On the Ranges of Nonlinear Set-Valued Accretive Operators in Banach Spaces

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Let $E$ be a real Banach space and $A: D(A) \subset E \to 2^E$, $B: D(B) \subset E \to 2^E$ be two set-valued accretive operators. We study the range problems of $A + \lambda I$ and $A + B + \lambda I$ for $\lambda > 0$. The results obtained in this paper generalize some other results.

1. INTRODUCTION

Accretive operators were introduced by F. E. Browder and T. Kato independently, and have been extensively studied by many authors. The $m$-accretive operators or accretive operators satisfying the range condition of the Crandall–Liggett exponential formula [11] play an important role in nonlinear semigroups, differential equations, and fully nonlinear partial differential equations. See [2, 5, 12, 16, 18, 17, 18, 21]. A well-known result of Martin [18] shows that a continuous accretive operator defined on a Banach space is $m$-accretive. And it is also known by [6] that a maximal accretive operator need not be $m$-accretive. Fitzgibbon [14] proved that Martin’s result is true for weak continuous accretive operators, and recently D. Bothe [3] proved that this result is still true for an upper semicontinuous accretive operator with compact convex value. But we do not know whether it is still true with the compact convex value replaced by a bounded closed convex value. The subject of the present paper is to give some further sufficient conditions such that a multivalued accretive is $m$-accretive or satisfies the range condition of the Crandall–Liggett exponential formula. This paper is organized as follows. In Section 2, we study the approximate solutions of differential inclusions associated with set

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valued accretive operators. In Section 3, we study the range of a given accretive operator defined on a cone or on the whole space. In Section 4, we study the range of sum of two accretive operators.

2. DIFFERENTIAL INCLUSIONS ASSOCIATED WITH ACCRETIVE OPERATORS

Let \( E \) be a real Banach space. An operator \( A: D(A) \subseteq E \to 2^E \) is said to be accretive if \( \|x - y\| \leq \|x - y + \lambda(u - v)\|, \forall x, y \in D(A), u \in Ax, v \in Ay, \lambda > 0 \); or equivalently, \( (x - y, u - v) \geq 0 \), where \( (\cdot, \cdot) \) is the semi-inner product defined by

\[
(x, y)_+ = \lim_{h \to 0^+} \frac{\|x + hy\| - \|x\|}{h}, \quad (x, y)_- = \lim_{h \to 0^+} \frac{\|x\| - \|x - hy\|}{h}.
\]

A subset \( P \subseteq E \) is said to be a cone if \( P \) is closed convex, \( P \cap (-P) = (0) \), and \( \lambda P \subseteq P, \forall \lambda > 0 \); there is an order "\( \leq \)" induced by \( P \), \( x \leq y \) if and only if \( x - y \in P \) for \( x, y \in E \). Let \( A, B \) be two bounded subsets of \( E \).

**Lemma 2.1.** Let \( E \) be a real Banach space, \( P \subseteq E \) a cone, \( A: P \to 2^E \) a Hausdorff continuous set-valued operator with bounded closed values, i.e., \( \lim_{y \to x} H(Ay, Ax) = 0, x \in P \). Suppose

\[
\sup_{a \in Ax} \lim_{h \to 0^+} \frac{1}{h} d(x + hu, P) = 0, \quad \forall x \in P;
\]

then for each \( x_0 \in P, s_0 \in R \), there exist \( \delta_0 > 0, M > 0, r > 0 \), for each \( \epsilon < 1 \), there exists a sequence \( t_i, 0 \leq i < +\infty \), and \( x_\epsilon(t): [s_0, s_0 + \delta_0] \to P \), satisfies the following conditions

(a) \( t_0 = s_0 < t_1 < \cdots < t_i \leq s_0 + \delta_0, t_i - t_{i-1} \leq \epsilon, 1 \leq i < +\infty \).

(b) \( x_\epsilon(t_0) = x_0, \) and \( \|x_\epsilon(t) - x_\epsilon(s)\| \leq M|t - s|, t, s \in [s_0, s_0 + \delta_0] \).

(c) \( \|x_\epsilon(t_{i-1}) - x_0\| \leq r_0, x_\epsilon(t) \) is linear on \( [t_{i-1}, t_i] \), and

\[
d(x_\epsilon(t), Ax_\epsilon(t_{i-1})) \leq \epsilon, t \in (t_{i-1}, t_i), 1 \leq i < +\infty.
\]

(d) If \( x \in P, \|x - x_\epsilon(t_{i-1})\| \leq M(t_i - t_{i-1}), \) then \( H(Ax, Ax_\epsilon(t_{i-1})) \leq \epsilon \).

**Proof.** For \( x_0 \in P \), since \( Ax_0 \) is bounded and \( A \) is Hausdorff continuous, so there exist \( r > 0, M > 1 \), such that

\[
\|y\| \leq \frac{M}{2}, \quad \forall y \in Ax, \quad \|x - x_0\| \leq r_0.
\]
Let $\delta_0 = \frac{r}{M}$. Suppose $x_i(t)$ is defined on $[s_0, t_{i-1}]$, $t_{i-1} < s_0 + \delta_0$, and satisfies (a), (b), (c). Choose $\alpha_i \in [0, \epsilon]$, such that $\alpha_i$ is the largest number satisfying:

1. $t_{i-1} + \alpha_i \leq s_0 + \delta_0$.
2. If $x \in P, \|x - x(t_{i-1})\| \leq M\alpha_i$, then $H(Ax, Ax(t_{i-1})) \leq \epsilon$.
3. There exists $u_{i-1} \in Ax(t_{i-1})$, such that $d(x(t_{i-1}) + \alpha_i u_{i-1}, P) \leq \frac{\epsilon}{2} \alpha_i$.

By the assumptions in Lemma 2.1, we know such $\alpha_i$ exists. Now, let $t_i = t_{i-1} + \alpha_i$. By (3), choose $x_i(t_i) \in P$, such that

$$\|x_i(t_{i-1}) + \alpha_i u_{i-1} - x_i(t_i)\| \leq \epsilon \alpha_i.$$ 

We define $x_i(t)$ on $[t_{i-1}, t_i]$ as the following:

$$x_i(t) = \frac{x_i(t_i) - x_i(t_{i-1})}{t_i - t_{i-1}} (t - t_{i-1}) + x_i(t_{i-1}), \quad t \in [t_{i-1}, t_i].$$

It is easy to see that $x_i(t)$ satisfies (a), (b), (c), and (d).

To complete the proof of Lemma 2.1, we need to show $\lim_{t \to s_0} t_i = s_0 + \delta_0$. If this is not true, then $\lim_{t \to s_0} t_i = \beta < s_0 + \delta_0$. Since $\|x_i(t_i) - x_i(t_i)\| \leq M\|t_i - t_i\|$, so $\lim_{t \to s_0} x_i(t_i) = z \in P$.

$A$ is Hausdorff continuous at $z$, so there exists $h_0 > 0$, such that

$$H(Ay, Az) \leq \frac{\epsilon}{8}, \quad \|y - z\| \leq 2h_0 M. \quad (2.1)$$

And there exists $h_1 < \min(\epsilon, h_0, s_0 + \delta_0 - \beta), h_1 > 0$, such that

$$d(x + h_1 u, P) \leq h_1 \frac{\epsilon}{8}, \quad \forall u \in Az. \quad (2.2)$$

Choose $N$ sufficiently large, such that

$$\beta - t_{i-1} < h_1, \|x_i(t_{i-1}) - z\| \leq h_1 M, \quad \forall i \geq N. \quad (2.3)$$

Since $t_{i-1} + h_1 < s_0 + \delta_0$, so (1) holds for $h_1$. If $y \in P, \|y - x_i(t_{i-1})\| \leq Mh_1, i \geq N$, then

$$\|y - z\| \leq \|y - x_i(t_{i-1})\| + \|x_i(t_{i-1}) - z\| \leq 2h_1 M < 2h_0 M.$$
From (2.1)–(2.3), we get
\[ H(Ay, Ax_{\varepsilon}(t_{i-1})) \leq \frac{\varepsilon}{4}, \quad i \geq N. \]

This means that (2) holds for \( h_i \) and \( i \geq N \), but \( \alpha_i \) is the largest number such that (1), (2), and (3) hold, so we must have
\[ d(x_i(t_{i-1}) + h_i u, P) > h_i \frac{\varepsilon}{2}, \quad \forall u \in Ax_{\varepsilon}(t_{i-1}), \quad i \geq N. \quad (2.4) \]

For each \( v \in Az \), since \( \lim_{i \to \infty} H(Ax_{\varepsilon}(t_{i-1}), Az) = 0 \), so there exists \( u_{i-1} \in Ax_{\varepsilon}(t_{i-1}) \), such that \( \lim u_{i-1} = v \).

It follows from (2.4) that
\[ d(z + h_i v, P) \geq h_i \frac{\varepsilon}{2}, \quad \forall v \in Az. \quad (2.5) \]

A contradiction to (2.2). We complete the proof.

**Theorem 2.1.** Let \( E \) be a real Banach space, \( P \subset E \) a cone, \( A: D(A) = P \to 2^E \) a set-valued accretive operator with bounded closed values. Suppose

(i) \( A \) is Hausdorff continuous;

(ii) \( \sup_{u \in Ax} \lim_{h \to 0^+} \frac{1}{h}d(x - hu, P) = 0, \quad \forall x \in P. \)

Then for each \( x_0 \in P \), and a positive sequence \( \{\varepsilon_n\}_{1}^{+\infty} \) with \( \lim_{n \to \infty} \varepsilon_n = 0 \), there exist \( x_{\varepsilon}(t): [0, \eta_{\varepsilon}] \to P \) satisfying the following conditions

1. \( x_{\varepsilon}(0) = x_0, n \geq 1. \)
2. \( \lim_{n \to \infty} \eta_{\varepsilon} = +\infty. \)
3. \( x_{\varepsilon}'(t) \) is continuous except a subset with zero measure.
4. \( d(x_{\varepsilon}'(t), -Ax_{\varepsilon}(t)) \leq 2 \varepsilon_n, \quad \text{a.e. } t \in [0, \eta_{\varepsilon}). \)
5. \( \lim_{n \to \infty} x_{\varepsilon}(t) = x(t), \quad t \in [0, +\infty) \) and the convergence is uniform on any compact interval.

**Proof.** For \( x_0 \in P \), and \( \varepsilon_n > 0 \). By Lemma 2.1, there exists \( \delta_0 > 0 \), and \( x_{\varepsilon}(t): [0, \delta_0] \to P \) satisfies (a), (b), (c), and (d) in Lemma 2.1 with \( A \) replaced by \( -A \). It follows from (c) and (d) of Lemma 2.1 that we have
\[ d(x_{\varepsilon}'(t), -Ax_{\varepsilon}(t)) \leq 2 \varepsilon_n, \quad \text{a.e. } t \in (0, \delta_0). \quad (2.6) \]

Suppose we have extended \( x_{\varepsilon}(t) \) to a maximal interval \( [0, \eta_{\varepsilon}] \), and (2.6) holds for almost all \( t \in (0, \eta_{\varepsilon}). \) We prove \( \lim_{n \to \infty} \eta_{\varepsilon} = +\infty. \) If this is not
true, there exist $\eta_{q_k}$, $k \geq 1$, such that

$$\lim_{n_k \to +\infty} \eta_{q_k} = s_0 < +\infty. \quad (2.7)$$

Let $\varphi_{n,m}(t) = \|x_{e_n}(t) - x_{e_m}(t)\|$. We have

$$\varphi_{n,m}(t) D^+ \varphi_{n,m}(t) \leq (x_{e_n}(t) - x_{e_m}(t), x'_e(t) - x'_e(t))_+,$$

where $D^+ \varphi_{n,m}(t) = \lim_{h \to 0^+} \frac{(\varphi_{n,m}(t) - \varphi_{n,m}(t - h))}{h}$. (See [11, p. 36]). By (2.6) and the accretivity of $A$, we have

$$\varphi_{n,m}(t) D^+ \varphi_{n,m}(t) \leq 2(\epsilon_n + \epsilon_m) \varphi_{n,m}(t), \quad \text{a.e. } t \in [0, \min(\eta_n, \eta_m)). \quad (2.8)$$

Hence we have $\lim_{n,m \to +\infty} \varphi_{n,m}(t) = 0, t \in [0, \lim_{n,m \to +\infty} \eta_n)$.

Denote by $x(t) = \lim_{n \to +\infty} x_{e_n}(t), t \in [0, s_0]$; then it is obvious that

$$\lim_{n_k \to +\infty} x_{e_{n_k}}(t) = x(t), \quad t \in [0, s_0). \quad (2.9)$$

Let $\psi_n(t) = \|x_{e_n}(t) - x_{e_n}(t)|, h > 0; then we have

$$\psi_n(t) D^+ \psi_n(t) \leq (x_{e_n}(t + h) - x_{e_n}(t), x'_e(t + h) - x'_e(t))_+, \quad \text{a.e. } t, t + h \in (0, \eta_n).$$

From (2.6) and the accretivity of $A$, we get

$$\psi_n(t) D^+ \psi_n(t) \leq 4\epsilon_n \psi_n(t).$$

Letting $n \to +\infty$, we get the following:

$$\|x(t + h) - x(t)\| \leq \|x(h) - x(0)\|, \quad t, h > 0, \quad t + h < s_0. \quad (2.10)$$

So $\lim_{t \to s_0^-} x(t) = z$ exists. $Az_0$ is bounded; therefore it follows from (2.6), (2.9), and the Hausdorff continuity of $A$ that $(x_{e_{n_k}}(t))$ is bounded for sufficiently large $n_k$ and $t$ closes to $\eta_{n_k}$.

Therefore $\lim_{t \to \eta_{n_k}} x_{e_{n_k}}(\eta_{n_k})$ exists for $n_k$ sufficiently large. By Lemma 2.1, there exists $\delta_{q_k} > 0, x_{e_n}(t)$ can be extended to $[0, \eta_{n_k} + \delta_{q_k})$ for $n_k$ sufficiently large, and (2.6) still holds for almost all $t \in [0, \eta_{n_k} + \delta_{q_k})$.

This is a contradiction to that $[0, \eta_{n_k})$ is the maximal interval. So we must have $\lim_{n \to +\infty} \eta_n = +\infty.$
By using the same argument as in Lemma 2.1 and Theorem 2.1, we get

**Theorem 2.2.** Let $A: D(A) = E \rightarrow 2^E$ be a Hausdorff continuous accretive operator with bounded closed values. Then for each $x_0 \in E$, and a positive sequence $\{\epsilon_n\}$ with $\lim_{n \rightarrow \infty} \epsilon_n = 0$, there exists $x_{\epsilon_n}(t): [0, \eta_n) \rightarrow E$ satisfying the following conditions

1. $x_{\epsilon_n}(0) = x_0$, $n \geq 1$.
2. $\lim_{n \rightarrow \infty} \eta_n = +\infty$.
3. $x_{\epsilon_n}'(t)$ is continuous except a subset with zero measure.
4. $d(x_{\epsilon_n}'(t), -Ax_{\epsilon_n}(t)) \leq 2 \epsilon_n$, a.e. $t \in [0, \eta_n)$.
5. $\lim_{n \rightarrow \infty} x_{\epsilon_n}(t) = x(t)$, $t \in [0, +\infty)$, and the convergence is uniform on any compact interval.

**Theorem 2.3.** Let $A: D(A) = E \rightarrow 2^E$ be a Hausdorff continuous accretive operator with compact values. Then the following differential inclusion

$$x'(t) \in -Ax(t), \quad \text{a.e. } t \in [0, +\infty),$$

$$x(0) = x_0 \in E.$$  \hspace{1cm} (*E2.1*)

has only one solution.

**Proof.** Let $\{\epsilon_n\}$ be a positive sequence with $\lim_{n \rightarrow \infty} \epsilon_n = 0$.

By Theorem 2.2, there exists $x_{\epsilon_n}(t): [0, \eta_n) \rightarrow E$ satisfying (1)–(5) in Theorem 2.2. Let $x(t) = \lim_{n \rightarrow \infty} x_{\epsilon_n}(t)$. We prove $x(t)$ is a solution of (*E2.1*). For any $T > 0$, since $x_{\epsilon_n}(t) \rightarrow x(t)$ uniformly on $[0, T]$, the Hausdorff continuity of $A$ and the compactness of $Ax$ for $x \in E$ imply that $\{x_{\epsilon_n}(t)\}$ must have a subsequence $\{x_{\epsilon_{m_k}}(t)\}$ converge uniformly on $[0, T]$ except a zero measure subset.

In fact, there exist $\delta_0 > 0$ and $t_n \in [0, T]$, $t_n \rightarrow t_0$, such that

$$\|x_{\epsilon_n}'(t_n) - x_{\epsilon_{m_k}}'(t_m)\| \geq \delta_0, \quad n \neq m. \quad (2.11)$$

Since $\lim_{n \rightarrow \infty} H(Ax_{\epsilon_n}(t_n), Ax(t_0)) = 0$, so there exist $y_n \in Ax(t_0)$, such that

$$\lim_{n \rightarrow \infty} \|x_{\epsilon_n}'(t_n) + y_n\| = 0.$$

From (2.11), we get $\lim_{n, m \rightarrow \infty} \|y_n - y_m\| \geq \delta_0$, a contradiction to that $Ax(t_0)$ is compact. So there exist $x_{\epsilon_{m_k}}'(t) \rightarrow y(t)$ uniformly and $y(t) \in Ax(t)$ for almost all $t$.

By (3) of Theorem 2.2, we know $x'(t) = y(t)$ for almost all $t \in [0, T]$, and $x(t)$ is a solution of (*E2.1*). The uniqueness is obvious. We complete the proof.
THEOREM 2.4. Let \( P \subset E \) be a cone and \( A: P \to 2^E \) an accretive operator. Suppose

(i) \( A \) is Hausdorff continuous with weakly compact values,

(ii) \( \sup_{u \in Ax} \lim_{h \to 0^+} \frac{1}{h} d(x - hu, P) = 0, \ x \in P, \)

(iii) \( (x - y, u - v)_\gamma \geq C\|x - y\|^2, x, y \in P, u \in Ax, v \in Ay, C > 0 \) is a constant.

Then for \( \epsilon_n \to 0^+ \), there exist \( x_0 \in P, x_{\epsilon_n}(t): [0, \eta_n) \to P, n = 1, 2, \ldots, \) satisfying

1. \( x_{\epsilon_n}(0) = x_0, \forall n \geq 1. \)
2. \( \lim_{n \to +\infty} \eta_n = +\infty. \)
3. \( x_{\epsilon_n}'(t) \) is continuous except a zero measure subset.
4. \( d(x_{\epsilon_n}'(t), -Ax_{\epsilon_n}(t)) \leq 2\epsilon_n, \text{ a.e. } t \in [0, \eta_n). \)
5. \( x_{\epsilon_n}(t) \to x_0 \) uniformly on any compact intervals.

Proof. For each \( y \in P, \) and \( \epsilon_n \to 0^+ \), by Theorem 2.1, there exists \( x_{\epsilon_n}(t): [0, \eta_n) \to P \) with \( x_{\epsilon_n}(0) = y \) and satisfies (1)–(5) in Theorem 2.1.

For any \( \delta_n \to 0^+ \), if \( y_{\delta_n}(t): [0, \eta_n') \to P \) with \( y_{\delta_n}(0) = y \) and also satisfies (1)–(5) in Theorem 2.1, denote by \( y(t) = \lim_{n \to +\infty} y_{\delta_n}(t) \). Then it is easy to show that

\[ y(t) = x(t), \quad \forall t \in [0, +\infty). \quad (2.12) \]

This means that \( x(t) \) in Theorem 2.1 is unique and does not depend on the choice of sequence \( (\epsilon_n) \).

Now, we fix a sequence \( \epsilon_n \to 0^+ \). For giving \( T > 0 \), we define an operator \( F_T: P \to P \) as the following

\[ F_T y = x(T, y), \quad \forall y \in P, \quad (2.13) \]

where \( x(t, y) \) is obtained by Theorem 2.1 with \( x(0, y) = y \).

It follows from (2.12) that (2.13) is well defined. Now for \( y, z \in P, \) by Theorem 2.1, there exist \( x_{\epsilon_n}(t) \to x(t, y), w_{\epsilon_n}(t) \to x(t, z) \) satisfying (1)–(5) in Theorem 2.1.

Let \( \varphi_n(t) = \|x_{\epsilon_n}(t) - w_{\epsilon_n}(t)\| \). We have

\[ \varphi_n(t) D^- \varphi_n(t) \leq (x_{\epsilon_n}(t) - w_{\epsilon_n}(t), x_{\epsilon_n}'(t) - w_{\epsilon_n}'(t)). \]

It follows from (4) of Theorem 2.1 that

\[ \varphi_n(t) D^- \varphi_n(t) \leq 4\epsilon_n \varphi_n(t) - C\varphi_n^2(t), \]

\[ \varphi_n(t) \leq e^{-C\varphi_n(0)} + 4\epsilon_n te^{-Ct}. \]
Letting $n \to \infty$, we get
\[
\|x(t,y) - x(t,z)\| \leq e^{-CT}\|y - z\|.
\] (2.14)

Therefore, it follows from (2.14) that $\|F_T y - F_T z\| \leq e^{-CT}\|y - z\|$, $\forall y, z \in P$.

By Banach’s contraction principle, there exists a unique $x_0 \in P$, such that
\[F_T x_0 = x_0.\]

We know from (2.12) that $F_T F_s x = F_s F_T x$, $\forall S > 0, x \in P$.

So we have $F_T F_s x_0 = F_s x_0$, $\forall S > 0$. Therefore we must have $F_s x_0 = x_0$, so $x(s, x_0) = x_0$, $\forall S > 0$. By Theorem 2.1, we know Theorem 2.4 is true.

3. THE RANGES OF ACCRETIVE OPERATORS

In this section, $A: D(A) \subseteq E \to 2^E$ is a set valued operator. We give some sufficient conditions such that $D(A) \subseteq (A + \lambda I)D(A)$. Consequently, it follows directly from [11] that $-A$ generates a contraction semigroup.

**Theorem 3.1.** Let $E$ be a real Banach space, $P \subset E$ a cone, and $A: D(A) = P \to 2^E$ an accretive operator. Suppose

(i) $A$ is Hausdorff continuous with weakly compact values.

(ii) $\sup_{u \in A^x} \lim_{h \to 0^+} \frac{1}{h} d(x - hu, P) = 0$, $\forall x \in P$.

Then $P \subseteq (A + \lambda I)(P)$, $\forall \lambda > 0$.

**Proof.** For giving $\lambda > 0$, and each $z_0 \in P$, we prove $z_0 \in (A + \lambda I)(P)$. Let $Bx = Ax + \lambda x - z_0$, $\forall x \in P$. Then $B$ is accretive. Moreover,

\[(x - y, u - v) \geq \lambda \|x - y\|^2, \quad \forall x, y \in P, \quad u \in Bx, \quad v \in By.\]

$B$ is also Hausdorff continuous with weakly compact values and satisfies
\[
\sup_{u \in Bx} \lim_{h \to 0^+} \frac{1}{h} d(x - hu, P) = 0, \quad \forall x \in P.
\]

By Theorem 2.4, for $\epsilon_n \to 0^+$, there exist $x_0 \in P$, $x_\epsilon(t) : [0, \eta) \to P$ satisfying (1)–(5) in Theorem 2.4 and $\lim_{n \to \infty} x_\epsilon(t) = x_0, t \in [0, +\infty)$.

Inasmuch as $Bx_\epsilon(t)$ is weakly compact for each $t \in [0, \eta)$, so $(x_\epsilon(t))$ is weakly relatively compact in $L^2(J, E)$ for any compact interval $J$. See [12, Corollary 2.6].
Let $x'_\epsilon(.)$ converge weakly to $y(.)$ in $L^1(0, T; E)$, $T > 0$.

Since $x'_\epsilon(t) = x_0 + \int_0^t x'_\epsilon(s) \, ds$, therefore $x_0 = x_0 + \int_0^t y(s) \, ds$, $t \in [0, T]$.

Consequently, $y(t) = 0$; the Hausdorff continuity of $B$ implies that $0 \in Bx_0$.

Therefore $z_0 \in Ax_0 + \lambda x_0$, We complete the proof.

**COROLLARY 2.1.** Let $E$ be a real Banach space, $P \subset E$ a cone, and $A$: $D(A) = P \to 2^E$ an accretive operator. Suppose

(i) $A$ is Hausdorff continuous with weakly compact values.

(ii) For each $x \in P$, there exist $\beta(x) > 0$, $u \leq \beta(x)x$, $\forall u \in Ax$.

Then $P \subseteq (A + \lambda I)(P), \forall \lambda > 0$.

**Proof.** For each $x \in P$, $u \in Ax$. Since $u \leq \beta(x)x$, so there exists $z_u \in P$, such that $\beta(x)x - u = z_u$. In addition, $x - hu = (1 - h\beta(x))x + hz_u$. For $h$ sufficiently small, we have $1 - h\beta(x) > 0$, and $x - hu \in P$.

Therefore $d(x - hu, P) = 0$ for sufficiently small $h > 0$.

So Corollary 3.1 follows from Theorem 3.1. We complete the proof.

**Remark.** Corollary 3.1 was proved in [7] under the assumption that $A$ has compact values.

By adapting the same argument as in Theorem 2.4 and Theorem 3.1, one can easily prove the following:

**THEOREM 3.2.** Let $A$: $D(A) = E \to 2^E$ be a Hausdorff continuous accretive operator with weakly compact values. Then $(A + \lambda I)E = E, \forall \lambda > 0$.

When $E$ is a Hilbert space, the concept “accretive” is the same as “monotone.” We have the following result.

**THEOREM 3.3.** Let $H$ be a real Hilbert space, $P \subset H$ a cone, $A$: $D(A) = P \to 2^H$ a monotone operator. Suppose

(i) $A$ is Hausdorff continuous with closed bounded values.

(ii) There exists a proper lower semi-continuous convex function $\varphi$: $D(\varphi) \to (-\infty, +\infty], P \subseteq D(\varphi)$ and $Ax \subset \partial \varphi(x), \forall x \in P$.

(iii) $\sup_{x \in Ax} \lim_{h \to 0} \frac{1}{h}d(x - hu, P) = 0, \forall x \in P$.

Then $P \subseteq (A + \lambda I)P, \forall \lambda > 0$.

**Remark.** Assumption (ii) is equivalent to that $A$ is cyclically monotone. See [2].

**Proof.** For each $\lambda > 0$, $z_0 \in P$. Let $Bx = Ax + \lambda x - z_0$, $x \in P$. By assumptions (i) and (iii), it is easy to see that $B$ satisfies all conditions of
Theorem 2.4. So for \( e_n \to 0^+ \), there exist \( x_0 \in P \), \( x_{e_n}(t) : [0, \eta_n) \to P \) satisfying (1)–(5) in Theorem 2.4 and \( \lim_{n \to \infty} x_{e_n}(t) = x_0 \).

Since \( Ax_0 \) is bounded, we may assume \( x'_{e_n}(.) \) converge weakly to \( y(.) \) in \( L^2(0, T; H) \) for any \( T > 0 \).

Note that \( x_{e_n}(t) = x_0 + \int_0^t x'_{e_n}(s) \, ds \), so \( x_0 = x_0 + \int_0^t y(s) \, ds \), i.e., \( y(t) = 0 \), a.e. \( t \in [0, +\infty) \).

In the following, we prove that \( x'_{e_n}(.) \to 0 \) strongly in \( L^2(0, T; H) \). By (4) of Theorem 2.4, there exist \( u_n(t) \in Ax_{e_n}(t), w_n(t) \in E, \|w_n(t)\| \leq 2 e_n \), such that

\[
x'_{e_n}(t) = -u_n(t) - \lambda x_{e_n}(t) + z_0 + w_n(t), \quad \text{a.e.} \quad (0, \eta_n). \tag{3.1}
\]

For simplicity, we denote by \( x_{e_n}(t) = x_n(t), x'_{e_n}(t) = x_n'(t) \). Times (3.1) by \( x_n'(t) \) and integrate on \([0, T]\), and we get

\[
\int_0^T \left\| x_n'(s) \right\|^2 \, ds = -\int_0^T \left[ (u_n(s) - \lambda x_n(s) + z_0 + w_n(s), x_n(s)) \right] \, ds.
\tag{3.2}
\]

By assumption (ii), we have

\[
\int_0^T (u_n(s), x_n'(s)) \, ds = \varphi(x_n(T)) - \varphi(x_n(0)),
\]

\[
\int_0^T (x_n(s), x_n'(s)) \, ds = \frac{1}{2} \left( \|x_n(T)\|^2 - \|x_n(0)\|^2 \right).
\]

Since \( x_n(0) = x_0 \), letting \( n \to \infty \) in (3.2), we get

\[
\lim_{n \to \infty} \int_0^T \left\| x_n'(s) \right\|^2 \, ds = -\lim_{n \to \infty} \varphi(x_n(T)) + \varphi(x_0) \leq 0.
\]

So \( x_n'(.) \) converge strongly to 0 in \( L^2(0, T; H) \). Therefore \( z_0 \in Ax_0 + \lambda x_0 \).

**Theorem 3.4.** Let \( A : D(A) = E \to 2^E \) be an upper semicontinuous accretive operator with closed bounded convex values and \( E^* \) be uniformly convex. Then \( (A + \lambda I)E = E, \forall \lambda > 0 \).

**Proof.** Given \( z_0 \in E \), let \( Bx = Ax + \lambda x - z_0, \quad x \in E \). For each \( \varepsilon > 0 \), there exists a locally Lipschitz operator \( f_\varepsilon(x) : E \to E^* \), for each \( x \in E \), there exist \( y \in E, u \in By \), such that

\[
\|x - y\| < \varepsilon, \|f_\varepsilon(x) - u\| < \varepsilon.
\tag{3.3}
\]

(See [1]).
Consider the following initial value problem:

\[ x'(t) = -f_a(x(t)), \quad (E3.1) \]
\[ x(0) = x_0 \in E. \]

For \( \varepsilon < 1 \), there exists \( \delta_0 > 0 \), such that (E3.1) has only one solution \( x_\varepsilon(t) \) on \([0, \delta_0]\). See [12]. One can easily show that \( \lim_{\varepsilon \to 0} x_\varepsilon(t) = x(t) \) exists. Since \( Ax_0 \) is bounded, we may assume \( (x'_\varepsilon(t)) \) is bounded on \([0, \delta_0]\) by choosing \( \delta_0 \) small enough.

Let \( x'_\varepsilon(.) \) converge to \( y(.) \) weakly in \( L^2(0, T; E) \). The convergence theorem in [1, p. 60] implies that \( y(t) \in -Bx(t) \) for almost all \( t \in [0, \delta_0] \). It is easy to prove that \( x'(t) = y(t) \), a.e. \( t \in (0, \delta_0) \). So \( x(t) \) is the solution of the following differential inclusion

\[ x'(t) \in Bx(t), \quad t \in [0, \delta_0) \]
\[ x(0) = x_0 \in E. \quad (E3.2) \]

It is obvious that the solution of (E3.2) is unique and can be extended to \([0, +\infty)\). So we assume that the solution of (E3.2) exists on \([0, +\infty)\). Now, for any \( T > 0 \), we define \( F_T : E \to E \) as the following

\[ F_T y = x(T, y), \quad y \in E, \]

where \( x(t, y) \) is the solution of (E3.2) with \( x(0, y) = y \).

Then we have

\[ \|F_T y - F_T z\| \leq e^{-\lambda T}\|y - z\|, \quad \forall y, z \in E. \]

So \( F_T \) has a unique fixed point \( y_0 \), and \( y_0 \in -By_0 \). This completes the proof.

4. THE RANGES OF SUM OF TWO ACCRETIVE OPERATORS

In this section, let \( A : D(A) \subseteq E \to 2^E \), \( B : D(B) \subseteq E \to 2^E \) be two accretive operators satisfying \( D(A) \subseteq (A + \lambda I)D(A) \) and \( D(B) \subseteq (A + \lambda I)D(B) \), respectively, for \( \lambda > 0 \). We prove under some conditions that \( D(A) \cap D(B) \subseteq (A + B + \lambda I)(D(A) \cap D(B)) \) for \( \lambda > 0 \). In the following, we always denote by

\[ J_\lambda = (I + \lambda A)^{-1}, \quad A_\lambda = \frac{1}{\lambda}(I - J_\lambda), \quad \forall \lambda > 0. \]
Theorem 4.1. Let $\alpha: DA \subseteq E \to 2^E$ be a $m$-accretive operator, $B: D(B) = E \to 2^E$ an upper semicontinuous accretive operator with closed bounded convex values. Suppose $E^*$ is uniformly convex; then $A + B$ is $m$-accretive.

Remark. Theorem 4.1 is a generalization of Theorem 1 in [3] without a compact condition.

Proof. It is well known that $A_\alpha$ is accretive and Lipschitz. Theorem 3.4 implies that $A_{\lambda + B}$ is $m$-accretive. So the following differential inclusion

$$
\begin{align*}
\dot{x}(t) &= -A_\alpha x(t) - Bx(t) - x(t) + z_0, \quad t > 0 \\
x(0) &= x_0 \in D(A)
\end{align*}
$$

(E 4.1)

has only one solution $x(t)$ by the Crandall–Liggett exponential formula and the reflexivity of $E$. And it is easy to show that $(x_\lambda(t))$ is bounded. Since $B$ is upper semicontinuous at $x_0$, so we may choose $T > 0$ small enough such that $Bx(t)$ is bounded on $[0, T]$; consequently $A_{\lambda} x(t)$ is bounded on $[0, T]$. So $\lim_{\lambda \to 0^+} x(t) = x(t)$ exists on $[0, T]$.

Let $y(t) \in Bx(t)$, such that $\dot{x}(t) = -A_\lambda x(t) - y(t) - x(t) + z_0$, a.e. $t \in [0, T]$. Without loss of generality, we may assume $y(x)$, $x(t)$ converge weakly to $y(x)$ and $z(x)$, respectively, in $L^1(0, T; E)$ (otherwise, taking subsequences).

The convergence theorem in [1, p. 60] implies that $y(t) \in Bx(t)$. So $A_{\lambda} x(t)$ converge weakly to $-y(t) - x(t) + z_0 - z(t)$ in $L^1(0, T; E)$.

Since $(J_{\lambda} x(t) - x, A_{\lambda} x(t) - u)_+ \geq 0$, $\forall x \in DA$, $u \in Ax$. Keeping in mind that $E^*$ is uniformly convex, by letting $\lambda \to 0^+$, we get

$$(x(t) - x, -y(t) - x(t) + z_0 - z(t) - u)_+ \geq 0, \quad x \in DA, u \in Ax.$$

So $-y(t) - x(t) + z_0 - z(t) \in Ax(t)$. It is easy to show that $x(t)$ is the unique solution of the following differential inclusion

$$
\begin{align*}
x'(t) &\in -(A + B + I)x(t) + z_0, \quad t \in [0, T], \\
x(0) &= x_0 \in D(A)
\end{align*}
$$

(E 4.2)

Since $A + B$ is accretive, the solution of (E 4.2) can be extended to $[0, +\infty)$. So we always assume that the solution of (E 4.2) exists on $[0, +\infty)$.

Now, for any $T > 0$, we define $F_T: DA \to DA$ as the following

$$F_T y = x(T, y), \quad y \in DA.$$
where \( x(t, y) \) is the unique solution of (E4.2) with \( x(0, y) = y \). We can easily prove that

\[
\| F_T y - F_T z \| \leq e^{-T} \| y - z \|, \quad \forall y, z \in D(A). \tag{4.1}
\]

By (4.1), we can extend \( F_T \) to \( D(A) \) by letting

\[
\tilde{F}_T x = \lim_{n \to \infty} F_T y_n, \quad y_n \in D(A), \quad y_n \to x \in D(A).
\]

It is easy to see that \( \tilde{F}_T \) satisfies

\[
\| \tilde{F}_T y - \tilde{F}_T z \| \leq e^{-T} \| y - z \|, \quad y, z \in D(A). \tag{4.2}
\]

So \( \tilde{F}_T \) has a unique fixed point \( x_0 \in D(A) \). We prove that \( x_0 \in D(A) \). Since \( \tilde{F}_T \tilde{F}_S x_0 = \tilde{F}_S \tilde{F}_T x_0, \tilde{F}_S x_0 = x_0, \forall S > 0 \). By the definition of \( \tilde{F}_S \), there exist \( y_n \in D(A) \) and solutions \( x(s, y_n) \) of (E4.2), \( n = 1, 2, \ldots \), such that \( y_n \to x_0 \) and \( x(s, y_n) \to x_0 \). Let \( \phi_n(s) = \| x(s + h, y_n) - x(s, y_n) \| \). We have

\[
\phi_n(s) \leq e^{-s} \phi_n^2(s).
\]

So \( \| x'(s, y_n) \| \leq e^{-s} \| x'(0, y_n) \|, s > 0 \). Taking \( s_n \to \infty, x'(s_n, y_n) \in -Ax(s_n, y_n) - Bx(s_n, y_n) - x(s_n, y_n) + z_0 \) such that \( x'(s_n, y_n) \to 0 \) as \( n \to \infty \).

Notice that (4.2) holds for all \( T > 0 \), so \( x(s, y_n) \) converge to \( x_0 \) uniformly on \( [0, +\infty) \). Let \( b_n \in Bx(s_n, y_n), a_n \in Ax(s_n, y_n) \), such that

\[
x'(s_n, y_n) = -a_n - b_n - x(s_n, y_n) + z_0, n = 1, 2, \ldots.
\]

By the upper semicontinuity of \( B \) we know that \( Bx(s_n, y_n) \) is bounded, so we may assume \( b_n \) converges weakly to \( b_0^* \); then by \( m \)-accretivity of \( A \) we have \( a_n \) converging weakly to \( -b_0 - x_0 + z_0 \in Ax_0 \). \( Bx_0 \) is closed convex, so \( b_0 \in Bx_0 \).

Therefore we have \( z_0 \in Ax_0 + Bx_0 + x_0 \). This completes the proof.

Remark. Our proof of Theorem 4.1 is different from that of Theorem 1 in [3].

**Theorem 4.2.** Let \( A: D(A) \subseteq E \to 2^E \) be an \( m \)-accretive operator and \( B: D(B) = E \to 2^E \) a Hausdorff continuous accretive operator with weakly compact values. Suppose \( E^* \) is uniformly convex; then \( A + B \) is \( m \)-accretive.

Proof. It is the same as that of Theorem 4.1. We omit the details.

Remark. When \( B \) is single valued, we get Barbu's well known result. See Theorem 3.2 of [2, Chap. 3].
Theorem 4.3. Let $P \subseteq E$ be a cone, $A$: $D(A) \subseteq P \to 2^P$ an accretive operator satisfying $(A + \lambda I)(D(A)) = P$, $\forall \lambda > 0$, and $B$: $D(B) = P \to 2^E$ an accretive operator. Suppose

(i) $B$ is Hausdorff continuous with weakly compact values.

(ii) For each $x \in P$, $\sup_{n \in \mathbb{N}} \lim_{h \to 0^+} \frac{1}{h}d(x - hu, P) = 0$.

(iii) $E^*$ is uniformly convex.

Then $P \subseteq (A + B + \epsilon I)(D(A))$, $\forall \epsilon > 0$.

Proof. Since $(A + \lambda I)(D(A)) = P$, we have $(I + \lambda A)(D(A)) = P$ for all $\lambda > 0$. For each $\epsilon > 0$, $z_0 \in P$, consider the following equation:

$$
\begin{align*}
x'(t) &\in -(A_\lambda + B + \epsilon I)x(t) + z_0, \quad t > 0, \\
x(0) &\in z_0 \in D(A).
\end{align*}
$$

(E 4.3)

Since $A_\lambda x \leq \frac{1}{2}x$, $x \in P$, and $A_\lambda + B + \epsilon I - z_0$ is accretive and Hausdorff continuous with weakly compact values. By Theorem 3.1, we have

$$P \subseteq (A_\lambda + B + \epsilon I - z_0 + \eta I)(D(A)), \quad \eta > 0.$$

So (E 4.3) has a unique solution $x_\lambda(t)$ by the Crandall–Liggett exponential formula [11] and the reflexivity of $E$. It is easy to prove that $(x_\lambda(t))_{\lambda > 0}$ is bounded for $t \in [0, +\infty)$. $B$ is Hausdorff continuous with bounded values, so we may take $T > 0$ small enough such that $(Bx_\lambda(t))$ is bounded on $t \in [0, T]$. Consequently $A_\lambda x_\lambda(t)$ is bounded on $[0, T]$.

By standard argument one can easily show that $\lim_{\lambda \to 0^+} x_\lambda(t) = x(t)$ exists uniformly for $t \in [0, T]$ and $x(t)$ is the unique solution of the following differential inclusion.

$$
\begin{align*}
x'(t) &\in -(A + B + \epsilon I)x(t) + z_0, \quad t \in [0, T], \\
x(0) &\in x_0 \in D(A).
\end{align*}
$$

(E 4.4)

We extend the solution of (E 4.4) to $[0, +\infty)$. Now we define an operator $F_T$: $D(A) \to D(A)$ as the following

$$F_Tx_0 = x(T, x_0), \quad \forall x_0 \in D(A),$$

where $T > 0$ is any given number and $x(t, x_0)$ is the unique solution of (E 4.4) with initial value $x_0$. It is easy to check that $\|F_Tx_0 - F_Ty_0\| \leq e^{-\epsilon T}\|x_0 - y_0\|$, $\forall x_0, y_0 \in D(A)$. 

So $F_T$ can be extended to $\overline{D(A)}$ as the following:

$$\tilde{F}_T y = \lim_{n \to \infty} F_T y_n, \quad y_n \in D(A), \quad y_n \to y \in \overline{D(A)}.$$  

$\tilde{F}_T$ is a contraction mapping on $\overline{D(A)}$, so it has a unique fixed point $x_0$ in $\overline{D(A)}$. It is the same as the proof of Theorem 4.1 to show that $x_0 \in D(A)$ and $z_0 \in (A + B + \epsilon I)x_0$. We complete the proof.

**Theorem 4.4.** Let $A: D(A) \subseteq P \to 2^P$, $B: D(B) \subseteq P \to 2^P$ be two accretive operators, $P \subseteq E$ a cone. Suppose

(i) $(A + \lambda I)D(A) = P$, $(B + \lambda I)D(B) = P$, $\forall \lambda > 0$.

(ii) $B$ is locally bounded. $E^*$ is uniformly convex.

Then $(A + B + \epsilon I)(D(A) \cap D(B)) = P$, $\forall \epsilon > 0$.

**Proof.** For each $z_0 \in P$, one can easily check that $A_A - z_0$ is Lipschitz continuous and accretive and satisfies

$$\lim_{h \to 0^+} \frac{1}{h} d(x - hA_h x + hz_0, P) = 0.$$  

So by Theorem 4.3, $P \subseteq (A_A + B - z_0 + \epsilon I)D(B)$, $\forall \epsilon > 0$. Hence the well known Crandall–Liggett exponential formula implies that

$$x'(t) \in -(A_A + B + \epsilon I)x(t) + z_0, \quad t > 0$$  

$$x(0) = x_0 \in D(A) \cap D(B)$$  

(E 4.5)

has only one solution, $x(t)$.

Since $B$ is locally bounded, so we can choose $T > 0$ sufficiently small such that $(Bx(t))_h > 0$ is bounded on $[0, T]$. Let $\phi_{h, \mu}(t) = \|x(t) - x_\mu(t)\|$. By standard argument, we can show that $\lim_{h \to 0^+} \phi_{h, \mu}(t) = 0$. Now, let $x(t) = \lim_{h \to 0^+} x_h(t)$. It is easy to prove that $x(t)$ is the unique solution of the following equation:

$$x'(t) \in -(A + B + \epsilon I)x(t) + z_0, \quad t \in [0, T]$$  

$$x(0) = x_0 \in D(A) \cap D(B).$$  

(E 4.6)

We extend $x(t)$ to $[0, +\infty)$. Then by adapting the same method as in the proof of Theorem 4.3, we get $z_0 \in (A + B + \epsilon I)(D(A) \cap D(B))$. This completes the proof.

Using the same argument as in Theorem 4.4, we get the following result.

**Theorem 4.5.** Let $A: D(A) \subseteq E \to 2^E$, $B: D(B) \subseteq E \to 2^E$ be two $m$-accretive operators, $E$ being uniformly convex. Suppose $D(A) \cap D(B)$ is nonempty and $B$ is locally bounded. Then $A + B$ is $m$-accretive.

**Remark.** The proofs of Theorem 3.5, and Theorem 3.6 in [2, Chap. 2] cannot be applied to our results.
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