Opial-type Integral Inequalities Involving Several Higher Order Derivatives

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Submitted by J. L. Brenner

Received February 17, 1989

In this paper, we have established for the first time Opial-type integral inequalities involving a function and several higher order derivatives, which contain some known results of P. R. Beesack and K. M. Das (Pacific J. Math. 26, 1968, 215-232) and K. M. Das (Proc. Amer. Math. Soc. 22, 1969, 258-261). Some discrete Opial-type inequalities will also be studied. In the special cases they are similar to those of J. S. W. Wong (Canad. Math. Bull. 10, 1967, 115-118) and C. M. Lee (Canad. Math. Bull. 11, 1968, 73-77). Finally two examples are given to illustrate the application of our inequalities in differential equations.

In 1960, Opial [1] presented an interesting inequality which is called Opial's inequality afterwards. Because Opial-type inequalities have important applications in the theory of ordinary differential equations and boundary value problems, they have been given considerable attention. A large number of papers have appeared in the literature, for example, P. R. Beesack and K. M. Das [3], K. M. Das [4], C. M. Lee [5], D. S. Mitrinović [7], W. Z. Chen [8], etc.

In 1968, D. Willett [2] considered the Opial-type inequality which involves a higher order derivative. Later K. M. Das [3] found a more general result. But all generalizations of Opial-type inequalities only involve a higher order or one order derivative at the left side of the inequality until now. In this paper, Opial-type integral inequalities involving several higher order derivatives are newly established, which contain some known results of Beesack and Das [3], Das [4]. By establishing a discrete inequality which is similar to the Taylor expansion with the integral remainder term, we obtain a discrete analogue of Theorem 1. In the special cases they are similar to those of J. S. W. Wong [6] and C. M. Lee [5]. Finally we study the new application of our inequalities in differential equations.

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To state conveniently, we define

\[ p = \sum_{i=0}^{n} p_i, \quad Q = \prod_{i=0}^{n-1} [(n-i-1)!]^{p_i}, \]

\[ R_i(x) = \int_{a}^{x} (x-t)^{(n-i-1)(p/(p-1))} r^{-1/(p-1)} \, dt, \]

\[ K(h, r) = \left[ \int_{a}^{b} h^{p/(p-p_n)} r^{p_n/(p-p_n)} \prod_{i=0}^{n-1} R_i^{p_i/(p-p_n)} \, dx \right]^{(p-p_n)/p}. \]

**Theorem 1.** Let \( p_i (i = 0, 1, \ldots, n) \) be nonnegative real numbers satisfying \( \sum_{i=0}^{n} p_i > p_n > 0, \ p > 1, \) and let \( u(x) \) be of class \( c^n \) on \( a \leq x \leq b \) satisfying \( u(a) = u'(a) = \ldots = u^{(n-1)}(a) = 0. \) Suppose that \( h(x), \ r(x) \) are positive functions on \( a < x < b, \) such that \( R_i(x) < +\infty, \ K(h, r) < +\infty. \) Then we have

\[ \int_{a}^{b} h^{p/(p-p_n)} r^{p_n/(p-p_n)} \prod_{i=0}^{n-1} \left| u^{(i)} \right|^{p_i} dx \leq \left( \frac{p_n}{p} \right)^{p_n/p} K(h, r) \int_{a}^{b} r^{p/(p-p_n)} \left| u^{(n)} \right|^{p} dx. \]

**Proof:** From the initial value condition of \( u \) and the Taylor expansion with the integral remainder term, we have

\[ u^{(i)}(x) = \frac{1}{(n-i-1)!} \int_{a}^{x} (x-t)^{n-i-1} u^{(n)}(t) \, dt, \quad i = 0, 1, \ldots, n-1. \]

For \( a \leq x \leq b, \) using the Holder inequality we obtain

\[ |u^{(i)}(x)| \leq \frac{1}{(n-i-1)!} \int_{a}^{x} (x-t)^{n-i-1} |u^{(n)}| \, r^{-1/p} r^{1/p} \, dt \]

\[ \leq \frac{1}{(n-i-1)!} \left[ \int_{a}^{x} (x-t)^{(n-i-1)p/(p-1)} r^{-1/(p-1)} \, dt \right]^{(p-1)/p} \]

\[ \times \left[ \int_{a}^{x} r \left| u^{(n)} \right|^p \, dt \right]^{1/p} \]

\[ = \frac{1}{(n-i-1)!} R_i^{(p-1)/p} \left[ \int_{a}^{x} r \left| u^{(n)} \right|^p \, dt \right]^{1/p}. \]

We define

\[ y(x) = \int_{a}^{x} r \left| u^{(n)} \right|^p \, dt. \]

Obviously, here we have
Applying (2), (3), and the Hölder inequality, we obtain

$$\int_a^b h |u^{(n)}|^p \prod_{i=0}^{n-1} |u^{(i)}|^p \, dx$$

$$\leq \int_a^b h r^{p,n/p} y^{p,n/p} \prod_{i=0}^{n-1} \left[ \frac{1}{(n-i-1)!} \right] \prod_{i=0}^{n-1} R_i^{p,(p-1)/(p-1)} y^{p,n/p} \, dx$$

$$- \frac{1}{Q} \int_a^b h r^{p,n/p} \left( \prod_{i=0}^{n-1} R_i^{p,(p-1)/(p-1)} \right) y^{(1/p)} \sum_{i=0}^{n-1} p_i y^{p,n/p} \, dx$$

$$- \frac{1}{Q} \int_a^b h r^{p,n/p} \left( \prod_{i=0}^{n-1} R_i^{p,(p-1)/(p-1)} \right) y^{(p-p_n)/p_n} y^{p,n/p} \, dx$$

$$\leq \frac{1}{Q} \left[ \int_a^b h^{p,(p-p_n)/(p-p_n)} y^{(p-p_n)/(p-p_n)} \left( \prod_{i=0}^{n-1} R_i^{p,(p-1)/(p-1)} \right) \, dx \right]^{(p-p_n)/p_n}$$

$$\times \left[ \int_a^b y^{(p-p_n)/(p-p_n)} y^{p,n/p} \, dx \right]^{p,n/p}$$

$$= \frac{1}{Q} K(h, r) \left[ \int_{y(a)}^{y(b)} y^{(p-p_n)/(p-p_n)} \, dy \right]^{p,n/p}$$

$$= \frac{1}{Q} K(h, r) \left( \frac{p_n}{p} \right)^{p,n/p} y(b)$$

$$= \left( \frac{p_n}{p} \right)^{p,n/p} K(h, r) \int_a^b r |u^{(n)}|^p \, dx.$$

This completes the proof of Theorem 1.

In the case when $r = 1$, Theorem 1 is interesting and practical. For convenient use, we list it.

**Corollary 1.** If the conditions hold in Theorem 1 and $r(x) = 1$, then we have

$$\int_a^b h |u^{(n)}|^p \prod_{i=0}^{n-1} |u^{(i)}|^p \, dx$$

$$\leq \left( \frac{p_n}{p} \right)^{p,n/p} \frac{Ew(h)}{Q} \int_a^b |u^{(n)}|^p \, dx,$$  \hspace{1cm} (4)
where
\[
E = \prod_{i=0}^{n-1} \left( \frac{p-1}{p(n-i)-1} \right)^{p_i(p-1)/p}
\]
\[
w(h) = \left[ \int_a^b h^{p/(p-p_i)}(x-a)^{(p/(p-p_i))} \sum_{i=0}^{n-1} \frac{1}{(n-i)p_i-1} \, dx \right]^{(p-p_n)/p}.
\]

**Remark 1.** In the special case when \(n = 1\), Theorem 1 reduces to the inequality
\[
\int_a^b h \, |u|^{p_0} \, |u'|^{p_1} \, dx \leq \left( \frac{p_1}{p_0 + p_1} \right)^{p_1/(p_0 + p_1)} K(h, r) \int_a^b r \, |u'|^{p_0 + p_1} \, dx,
\]
where
\[
K(h, r) = \left[ \int_a^b h^{p_0 + p_1 / (p_0 + p_1)} r^{-p_1 / p_0} \left( \int_a^x r^{-1/(p_0 + p_1 - 1)} \, dt \right)^{p_0 + p_1 - 1} \, dx \right]^{p_0 / (p_0 + p_1)}.
\]
This is Theorem 1 of Beesack and Das [3].

**Remark 2.** It is interesting to note that in the special case when \(h(x) = 1\), \(p_0 > 0\), and \(p_i = 0\), \(i = 1, 2, \ldots, n - 1\), our Corollary 1 reduces to the inequality
\[
\int_a^b |u|^{p_0} \, |u^{(n)}|^{p_n} \, dx \leq \left( \frac{p_n^{p_0 + p_n}}{(p_0 + p_n)(n!)^{p_0}} \right)^{p_n / (p_0 + p_n - 1)} \left( \frac{n(p_0 + p_n - 1)}{n(p_0 + p_n) - 1} \right)^{p_0 / (p_0 + p_n)}
\times (b-a)^{n p_0} \int_a^b |u^{(n)}|^{p_0 + p_n} \, dx,
\]
which is the theorem of Das [4].

To study a discrete analogue of Theorem 1, we need to establish two lemmas.

As usual, we define
\[
\Delta y_i = y_{i+1} - y_i,
\]
\[
\Delta^k y_i = \Delta^{k-1} y_{i+1} - \Delta^{k-1} y_i,
\]
and provide
\[
\sum_{i \in \emptyset} y_i = 0, \quad \prod_{i \in \emptyset} y_i = 1,
\]
where \(\emptyset\) is the empty set.
**Lemma 1.** Let \( \{u_i\}, \{v_i\} \) be two real sequences. The following identity holds:

\[
\sum_{i=0}^{n-1} v_i A u_i = u_n v_n - u_0 v_0 - \sum_{i=0}^{n-1} u_{i+1} A v_i.
\]  

(5)

**Proof.** For every natural number \( i \),

\[
A(u_i v_i) = u_{i+1} v_{i+1} - u_i v_i
\]

That is, \( v_i A u_i = A(u_i v_i) - u_{i+1} A v_i \).

Substituting \( i = 0, 1, 2, ..., n - 1 \) and adding in the last equality, we have

\[
\sum_{i=0}^{n-1} v_i A u_i = \sum_{i=0}^{n-1} A(u_i v_i) - \sum_{i=0}^{n-1} u_{i+1} A v_i
\]

\[
= u_n v_n - u_0 v_0 - \sum_{i=0}^{n-1} u_{i+1} A v_i.
\]

The proof of Lemma 1 is complete.

The Taylor expansion with the integral remainder term plays an important role in the proof of Theorem 1. No expansion can be used for the discrete case, so we deduce an inequality which is similar to the Taylor expansion with the integral remainder term.

**Lemma 2.** Let \( \{u_i\}, \{A u_i\}, \ldots , \{A^k u_i\} \) be nonnegative sequences and satisfy \( u_0 = A u_0 = \cdots = A^{k-1} u_0 = 0 \), then

\[
u_n \leq \frac{1}{(m-1)!} \sum_{j=0}^{n-1} (n-j-1)^{m-1} A^m u_j, \quad m \leq k
\]  

(6)

\[
A^i u_n \leq \frac{1}{(k-i-1)!} \sum_{j=0}^{n-1} (n-j-1)^{k-i-1} A^k u_j, \quad 0 \leq i \leq k-1.
\]  

(7)

**Proof.** When \( m = 1 \), (6) holds obviously. We now suppose that (6) holds when \( m = t \), where \( 1 \leq t \leq k-1 \). From inductive assumption, Lemma 1, and \( A^t u_0 = 0 \), we have

\[
u_n \leq \frac{1}{(t-1)!} \sum_{j=0}^{n-1} (n-j-1)^{t-1} A^t u_j
\]

\[
\leq \frac{1}{(t-1)!} \sum_{j=0}^{n-1} A^t u_j
\]

\[
= \frac{(n-j-1)^{t-1} + (n-j-1)^{t-2} (n-j) + \cdots + (n-j-1)(n-j)^{t-2} + (n-j)^{t-1}}{t}
\]
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This completes the proof of (6). Substituting \( A'u_n \) for \( u_n \) and setting \( m = k - i \), we can get (7). The proof of Lemma 2 is complete.

Similarly, we define

\[
P = \sum_{i=0}^{k} p_i, \quad Q = \prod_{i=0}^{k-1} [(k - i - 1)!]^p_i,
\]

\[
R_{ix} = \sum_{j=0}^{x-1} (x - j - 1)^{(k - i - 1)p/(p - 1)} r_j^{-1/(p - 1)},
\]

\[
w_n(h, r) = \left[ \sum_{x=0}^{n-1} h_x (p - p_x)^r_x - p/(p - p_x) \prod_{i=0}^{k-1} R_{ix}^{r_x/(p - p_x)} \right]^{(p - p_x)/p}.
\]

**THEOREM 2.** Let \( p_i \geq 0, i = 0, 1, ..., k - 1, \ p > p_k > 0, \ p > 1, \ \{u_x\}, \ \{Au_x\}, ..., \{A^k u_x\} \) be nonnegative sequences and satisfy \( u_0 = Au_0 = \cdots = A^{k-1}u_0 = 0, \ h_x \geq 0, \ r_x > 0, \) then

\[
\sum_{x=0}^{n-1} h_x (A^k u_x)^p x \prod_{i=0}^{k-1} (A^i u_x)^{p_i} \leq \left( \frac{p_k}{p} \right)^{p_k/p} w_n(h, r) \sum_{x=0}^{n-1} r_x (A^k u_x)^p.
\]  

**Proof.** For every natural number \( j \), define

\[
y_j = \sum_{x=0}^{j-1} r_x (A^k u_x)^p,
\]

then we have

\[
y_0 = 0, \quad Ay_j = r_j (A^k u_j)^p, \quad A^k u_j = r_j^{-1/p} (Ay_j)^{1/p}.
\]
From Lemma 2, (9), and the Hölder inequality, we obtain

\[
\Delta' u_n \leq \frac{1}{(k-i-1)!} \sum_{j=0}^{n-1} (n-j-1)^{k-i-1} \Delta' u_j
\]

\[
= \frac{1}{(k-i-1)!} \sum_{j=0}^{n-1} (n-j-1)^{k-i-1} r_j^{-1/p} (\Delta y_j)^{1/p}
\]

\[
\leq \frac{1}{(k-i-1)!} \left[ \sum_{j=0}^{n-1} (n-j-1)^{(k-i-1)p/(p-1) r_j^{-1/(p-1)}} \right]^{(p-1)/p} 
\times \left( \sum_{j=0}^{n-1} \Delta y_j \right)^{1/p}
\]

\[
= \frac{1}{(k-i-1)!} R_{m}^{(p-1)/p} y_n^{1/p}.
\]  

Using (9), (10), and the Hölder inequality, we have

\[
\sum_{x=0}^{n-1} h_x (\Delta^k u_x)^{p_k} \prod_{i=0}^{k-1} (\Delta' u_x)^{p_i}
\]

\[
\leq \sum_{x=0}^{n-1} h_x r_x^{-p_k/p} (\Delta y_x)^{p_k/p} \prod_{i=0}^{k-1} \left[ \frac{1}{(k-i-1)!} R_{i_x}^{(p-1)/p} y_x^{1/p} \right]^{p_i}
\]

\[
= \frac{1}{Q} \sum_{x=0}^{n-1} h_x r_x^{-p_k/p} \prod_{i=0}^{k-1} R_{i_x}^{(p-1)/p} y_x^{(p-1)/p} (\Delta y_x)^{p_k/p}
\]

\[
\leq \frac{1}{Q} \left[ \sum_{x=0}^{n-1} h_x^{(p-1)/p} r_x^{-p_k/(p-1)} \prod_{i=0}^{k-1} R_{i_x}^{(p-1)/p} (p-1) y_x^{(p-1)/p} \right]^{(p-1)/p}
\]

\[
\times \left( \sum_{x=0}^{n-1} y_x^{(p-1)/p} (\Delta y_x)^{p_k/p} \right)
\]

\[
\leq \frac{w_n(h, r)}{Q} \left( \sum_{x=0}^{n-1} \int_{y_x}^{y_x+1} t^{(p-1)/p} \frac{p_k}{p_k} dt \right)^{p_k/p}
\]

\[
= \frac{w_n(h, r)}{Q} \left( \int_{0}^{y_n} t^{(p-1)/p} \frac{p_k}{p_k} dt \right)^{p_k/p}
\]

\[
= \frac{w_n(h, r)}{Q} \left( \frac{p_k}{p} \right)^{p_k/p} y_n
\]

\[
= \left( \frac{p_k}{p} \right)^{p_k/p} \frac{w_n(h, r)}{Q} \sum_{x=0}^{n-1} r_x (\Delta^k u_x)^p.
\]

The proof of Theorem 2 is complete.
This theorem is also the first discrete inequality that involves a function and several higher order differences in the literature.

**Corollary 2.** If the conditions hold in Theorem 2, then

\[
\sum_{x=0}^{n-1} (\Delta^k u_x)^{p_k} \prod_{i=0}^{k-1} (\Delta^i u_x)^{p_i} \leq \frac{p_k^{p/p}}{pQ} \prod_{i=0}^{k-1} \left[ \frac{p-1}{(k-i) p-1} \right]^{(p-1)p_i/p} \left[ \frac{p-p_k}{\sum_{i=0}^{k-1} (k-i) p_i} \right]^{(p-p_k)/p} \\
\times n^{\sum_{i=0}^{k-1} (k-i) p_i} \sum_{x=0}^{n-1} (\Delta^k u_x)^{p}.
\]

(11)

**Proof.** In Theorem 2 let \( r_x = h_x \equiv 1, x = 0, 1, \ldots, n \), then

\[
R_{lx} \leq \frac{p-1}{(k-i) p-1} x^{((k-i) p-1)/(p-1)},
\]

\[
w_n = \frac{n-1}{\prod_{i=0}^{k-1} R_{ix}^{(p-1)/(p-p_k)}} \left[ \frac{p-1}{(k-i) p-1} \right]^{(p-1)p_i/(p-p_k)} \left[ \sum_{x=0}^{n-1} x^{((k-i) p-1)/(p-p_k)} \right]^{(p-p_k)/p} \\
\leq \left[ \sum_{x=0}^{n-1} \prod_{i=0}^{k-1} \left( \frac{p-1}{(k-i) p-1} \right)^{p_i/(p-p_k)} \left[ \sum_{x=0}^{n-1} x^{((k-i) p-1)/(p-p_k)} \right]^{(p-p_k)/p} \\
\times \left[ \sum_{x=0}^{n-1} x^{(p/(p-p_k)) \sum_{i=0}^{k-1} (k-i) p_i-1} dt \right]^{(p-p_k)/p} \right]^{(p-1)p/p} \\
= \prod_{i=0}^{k-1} \left( \frac{p-1}{(k-i) p-1} \right)^{(p-1)p_i/p} \\
\times \left[ \frac{p-p_k}{p \sum_{i=0}^{k-1} (k-i) p_i} \right]^{(p/p-p_k) \sum_{i=0}^{k-1} (k-i) p_i} \\
\times n^{\sum_{i=0}^{k-1} (k-i) p_i} \\
= \prod_{i=0}^{k-1} \left[ \frac{p-1}{(k-i) p-1} \right]^{(p-1)p_i/p} \left[ \frac{p-p_k}{p \sum_{i=0}^{k-1} (k-i) p_i} \right]^{(p-p_k)/p} \times n^{\sum_{i=0}^{k-1} (k-i) p_i}.
\]

We put them into (8) and derive the desired inequality (11).
Remark 3. In the above Corollary 2 if $k = 1$, from (11) we have

$$\sum_{j=0}^{n-1} (u_j)^{p_0} (\Delta u_j)^{p_1} \leq \frac{p_1^{p_1/(p_0 + p_1)}}{p_0 + p_1} n^{p_0} \sum_{j=0}^{n-1} (\Delta u_j)^{p_0 + p_1}. \quad (12)$$

This is an inequality which is similar to those of Wong [6] and Lee [5]. In my opinion, P. R. Beesack's comment that inequality (12) was superior to Lee's inequality [5] in Zbl.399.26010 is not correct. Because when $p, q \geq 1$, the inequality of Lee [5] is

$$\sum_{j=0}^{n-1} (u_{i+1})^p (\Delta u_i)^q \leq \frac{q(n+1)^p}{p+q} \sum_{i=0}^{n-1} (\Delta u_i)^{p+q}. \quad (13)$$

Let $p_0 = p, p_1 = q$, although the right side of (12) is less than that of (13), yet the left side of (12) may be less than that of (13) for $u_i \leq u_{i+1}$. Neither result can be compared.

Remark 4. In the above Corollary 2, let $p_0 > 0, p_i = 0, i = 1, 2, \ldots, k - 1$. From (11) we have

$$\sum_{j=0}^{n-1} u_j^{p_0} (\Delta^k u_j)^{p_k} \leq \frac{p_k^{p_k/(p_0 + p_k)}}{(p_0 + p_k)(k!)^{p_0}} \left[ \frac{k(p_0 + p_k - 1)}{k(p_0 + p_k) - 1} \right]^{(p_0 + p_k - 1)/(p_0 + p_k)} \times n^{p_0} \sum_{j=0}^{n-1} (\Delta^k u_j)^{p_0 + p_k},$$

which is a discrete analogue of the inequality of Das [4].

Now we give two examples to illustrate the application of our inequalities in differential equations.

**Example 1.** Suppose the following initial value problem

$$2y^{(n)} + \sum_{j=0}^{k} h_j(x) \prod_{i=0}^{n-1} (y^{(i)})^{p_i} = 0, \quad (14)$$

$$y(a) = y'(a) = \ldots = y^{(n-1)}(a) = 0.$$

If $p_j = \sum_{i=0}^{n-1} p_i \geq 1, h_j(x)$ are continuous functions on $a \leq x \leq b$ such that $w(h_j) < +\infty, j = 1, 2, \ldots, k$, where

$$w(h_j) = \left[ \int_a^b |h_j(x)|^\beta (x-a)^\beta \Sigma_{i=0}^{n-2} (n-i-1) p_{i-1} \, dx \right]^{1/\beta},$$

$$\beta = \frac{p_j + 1}{p_j - p_{j(n-1)}},$$

then the solution to the problem (14) is $y(x) \equiv 0, a \leq x \leq b$. 

Proof. Equation (14) can be written

\[ 2y^{(n)} = -\sum_{j=1}^{k} h_j(x) \prod_{i=0}^{n-1} (y^{(i)})^{p_j}. \]

Multiplying \( y^{(n-1)} \) to both sides of the above equality, integrating both sides from \( a \) to \( x \), and using \( y^{(n-1)}(a) = 0 \), we get

\[ |y^{(n-1)}(x)|^2 = (y^{(n-1)}(x))^2 \]

\[ = -\sum_{j=1}^{k} \int_{a}^{x} h_j(s) \prod_{i=0}^{n-1} (y^{(i)})^{p_j} ds \]

\[ \leq \sum_{j=1}^{k} \int_{a}^{x} |h_j(s)| |y^{(n-1)}|^{1 + p_{j(n-1)}} \prod_{i=0}^{n-2} |y^{(i)}|^{p_j} ds. \]  

(15)

Applying Corollary 1 to the right side of (15), we have

\[ |y^{(n-1)}(x)|^2 \leq \sum_{j=1}^{k} \int_{a}^{x} |h_j(s)| |y^{(n-1)}|^{1 + p_{j(n-1)}} \prod_{i=0}^{n-2} |y^{(i)}|^{p_j} ds \]

\[ \leq \sum_{j=1}^{k} M_j \int_{a}^{x} |y^{(n-1)}|^{p_j + 1} ds, \]  

(16)

where

\[ M_j = \left( \frac{1 + p_{j(n-1)}}{1 + p_j} \right)^{(1 + p_{j(n-1)}/(1 + p_j))} \times \frac{\prod_{i=0}^{n-2} \left[ p_j/(p_j + 1) \right]^{p_j/(p_j + 1)} \left[ (n - i - 2)! \right]^{p_i/p_i} \cdot w(h_j). \]

Since \( y^{(n-1)}(a) = 0 \) and \( y^{(n-1)}(x) \) is continuous, we obtain an interval \([a, h] \subset [a, b] \) such that \( |y^{(n-1)}(x)| \leq 1 \) on \([a, h] \), then

\[ |y^{(n-1)}(x)|^{p_j + 1} \leq |y^{(n-1)}(x)|^2, \quad x \in [a, h]. \]

Putting it into (16), we have

\[ |y^{(n-1)}(x)|^2 \leq \sum_{j=1}^{k} M_j \int_{a}^{x} |y^{(n-1)}|^{p_j + 1} ds \]

\[ \leq \left( \sum_{j=1}^{k} M_j \right) \int_{a}^{x} |y^{(n-1)}|^2 ds. \]

From the last inequality we get

\[ |y^{(n-1)}(x)| \equiv 0, \quad x \in [a, h], \]

that is, \( y^{(n-1)}(x) \equiv 0, \quad x \in [a, h] \).
Combining it with the identities
\[ y^{(i-1)}(x) = y^{(i-1)}(a) + \int_a^x y^{(i)}(s) \, ds, \quad i = 1, 2, \ldots, n-1. \]
we have
\[ y(x) \equiv y'(x) \equiv \cdots \equiv y^{(n-1)}(x) \equiv 0, \quad x \in [a, h]. \]

Especially, we have
\[ y(h) = y'(h) = \cdots = y^{(n-1)}(h) = 0. \]

Taking \( h \) as the left end point of the new interval and repeating the same argument as used above, we obtain a new interval \([h, h'] \subset [h, b]\) such that
\[ y(x) \equiv y'(x) \equiv \cdots \equiv y^{(n-1)}(x) \equiv 0, \quad x \in [h, h']. \]

Continuing in this way, we derive
\[ y(x) \equiv y'(x) \equiv \cdots \equiv y^{(n-1)}(x) \equiv 0, \quad x \in [a, b]. \]

Suppose the result is untenable, then there exists a point \( x_0, a < x_0 < b \), such that
\[ y(x) = y'(x) = \cdots = y^{(n-1)}(x) = 0, \quad x \in [a, x_0], \]
but in the right neighborhood of \( x_0 \) not all \( y(x), y'(x), \ldots, y^{(n-1)}(x) \) are identically vanishing. Since \( y^{(n-1)}(x_0) = 0 \) and \( y^{(n-1)}(x) \) is continuous, we obtain a non-degenerate interval \([x_0, h] \subset [a, b]\) such that \( |y^{(n-1)}(x)| \leq 1 \) on \([x_0, h_0]\). Imitating the same argument as used above, we derive \( y(x) \equiv y'(x) \equiv \cdots \equiv y^{(n-1)}(x) \equiv 0, \quad x \in [x_0, h_0] \), which is in contradiction with what we suppose. So
\[ y(x) \equiv y'(x) \equiv \cdots \equiv y^{(n-1)}(x) \equiv 0, \quad x \in [a, b] \]
is surely tenable.

**Example 2.** Suppose the following initial value problem
\[ 2y^{(n)} + f(x, y, y', \ldots, y^{(n-1)}) = 0, \quad \text{(17)} \]
\[ y(a) = y_0, \quad y'(a) = y'_0, \ldots, y^{(n-1)}(a) = y_0^{(n-1)}, \]
where \( f(x, s_1, s_2, \ldots, s_n) \) is defined on a domain \( B_0 \subset \mathbb{R}^{n+1} \) which contains the
initial point. If for any two points \((x, s_1, \ldots, s_n), (x, t_1, \ldots, t_n) \in B_0\), the following inequality holds

\[
|f(x, s_1, \ldots, s_n) - f(x, t_1, \ldots, t_n)| \leq \sum_{j=1}^{k} |h_j(x)| \prod_{i=0}^{n-1} |s_{i+1} - t_{i+1}|^{p_{ji}},
\]

(18)

where \(k \geq 1\), \(h_j\) and \(p_{ji}\) satisfy the conditions of Example 1, then the solution to the initial value problem (17) is unique.

**Proof.** Suppose \(y(x), u(x)\) are two arbitrary solutions to the problem (17). Let \(w(x) = y(x) - u(x)\), then

\[
w(a) = w'(a) = \ldots = w^{(n-1)}(a) = 0,
\]

\[
2w^{(n)}(x) = f(x, u, u', \ldots, u^{(n-1)}) - f(x, y, y', \ldots, y^{(n-1)}).
\]

Multiplying \(w^{(n-1)}(x)\) to both sides of the above equality and integrating both sides from \(a\) to \(x\), we obtain

\[
(w^{(n-1)}(x))^2 = \int_a^x w^{(n-1)}(s)[f(s, u, u', \ldots, u^{(n-1)}) - f(s, y, y', \ldots, y^{(n-1)})] \, ds.
\]

Taking the absolute values of both sides of the above equality and using the condition (18), we have

\[
|w^{(n-1)}(x)|^2 \leq \int_a^x |w^{(n-1)}(s)| \sum_{j=1}^{k} |h_j(s)| \prod_{i=0}^{n-1} |w^{(i)}|^{p_{ji}} \, ds
\]

\[
= \sum_{j=1}^{k} \int_a^x |h_j(s)| |w^{(n-1)}| \prod_{i=0}^{n-1} |w^{(i)}|^{p_{ji}} \, ds.
\]

Imitating the same argument as used in Example 1, we have \(w(x) \equiv 0\), that is, \(y(x) \equiv u(x)\).

**Remark 5.** In the special case when \(f(x, y, y', \ldots, y^{(n-1)}) = \sum_{i=0}^{n-1} a_i(x) y^{(i)} + a_n(x)\), the result of Example 2 is the uniqueness theorem for an \(N\)th order linear ordinary differential equation.

**References**