 |\rho(A)|^2}.
\]

(3.12)
then

\[ r < \frac{\lambda_{\text{min}} \left( (A + A^H)/2 \right)}{2 [\rho(A) \sin (\theta_k/2)]^2}, \quad 1 \leq k \leq 2N - 1, \]  

(3.13)

and by (3.11) it follows that

\[ 8r \sin^2 \left( \frac{\theta_k}{2} \right) \lambda_{\text{min}} \left( \frac{A + A^H}{2} \right) > 16r^2 \sin^4 \left( \frac{\theta_k}{2} \right) (\rho(A))^2, \]

\[ 1 - 4rz \sin^2 \left( \frac{\theta_k}{2} \right) < 1. \]

Thus the numerical solution of problem (1.1)-(1.4), given by (3.1),(3.4), remains bounded as \( N \to \infty \) with \( r \) satisfying (3.12). Summarizing, the following stability result has been established.

**THEOREM 6.** With the hypotheses and the notation of Theorem 5, if \( A \) satisfies the spectral condition (3.9), \( h = 1/N, \) \( r = k/h^2 \) satisfies (3.12), and \( F(x) \) is bounded, then the numerical solution \( \{U(i,j)\} \) of problem (1.1)-(1.4), given by Theorem 5, remains bounded as \( N \to \infty \).

**EXAMPLE 1.** Consider problem (1.1)-(1.4), where

\[
A = \begin{bmatrix}
1 & 1/2 & 1/2 \\
0 & 1/2 & 0 \\
0 & 0 & 1/4
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & -1 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{bmatrix}, \quad F(x) = (f_1(x), 0, 0)^T,
\]

where \( f_1(x) \) is a real valued function. Note that condition (2.20) holds with \( \lambda_0 = 1 \) because

\[
B + C = \begin{bmatrix}
2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Note that

\[
(B + C)^{-1}C = \begin{bmatrix}
1 & -1 & 0 \\
0 & 0 & -2 \\
0 & -2 & 0
\end{bmatrix},
\]

\[
\sigma((B + \lambda_0 C)^{-1}C) = \sigma((B + C)^{-1}C) = \left\{-1, \frac{1}{2}, 1 \right\}.
\]

The eigenvalue \( \lambda_0 = 1/2 \in \sigma((B + C)^{-1}C) \) satisfies (2.24) because

\[
(1 + \lambda_0) \alpha_0 - 1 = (1 + 1) \frac{1}{2} - 1 = 0.
\]

Let \( \theta_k \in I_k = |(k - 1)/(N - 1)|, k/(N - 1) \) be a solution of

\[ \tan ((N - 1)\theta_k) = \frac{\sin \theta_k}{(2N^2 - 1)[\cos \theta_k - 2N^2/(2N^2 - 1)]}, \quad 1 \leq k \leq N - 2. \]

Let

\[
G = C - \frac{1}{2}(B + C) = \begin{bmatrix}
0 & -1 & 0 \\
0 & -\frac{1}{2} & -1 \\
0 & -1 & -\frac{1}{2}
\end{bmatrix}.
\]
and note that

\[ \ker G = \text{Lin} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

is an invariant subspace of \( A \) because

\[ G^+ = \frac{2}{29} \begin{bmatrix} 0 & 0 & 0 \\ -10 & 3 & -6 \\ 8 & -14 & 1 \end{bmatrix} \]

satisfies \( GA \left( I - G^+ G \right) = 0 \).

Furthermore, \( F(i) = (f_1(x), 0, 0)^T \) lies in \( \ker G \). By Theorem 5(i), the vector function

\[ U(i, j) = \sum_{k=1}^{2N-2} \left( I - 4rA \sin^2 \left( \frac{{\theta}_k}{2} \right) \right) \{ (1 - N) \sin (i{\theta}_k) + N \sin ((i - 1){\theta}_k) \} d_k, \]

\[ d_k = \frac{2}{N} \sum_{k=1}^{2N-1} \{ (1 - N) \sin (i{\theta}_k) + N \sin ((i - 1){\theta}_k) \} \begin{bmatrix} f_1(i) \\ 0 \\ 0 \end{bmatrix}. \]

Furthermore, since

\[ \frac{A + A^H}{2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 4 \\ 1 & 4 & 0 \end{bmatrix}, \]

\[ \lambda_{\min} \left( \frac{A + A^H}{2} \right) = 0.015564, \]

the numerical solution \( \{U(i, j)\} \) remains bounded as \( N \to \infty \), for

\[ r < \frac{\lambda_{\min} \left( \left( A + A^H \right)/2 \right)}{2 \left| \rho(A) \sin (\theta_k/2) \right|^2} = \frac{0.015564}{2} = 0.007784. \]

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