Note

The Rotor Effect Can Alter The Chromatic Polynomial

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Let $G$ be a finite graph with vertex set $V(G)$, let $\theta$ be an automorphism of $G$, let $J \subseteq V(G)$ be an orbit of $\theta$, let $v$ be vertex in $J$, and let $P \subseteq \mathcal{P}(J)$ be a partition of $J$ into disjoint nonempty sets. Then $(G, \theta, J, v, P)$ is called a rotor of order $\text{Card } J$.

Let $G(P)$ denote the graph obtained from $G$ by contracting each block $B$ of $P$, together with the edges joining vertices of $B$ among themselves, to a single vertex. To the rotor $(G, \theta, J, v, P)$ we associate a function $\phi: J \to J$ called reflection, given by $\phi(\theta(v)) = \theta^{-i}(v)$. Then $\phi(P)$ is another partition of $J$, denoted $P'$. The rotor effect is the transformation that associates $G(P')$ to $G(P)$.

It is known that $G(P)$ and $G(P')$ have the same number of spanning trees [2, 4]. Moreover, for rotors of order at most 5, the dichromate is unaltered by the rotor effect [3]. It was hoped that this result could be extended to rotors of any order $k$, thereby implying a fortiori that not only the number of spanning trees but also the chromatic polynomial is unchanged by the rotor effect. We are going to give a counterexample for any $k > 5$.

Let us recall that if the chromatic polynomial $P(G, \lambda)$ of a graph having $n$ vertices is not 0, then the coefficient of $\lambda^{n-1}$ in $P(G, \lambda)$ is the number of adjacent pairs of vertices of $G$, id est

$$\text{Card } \{\{x, y\} \subseteq V(G) | \exists \text{ at least 1 edge joining } x \text{ and } y\}.$$  

This number can be called the adjacency number of $G$. We denote it by $a(G)$.

For $k > 5$ let us define the graph $G_k$ as follows. $V(G_k)$ has $2k$ elements, partitioned into 2 disjoint sets $I$ and $J$, each containing $k$ elements indexed by the integers modulo $k$: $I = \{x_i | i \in \mathbb{Z}_k\}$, $J = \{v_i | i \in \mathbb{Z}_k\}$. Each $x_i$ is adjacent with exactly 3 vertices: $v_i$, $v_{i+1}$ and $v_{i+2}$. No two vertices of $J$ are adjacent. $G_k$ has no loops or multiple edges. $G_8$ is pictured in Fig. 1.

Let $\theta$ be the automorphism of $G_k$ defined by $\theta(x_i) = x_{i+1}$, $\theta(v_i) = v_{i+1}$.  

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Let $P$ be the partition of $J$ in which $\{v_0, v_1, v_3\}$ is a block and the other blocks consist of single vertices. Consider the rotor $(G_k, \theta, J, v_0, P)$.

If $k = 6$, then $x_0$ is adjacent to all the three vertices of $\{v_0, v_1, v_3\}$. $x_3$ is adjacent to $v_3$ and $v_0$. All other vertices are adjacent to at most 1 vertex in $\{v_0, v_1, v_3\}$. On the other hand, no vertex is adjacent to all the three vertices of the reflected block $\{v_0, v_5, v_3\}$ of $P'$. $x_5$ is adjacent to $v_0$ and $v_5$; $x_3$ is adjacent to $v_0$ and $v_3$; $x_0$ is adjacent to $v_0$ and $v_4$. All other vertices are adjacent to at most 1 vertex in $\{v_0, v_5, v_3\}$. Since no two vertices of $\{v_0, v_1, v_3\}$ or of $\{v_0, v_5, v_3\}$ are adjacent, and all other blocks of $P$ or $P'$ are singletons, it follows that

$$a(G_6(P)) = a(G_6) - (3 - 1) - (2 - 1) = 18 - 2 - 1 = 15,$$

while

$$a(G_6(P')) = a(G_6) - 4(2 - 1) = 18 - 4 = 14.$$

If $k > 6$, the situation is even simpler. As above, $x_0$ is adjacent to every vertex in $\{v_0, v_1, v_3\}$. All other vertices are adjacent to at most 1 vertex of this block. On the other hand, no vertex is adjacent to all the three vertices of the reflected block $\{v_0, v_{k-1}, v_{k-3}\}$. $x_{k-1}$ is adjacent to $v_0$ and $v_{k-1}$; $x_{k-3}$ is adjacent to $v_0$ and $v_{k-3}$; $x_{k-4}$ is adjacent to $v_{k-1}$ and $v_{k-3}$. All other vertices are adjacent to at most 1 vertex in $\{v_0, v_{k-1}, v_{k-3}\}$. We have

$$a(G_k(P)) = a(G_k) - (3 - 1) = 3k - 2,$$

while

$$a(G_k(P')) = a(G_k) - 3(2 - 1) = 3k - 3.$$
For any $k > 5$, $a(G_k(P)) \neq a(G_k(P'))$. Since the chromatic polynomials of $G_k(P)$ and $G_k(P')$ are not 0, they must be different. We have therefore the following

**Theorem.** Let $k$ be any integer $> 5$. There is a rotor of order $k$ for which the chromatic polynomial is altered by the rotor effect.

**References**