General theorems for stability and boundedness for nonlinear functional discrete systems

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Abstract

We consider the nonlinear functional discrete system

\[ x(n+1) = G(n, x(s); 0 \leq s \leq n) \overset{\text{def}}{=} G(n, x(\cdot)) \]

and obtain sufficient conditions for stability properties of the zero solution and for the boundedness of solutions.

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1. Introduction

In this work we use Lyapunov functionals to prove general theorems about boundedness and stability of the solution of the nonlinear discrete system

\[ x(n + 1) = G(n, x(s); 0 \leq s \leq n) \overset{\text{def}}{=} G(n, x(\cdot)), \quad (1.1) \]

where \( G : \mathbb{Z}^+ \times \mathbb{R}^k \to \mathbb{R}^k \) is continuous in \( x \). When Lyapunov functionals are used to study the behavior of solutions of functional difference equations of the form of (1.1), we often end up with a pair of inequalities in the form of

\[ V(n, x(\cdot)) = W_1(x(n)) + \sum_{s=0}^{n-1} K(n, s)W_2(x(s)), \quad (1.2) \]

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\[ \Delta V(n, x(n)) \leq -W_i(x(n)) + F(n), \quad (1.3) \]

where \( V \) is a Lyapunov functional bounded below, \( x \) is the known solution of the functional difference equation, and \( K, F, \) and \( W_i, i = 1, 2, 3, \) are scalar positive functions.

The goal of this paper is to use Eqs. (1.2) and (1.3) to conclude boundedness of \( x(n) \) when \( F \) is bounded. Also, we obtain stability results about the zero solution of (1.1) when \( F = 0 \) and \( G(n, 0) = 0. \)

Recently, several authors (Medina [9–11], Islam and Raffoul [7], and Raffoul [12]) obtained stability and boundedness results of the solutions of discrete Volterra equations by means of representing the solution in terms of the resolvent matrix of the corresponding system of difference Volterra equations. Eloe et al. [6] and Elaydi et al. [5] used the notion of total stability and established results on the asymptotic behavior of the solutions of discrete Volterra system with nonlinear perturbation. Their work heavily depended on the summability of the resolvent matrix. For more results on stability of the zero solution of Volterra discrete system we refer the reader to Elaydi [4] and Agarwal and Pang [1]. This research is a continuation of the research initiated by Raffoul [12] and related to the work of Medina [9]. In this paper we extend some of the results in [2] and later in the paper, we apply the obtained theorems to obtain new results related to the behavior of solutions of Volterra discrete systems. For more on the calculus of difference equations, we refer the interested reader to [3].

We say that \( x(n) = x(n, n_0, \phi) \) is a solution of (1.1) with a bounded initial function \( \phi : [0, n_0] \to \mathbb{R}^k \) if it satisfies (1.1) for \( n > n_0 \) and \( x(j) = \phi(j) \) for \( j \leq n_0. \)

If \( D \) is a matrix or a vector, \( |D| \) means the sum of the absolute values of the elements.

**Definition 1.1.** Solutions of (1.1) are uniformly bounded if for each \( B_1 > 0 \) there is \( B_2 > 0 \) such that

\[
\{ n_0 \geq 0, \phi : [0, n_0] \to \mathbb{R}^k \text{ with } |\phi(n)| < B_1 \text{ on } [0, n_0], \ n > n_0 \}
\]

implies \( |x(n, n_0, \phi)| < B_2. \)

**Definition 1.2.** Solutions of (1.1) are uniformly ultimately bounded for bound \( B \) if there is \( B > 0 \) and if for each \( M > 0 \) there exists \( N > 0 \) such that

\[
\{ n_0 \geq 0, \phi : [0, n_0] \to \mathbb{R}^k \text{ with } |\phi(n)| < M \text{ on } [0, n_0], \ n > n_0, \ n > n_0 + N \}
\]

implies \( |x(n, n_0, \phi)| < B. \)

If \( G(n, 0) = 0 \) then \( x(n) = 0 \) is a solution of (1.1). In this case we state the following definitions.

**Definition 1.3.** The zero solution of (1.1) is stable if for each \( \varepsilon > 0 \) there is \( \delta = \delta(\varepsilon) > 0 \) such that

\[
\{ \phi : [0, n_0] \to \mathbb{R}^k \text{ with } |\phi(n)| < \delta \text{ on } [0, n_0], \ n \geq n_0 \}
\]

implies \( |x(n, n_0, \phi)| < \varepsilon. \) It is uniformly stable if \( \delta \) is independent of \( n_0. \)
Definition 1.4. The zero solution of (1.1) is uniformly asymptotically stable if it is US and there exists $\gamma > 0$ with the property that for each $\mu > 0$ there exists $N > 0$ such that
\[ \{ n_0 \geq 0, \phi : [0, n_0] \to \mathbb{R}^k \text{ with } |\phi(n)| < \gamma \text{ on } [0, n_0], \ n \geq n_0 + N \} \]
implies $|x(n, n_0, \phi)| < \mu$.

2. Boundedness and stability

In this section we are going to use (1.2) and (1.3) to conclude boundedness of the solution $x(n)$ of the functional difference equation
\[ x(n+1) = G(n, x(\cdot)), \] (2.1)
where $G$ is as described in (1.1), when $F$ is bounded and when $W_i(x) \to \infty$ as $|x| \to \infty$, $i = 1, 2, 3$.

Let $W_i : [0, \infty) \to [0, \infty)$ be continuous with $W_i(0) = 0$, $W_i(r)$ strictly increasing, and $W_i(r) \to \infty$ as $r \to \infty$, $i = 1, 2, 3$.

We are now ready to state the main results for this section.

Theorem 2.1. Let $\phi(n, s)$ be a scalar sequence for $0 \leq s \leq n < \infty$ and suppose that $\phi(n, s) \geq 0$, $\Delta_n \phi(n, s) \leq 0$, $\Delta_s \phi(n, s) \geq 0$, and there are constants $B$ and $J$ such that
\[ \sum_{n=0}^{n_0} \phi(n, s) \leq B \text{ and } \phi(0, s) \leq J. \]
Also, suppose that for each $n_0 \geq 0$ and each bounded initial function $\phi : [0, n_0] \to \mathbb{R}^k$ every solution $x(n) = x(n, n_0, \phi)$ of (2.1) satisfies
\[ W_1\left(|x(n)|\right) \leq V(n, x(\cdot)) \leq W_2\left(|x(n)|\right) + \sum_{s=0}^{n-1} \phi(n, s) W_3\left(|x(s)|\right) \] (2.2)
and
\[ \Delta V(n, x(\cdot)) \leq -\rho W_3\left(|x(n)|\right) + K \] (2.3)
for some constants $\rho$ and $K \geq 0$. Then solutions of (2.1) are uniformly bounded.

Proof. Let $H > 0$ and $|\phi(n)| < H$ on $[0, n_0]$, and set $V(n) = V(n, x(\cdot))$. Let $V(n^*) = \max_{0 \leq n \leq n_0} V(n)$. If $V(n) \leq V(n^*)$ for all $n \geq n_0$, then by (2.2) we have
\[ W_1\left(|x(n)|\right) \leq V(n) \leq V(n^*) \leq W_2\left(|x(n^*)|\right) + \sum_{s=0}^{n^*-1} \phi(n^*, s) W_3\left(|x(s)|\right) \]
\[ \leq W_2\left(|\phi(n^*)|\right) + \sum_{s=0}^{n^*-1} \phi(n^*, s) W_3\left(|\phi(s)|\right) \]
\[ \leq W_2(H) + BW_3(H), \]
from which it follows that
\[ |x(n)| \leq W^{-1}_1\left[W_2(H) + BW_3(H)\right]. \]
On the other hand, if $V(n) > V(n^*)$ for some $n \geq n_0$, then $V(n) = \max_{0 \leq s \leq n} V(s)$.
We multiply both sides of (2.3) by $\phi(n,s)$ and then sum from $s = n_0$ to $s = n - 1$; we obtain
\[
\sum_{s=n_0}^{n-1} (\Delta V(s)) \phi(n,s) \leq -\rho \sum_{s=n_0}^{n-1} \phi(n,s) W_3(\|x(s)\|) + KB.
\]
Summing by parts the left side we arrive at
\[
V(n) \phi(n,n) - V(n_0) \phi(n,n_0) - \sum_{s=n_0}^{n-1} V(s+1) \Delta_s \phi(n,s)
\]
\[
\leq -\rho \sum_{s=n_0}^{n-1} \phi(n,s) W_3(\|x(s)\|) + KB.
\]
Hence
\[
\rho \sum_{s=n_0}^{n-1} \phi(n,s) W_3(\|x(s)\|) \leq V(n) \phi(n,n) + V(n_0) \phi(n,n_0)
\]
\[
+ \sum_{s=n_0}^{n-1} V(s+1) \Delta_s \phi(n,s) + KB.
\]
(2.4)
Since $\Delta_s \phi(n,s) \geq 0$, for $V(n) = \max_{0 \leq x \leq n-1} V(s + 1)$ we have
\[
\sum_{s=n_0}^{n-1} V(s+1) \Delta_s \phi(n,s) \leq V(n) \sum_{s=n_0}^{n-1} \Delta_s \phi(n,s) = V(n) \left[ \phi(n,n) - \phi(n,n_0) \right].
\]
Thus, from inequality (2.4) we have
\[
\rho \sum_{s=n_0}^{n-1} \phi(n,s) W_3(\|x(s)\|) \leq V(n) \left[ \phi(n,n) - \phi(n,n_0) \right]
\]
\[
- V(n) \phi(n,n) + V(n_0) \phi(n,n_0) + KB
\]
\[
\leq V(n_0) \phi(n,n_0) + KB
\]
\[
\leq V(n_0) \phi(0,n_0) + KB \leq V(n_0) J + KB.
\]
(2.5)
In view of (2.2), we have
\[
V(n_0) \leq W_2(\|\phi(n_0)\|) + \sum_{s=0}^{n_0-1} \phi(n_0,s) W_3(\|\phi(s)\|) \leq W_2(H) + BW_3(H).
\]
As a result, inequality (2.5) yields
\[
\sum_{s=n_0}^{n-1} \phi(n,s) W_3(\|x(s)\|) \leq \frac{W_2(H) + BW_3(H)}{\rho} + \frac{KB}{\rho}.
\]
Now, inequality (2.2) implies that
\[ V(n) \leq W_2(|x(n)|) + \sum_{s=0}^{n_0-1} \psi(n,s)W_3(|x(s)|) + \sum_{s=n_0}^{n-1} \psi(n,s)W_3(|x(s)|) \]
\[ \leq W_2(|x(n)|) + BW_3(H) + \frac{W_2(H) + BW_3(H)}{\rho} + \frac{KB}{\rho} \]
\[ \leq W_2(|x(n)|) + D(H), \]
where
\[ D(H) = BW_3(H) + \frac{W_2(H) + BW_3(H)}{\rho} + \frac{KB}{\rho}. \]

Since \( W_3(r) \to \infty \) as \( r \to \infty \), there exists an \( L > 0 \) such that \( W_3(L) = K/\rho \). Now, by (2.3), if \( |x| > L \), then \( \Delta V < 0 \). Thus, \( V(n) \) attains its maximum when \( |x| \leq L \). Hence we have
\[ W_1(|x(n)|) \leq V(n) \leq W_2(|x(n)|) + D(H) \leq W_2(L) + DH. \]

Finally, from the above inequality we arrive at
\[ |x(n)| \leq W_1^{-1}[W_2(L) + D(H)]. \]

This completes the proof. \( \square \)

The next theorem extends Theorem 2.1.

**Theorem 2.2.** Let \( \psi_i(n,s) \) be a scalar sequence for \( 0 \leq s \leq n < \infty \) and suppose that \( \psi_i(n,s) \geq 0, \Delta_n \psi_i(n,s) \leq 0, \Delta_s \psi_i(n,s) \geq 0 \), and there are constants \( B_i \) and \( J_i \) such that \( \sum_{n=0}^{n} \psi_i(n,s) \leq B_i \) and \( \psi_i(0,s) \leq J_i \). Also, suppose that for each \( n_0 \geq 0 \) and each bounded initial function \( \phi : [0,n_0] \to \mathbb{R}^k \), every solution \( x(n) = x(n,n_0,\phi) \) of (2.1) satisfies
\[ W_1(|x(n)|) \leq V(n,x(s)) \leq W_2(|x(n)|) + \sum_{s=0}^{n-1} \psi_1(n,s)W_3(|x(s)|) \]
\[ + \sum_{s=0}^{n-1} \psi_2(n,s)W_4(|x(s)|) \] (2.6)
and
\[ \Delta V_{(2.1)}(n,x(s)) \leq -\rho_1 W_3(|x(n)|) - \rho_2 W_4(|x(n)|) + K \] (2.7)
for some constants \( \rho_i \geq 0, i = 1, 2, \) and \( K \geq 0 \). Then solutions of (2.1) are uniformly bounded.

**Proof.** We follow the proof of the previous theorem. Let \( V(n) = \max_{0 \leq s \leq n} V(s), n \geq n_0. \) If the max of \( V(n) \) occurs on \([0,n_0] \), then it is trivial. Multiply both sides of (2.7) by \( \psi_i(n,s) \) and then sum from \( s = n_0 \) to \( s = n - 1 \) to obtain
\[ \rho_i \sum_{s=n_0}^{n-1} \psi_i(n,s)W_3(|x(s)|) \leq V(n_0)J + KB, \quad i = 1, 2. \] (2.8)
For \( H > 0 \) and \(|\phi(n)| < H\), we have
\[
V(n_0) \leq W_2(H) + BW_3(H) + B_2W_4(H) \overset{\text{def}}{=} R(H)
\]
and
\[
\sum_{s=0}^{n_0-1} \phi_i(n, s) W_{i+2}(|x(s)|) \leq W_{i+2} B_i.
\]
Thus, inequality (2.8) yields
\[
\sum_{s=0}^{n-1} \phi_i(n, s) W_{i+2}(|x(s)|) \leq \frac{R(H)J_i + KB_i}{\rho_i} + W_{i+2} B_i \overset{\text{def}}{=} S_i(H).
\]
Now, by (2.6), if \(|x| > L\), then \(\Delta V < 0\). Thus, \(V(n)\) attains its maximum when \(|x| \leq L\) and hence
\[
W_1(|x(n)|) \leq V(n) \leq W_2(|x(H)| + S_1(H)) + S_2(H) \leq W_2[L + S_1(H)] + S_2(H).
\]
From the above inequality we obtain
\[
|x(n)| \leq W_1^{-1} [W_2[L + S_1(H)] + S_2(H)].
\]
This completes the proof. \(\square\)

In the next theorem we obtain boundedness and stability results about solutions and the zero solution of (1.1).

**Theorem 2.3.** Let \(\psi(n) \geq 0\) be a scalar sequence for \(n \geq 0\) and \(V \) and \(W_i, i = 1, 2\), be defined as before. Also, suppose that for each \(n_0 \geq 0\) and each bounded initial function \(\phi: [0, n_0] \rightarrow \mathbb{R}^k\), every solution \(x(n) = x(n, n_0, \phi)\) of (2.1) satisfies
\[
W_1(|x(n)|) \leq V(n, x(\cdot)) \leq \alpha W_2(|x(n)|) + \sum_{s=0}^{n-1} \psi(n - s - 1)W_2(|x(s)|)
\]
and
\[
\Delta V_{(2.1)}(n, x(\cdot)) \leq -\rho W_2(|x(n)|)
\]
for some constants \(\rho\) and \(\alpha > 0\).

(a) If \(\sum_{s=0}^{\infty} \psi(s) = B\), then solutions of (2.1) are uniformly bounded and the zero solution of (2.1) is uniformly stable.

(b) If \(\sum_{n=0}^{\infty} \sum_{s=n}^{\infty} \psi(s) = J\), then solutions of (2.1) are uniformly ultimately bounded and the zero solution of (2.1) is uniformly asymptotically stable.

**Proof.** Let \(H > 0\) and \(|\phi(n)| < H\) on \([0, n_0]\), and set \(V(n) = V(n, x(\cdot))\). By (2.10), \(V(n)\) is monotonically decreasing and hence, by (2.9), we have
\[ W_1(\|x(n)\|) \leq V(n) \leq V(n_0) \leq \alpha W_2(H) + W_2(H) \sum_{u=0}^{n_0-1} \psi(u) \leq W_2(H)(\alpha + B). \]

(2.11)

Let \( \epsilon > 0 \) be given. Choose \( H \) such that \( H < \epsilon \) and 
\[ W_2(H)(\alpha + B) < W_1(\epsilon). \]

Hence from (2.11) we have \( |x(n)| < \epsilon \) for \( n \geq n_0 \). Consequently, the zero solution of (2.1) is uniformly stable. Also, it follows from (2.11) that 
\[ |x(n)| < W_1^{-1}[W_2(H)(\alpha + B)]. \]

which implies that solutions of (2.1) are uniformly bounded.

Sum (2.10) from \( s = n_0 \) to \( s = n - 1 \) to obtain 
\[ -V(n_0) \leq V(n) - V(n_0) \leq -\rho \sum_{s=n_0}^{n-1} W_2(\|x(s)\|) \]

and hence 
\[ \sum_{s=n_0}^{n-1} W_2(\|x(s)\|) \leq \frac{V(n_0)}{\rho} \leq \frac{(\alpha + B)W_2(H)}{\rho}. \]

On the other hand, if we sum (2.9) from \( s = n_0 \) to \( s = n - 1 \), we arrive at 
\[ \sum_{s=n_0}^{n-1} V(s) \leq \frac{\alpha + B)W_2(H)}{\rho} + \sum_{s=n_0}^{n-1} \sum_{u=s}^{n-1} \psi(u - s - 1)W_2(\|x(s)\|) \]
\[ \leq \frac{(\alpha + B)W_2(H)}{\rho} + \sum_{s=n_0}^{n-1} \sum_{u=n_0}^{n-1} \psi(u - s - 1)W_2(\|x(s)\|) \]
\[ + \sum_{s=n_0}^{n-1} \sum_{u=n_0}^{n-1} \psi(u - s - 1)W_2(\|x(s)\|) \]
\[ \leq \frac{(\alpha + B)W_2(H)}{\rho} + \sum_{s=n_0}^{n-1} W_2(\|x(s)\|) \sum_{u=n_0}^{n-1} \psi(u - s - 1) \]
\[ + \sum_{s=n_0}^{n-1} W_2(\|x(s)\|) \sum_{u=s}^{n-1} \psi(u - s - 1) \]
\[ \leq \frac{(\alpha + B)W_2(H)}{\rho} + W_2(H) \sum_{s=n_0}^{n-1} \sum_{u=n_0}^{n-1} \psi(u - s - 1) + B \sum_{s=n_0}^{n-1} W_2(\|x(s)\|) \]
\[ \leq \frac{(\alpha + B)^2W_2(H)}{\rho} + W_2(H) \sum_{s=n_0}^{n-1} \sum_{u=s}^{n-1} \psi(u - s - 1) \]
\[ + \sum_{s=n_0}^{n-1} \sum_{u=s}^{n-1} \psi(u - s - 1). \]
\[ \leq \frac{(\alpha + B)^2 W_2(H)}{\rho} + W_2(H) \sum_{r=\xi}^{\infty} \sum_{r=\xi}^{\infty} \phi(r) \]
\[ \leq \frac{(\alpha + B)^2 W_2(H)}{\rho} + W_2(H) J \]
\[ \leq \left[ J + \frac{(\alpha + B)^2}{\rho} W_2(H) \right] \overset{\text{def}}{=} a W_2(H). \tag{2.12} \]

Since \( V(n) \) is positive and decreasing for all \( n \geq n_0 \geq 0 \), we have
\[ \sum_{s=n_0}^{n-1} V(s) \geq V(n)(n - n_0). \]

Let \( \epsilon > 0 \) be given. Then, for \( n \geq n_0 + a W_2(H) / W_1(\epsilon) \) we have form (2.9) and (2.12) that
\[ W_1\left( |x(n)| \right) \leq V(n) \leq \frac{a W_2(H)}{n - n_0} < W_1(\epsilon). \tag{2.13} \]

Hence, inequality (2.13) implies that
\[ |x(n)| \leq W_1^{-1}\left( \frac{a W_2(H)}{n - n_0} \right) < \epsilon. \]

From this we have the uniform ultimate boundedness and the uniform asymptotic stability.

\[ \blacksquare \]

3. **Volterra discrete systems**

In this section we apply Theorems 2.1 and 2.3 to establish stability and boundedness results about the nonlinear Volterra discrete system
\[ x(n + 1) = A(n)x(n) + \sum_{s=0}^{n} C(n, s) f(x(s)) + g(n, x(n)), \tag{3.1} \]

where \( A(n), C(n, s) \) are \( k \times k \) matrices, \( g(n, x(n)), f(x(n)) \) are \( k \times 1 \) vector functions with \( |g(n, x(n))| \leq N \) and \( |f(x(n))| \leq \lambda |x(n)| \) for some positive constants \( N \) and \( \lambda \).

In the case of \( f(x) = x \), Medina [9] showed that if the zero solution of the homogenous equation associated with (3.1) is uniformly asymptotically stable, then all solutions of (3.1) are bounded when \( C(n, s) = C(n - s) \) and \( g(n, x(n)) = g(n) \) is bounded. In proving his results, Medina used the notion of the resolvent matrix coupled with the variation of parameters formula. Also, Raffoul in [12] used Lyapunov functionals of convolution type coupled with the \( z \)-transform and obtained results about boundedness of solutions of (3.1) when \( g(n, x(n)) = g(n) \). Moreover, when \( f \) is linear in \( x \), unlike the case here, Eloe et al. in [6] used total stability and under suitable conditions, they showed that the zero solution of (3.1) is uniformly asymptotically stable when \( |g(n, x(n))| \leq \lambda(n)|x(n)| \). We remark that the notion of the resolvent cannot be used to obtain boundedness of solutions of (3.1), since the summation term in (3.1) is nonlinear.
Theorem 3.1. Suppose $A(n) = A$ is a $k \times k$ constant matrix, and $C^T(n, s) = C(n, s)$. Let $I$ be the $k \times k$ identity matrix. Also, suppose there exists positive constants $\rho$, $\mu$, and a constant $k \times k$ symmetric matrix $B$ such that

$$A^TBA - B = -\mu I,$$  \hfill (3.2)

$$\lambda |ATB| \sum_{s=0}^{n} |C(n, s)| + |B| \sum_{s=n}^{\infty} |C(n, s)| + N^2 - \mu \leq -\rho,$$  \hfill (3.3)

and

$$\lambda |ATB| \sum_{s=0}^{n} |C(n, s)| + \lambda^2 |B| \sum_{s=0}^{n} |C(n, s)| + \lambda - |B| \leq 0.$$  \hfill (3.4)

Then solutions of (3.1) are uniformly bounded.

Proof. Define the Lyapunov functional $V(n) = V(n, x(n, \cdot))$ by

$$V(n, x(\cdot)) = x^T(n)Bx(n) + |B| \sum_{j=0}^{n-1} \sum_{s=n}^{\infty} |C(s, j)|x^2(j),$$  \hfill (3.5)

where $x^2(j) = x^T(j)x(j)$. Then along solutions of (3.1) we have

$$\Delta V_{(3.1)}(n) \leq \left[ \lambda |ATB| \sum_{s=0}^{n} |C(n, s)| + |B| \sum_{s=n}^{\infty} |C(n, s)| + N^2 - \mu \right] x^2(n)$$

$$+ 2x^T(n)ATB x(n) + 2g^T(n, x(n))B \sum_{s=0}^{n} C(n, s)f(x(s))$$

$$+ \sum_{j=0}^{n-1} \sum_{s=n}^{\infty} |C(s, j)|x^2(j)$$

$$+ |B| \sum_{s=n+1}^{\infty} |C(n, s)|x^2(n) - |B| \sum_{s=0}^{n-1} |C(n, s)|x^2(s)$$

$$+ g^T(n, x(n))Bg(n, x(n)).$$  \hfill (3.6)

Using (3.2)–(3.4) and the fact that for any two real numbers $a$ and $b$, $2ab \leq a^2 + b^2$, Eq. (3.6) reduces to

$$\Delta V_{(3.1)}(n) \leq \left[ \lambda |ATB| \sum_{s=0}^{n} |C(n, s)| + |B| \sum_{s=n}^{\infty} |C(n, s)| + N^2 - \mu \right] x^2(n)$$

$$+ \left[ \lambda |ATB| \sum_{s=0}^{n} |C(n, s)| + \lambda^2 |B| \sum_{s=0}^{n} |C(n, s)| + \lambda - |B| \right]$$

$$\times \sum_{s=0}^{n} |C(n, s)|x^2(s) + |ATB|^2 x^2 + \lambda N^2 |B|^2 \sum_{s=0}^{n} |C(n, s)| + |g^T Bg|$$

$$\leq -\rho x^2(n) + K,$$
where $K = |A^T B|^2 + \lambda N^2 |B|^2 \sum_{s=0}^{n} |C(n, s)| + |g^T Bg|$. Thus, by Theorem 2.1, all solutions of (3.1) are uniformly bounded. □

In the next theorem, we use Theorem 2.3 to establish boundedness and stability results for (3.1) when $g(n, x(n))$ is identically zero.

**Theorem 3.2.** Suppose there is a function $\phi(n) \geq 0$ with $\Delta \phi(n) \leq 0$ for $n \geq 0$ such that

$$
\Delta_n \phi(n - s - 1) + \left| C(n, s) \right| \leq 0 \quad \text{for} \quad 0 \leq s < n < \infty.
$$

Also, suppose that for $n \geq 0$,

$$
\left| A(n) \right| + \left| C(n, n) \right| + \phi(0) \leq 1 - \rho \quad \text{for some} \quad \rho \in (0, 1).
$$

(a) If $\sum_{s=0}^{\infty} \phi(s) = B$, then solutions of (3.1) are uniform bounded and the zero solution of (3.1) is uniformly stable.

(b) If $\sum_{n=0}^{\infty} \sum_{s=n}^{\infty} \phi(s) = J$, then solutions of (3.1) are uniform ultimate bounded and the zero solution of (3.1) is uniformly asymptotically stable.

**Proof.** Define the Lyapunov functional $V(n) = V(n, x(n, \cdot))$ by

$$
V(n) = \left| x(n) \right| + \sum_{s=0}^{n-1} \phi(n - s - 1) \left| x(s) \right|, \quad n \geq 0. \tag{3.7}
$$

Then along solutions of (3.1) we have

$$
\Delta V_{(3.1)}(n) \leq \left( \left| A(n) \right| + \left| C(n, n) \right| + \phi(0) - 1 \right) \left| x(n) \right|
$$

$$
+ \sum_{s=0}^{n-1} \left( \Delta_n \phi(n - s - 1) + \left| C(n, s) \right| \right) \left| x(s) \right|
$$

$$
\leq -\rho \left| x(n) \right|,
$$

and the results follow from Theorem 2.3. □

4. Scalar case

In this section we consider the scalar Volterra discrete equation with nonlinear perturbation

$$
x(n + 1) = a(n)x(n) + \sum_{s=0}^{n} b(n, s)x(s) + g(n, x(n)), \quad n \geq 0, \tag{4.1}
$$

where $g(n, x(n))$ is as defined before. In Theorem 4.2 we will display a Lyapunov functional and make use of Theorem 2.2 and obtain boundedness and stability results for (4.1).

In the past and present literature on the stability of the zero solution of (4.1), see [3–7, 9–12], it has always been assumed that $|a(n)| < 1$, a condition that is very restrictive. Instead, we will use the sign of $b(n, n)$ and try to relax the severe condition $|a(n)| < 1$. As the reader will see, Theorem 2.1 cannot be used to arrive at any results regarding Eq. (4.1).
Proposition 4.1. Let $Q(n) = a(n) - H(n, n)$ and $\Delta_n H(n, s) = b(n, s)$. Then (4.1) is equivalent to

$$x(n + 1) = Q(n)x(n) + \Delta_n \left[ \sum_{s=0}^{n-1} H(n, s)x(s) \right] + g(n, x(n)).$$

We omit the proof.

Theorem 4.2. Suppose

$$\Delta_s \left| H(n, s) \right|, \Delta_s \sum_{u=n}^{\infty} \left| H(u, s) \right| \geq 0,$$

where $\Delta_n H(n, s) = b(n, s)$. Also, we assume that

$$\Delta_n \left| H(n, s) \right| \leq 0.$$

If

$$2 + \left| Q \right| - P \leq 0$$

and

$$Q^2(n) + \left| 1 - Q(n) \right| \sum_{x=0}^{n-1} \left| H(n, s) \right| + P \sum_{x=n+1}^{\infty} \left| H(n, s) \right| - 1 \leq -\mu$$

for some positive constants $P$ and $\mu$, then solutions of (4.1) are uniformly bounded.

Proof. Define the Lyapunov functional $V(n, x(\cdot))$ by

$$V(n, x(\cdot)) = \left( x(n) - \sum_{x=0}^{n-1} H(n, s)x(s) \right)^2 + P \sum_{x=0}^{n-1} \sum_{u=n}^{\infty} \left| H(u, s) \right| x^2(s).$$

Note that

$$V(n, x(\cdot)) \leq \left( 1 + \sum_{x=0}^{n-1} \left| H(n, s) \right| \right) x^2(n) + \left( 1 + \sum_{x=0}^{n-1} \left| H(n, s) \right| \right) \sum_{x=0}^{n-1} \left| H(n, s) \right| x^2(s)$$

$$+ P \sum_{x=0}^{n-1} \sum_{u=n}^{\infty} \left| H(u, s) \right| x^2(s).$$

Then along solutions of (4.2) and after some calculations, we have

$$\Delta V(n, x(\cdot)) \leq \left[ Q^2(n) + \left| 1 - Q(n) \right| \sum_{x=0}^{n-1} \left| H(n, s) \right| + P \sum_{x=n+1}^{\infty} \left| H(n, s) \right| - 1 \right] x^2(n)$$

$$+ 2\left| Q \right| N \left| x(n) \right| + \left[ 2 + \left| Q \right| - P \right] \sum_{x=n+1}^{\infty} \left| H(n, s) \right| x^2(s).$$
\[
+N^2 \left(1 + \sum_{s=0}^{n-1} |H(n,s)| \right)
\leq -\mu x^2(n) + \gamma|x(n)| + K \leq -\mu_1 x^2(n) - \mu_2 |x(n)| + K
\]
for some constants \(\mu_1, \mu_2 > 0\), where
\[
K = N^2 \left(1 + \sum_{s=0}^{n-1} |H(n,s)| \right).
\]
Let
\[
\varphi_1(n,s) = |H(n,s)| \quad \text{and} \quad \varphi_2(n,s) = \sum_{u=n}^{\infty} |H(u,s)|.
\]
Then all the conditions of Theorem 2.2 are satisfied, and hence solutions of (4.1) are uniformly bounded.

**References**