MATHEMATICS

ON THE CARDINALITY OF CLOSED SUBSETS

BY

S. MROWKA AND W. J. PERVIN *)

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It is well known [1] that the infinite closed subsets of βN are of cardinality 2^c. In contrast to this, we shall show (assuming the Continuum Hypothesis):

Theorem. No compact Hausdorff space has the property that all of its infinite closed subsets have cardinality c.

Proof. Let X be a compact Hausdorff space with the stated property. First we observe that X cannot contain any subspace which is the one point compactification of an infinite discrete space, for any set containing the ideal point would be closed. Now consider the derived sets $d(X)$, consisting of all limit points of X, and $d(d(X))=d^2(X)$. Both $d(X)$ and $d^2(X)$ are closed subsets of X, and so $d^2(X) \subseteq d(X)$. If $d^2(X) \neq d(X)$, then letting $x \in d(X) \backslash d^2(X)$, there is a closed neighborhood V of x such that $V \cap d(X)=\{x\}$. But then it is clear that V is the one point compactification of the infinite discrete space $V \backslash \{x\}$, and this is a contradiction. On the other hand, if $d^2(X)=d(X)$, then clearly $d(X)$ is infinite and closed. Consequently, by our assumption, its cardinality is c. It is clear that the first axiom of countability is not satisfied in the space $d(X)$ at any of its points, for otherwise we could construct a closed countable subset of $d(X)$ and hence of X by taking a convergent sequence and its limit. This, however, contradicts a theorem (utilizing the Continuum Hypothesis) of MRÓWKA [2] which states that a compact Hausdorff space of cardinality c must contain continuously many points at which the first axiom of countability is satisfied.

It is natural to ask, given a compact Hausdorff space of cardinality $m > \aleph_0$, must it contain infinite closed subsets of cardinality less than m? The example of βN shows that the answer is "no" for $m=2^c$. Assuming the Generalized Continuum Hypothesis, the answer is "yes" for all other cardinals $m > \aleph_0$; indeed, for $m=c$ this is the statement of the theorem and for $m > 2^c$ we note that the space must contain a closed infinite subset of cardinality no greater than 2^c (the closure of any infinite countable set).

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To conclude the paper we will show (without using the Continuum Hypothesis) that the theorem fails to be true without the assumption of compactness. We first need the following

Lemma. Let Y be a topological space with the property that every infinite closed subset of Y is of cardinality greater than or equal to c. Given a subset B of Y of cardinality c, there is a subset A of Y such that $B \subseteq A$, card $A = c$, and every infinite subset of B has continuously many accumulation points in A.

Proof. Let $\mathscr{E}$ be the class of all countably infinite subsets of B. Clearly, card $\mathscr{E} = c$. For every $E \in \mathscr{E}$, card $c(E) \geqslant c$, hence card $[c(E) \backslash E] \geqslant c$. We may then select a subset F_E of $c(E) \backslash E$ with card $F_E = c$. Let $A = B \cup \bigcup \{F_E : E \in \mathscr{E}\}$. Clearly, A has the required properties.

Theorem. These exists an infinite completely regular Hausdorff space X with the property that every infinite closed subset of X is of cardinality c.

Proof. Let Y be an infinite completely regular Hausdorff space with the property that every infinite closed subset of Y is of cardinality $\geqslant c$ (for example, we may take $Y = \beta N$). Let A be a subset of Y with cardinality c. We shall define a transfinite sequence A_λ, $\lambda < \Omega$ (where Ω is the first uncountable ordinal) of subsets of Y as follows. We let $A_0 = A$ and suppose that the sets A_λ have been defined for every $\lambda < \lambda_0$ so that card $A_\lambda = c$. If $B = \bigcup \{A_\lambda : \lambda < \lambda_0\}$, then card $B = c$ and, by the lemma, we may let A_{λ_0} be a subset of Y such that $B \subseteq A_{\lambda_0}$ card $A_{\lambda_0} = c$, and every infinite subset of B has continuously many accumulation points in A_{λ_0}. Finally, we let $X = \bigcup \{A_\lambda : \lambda < \Omega\}$. Clearly, X is a completely regular Hausdorff space whose cardinality is c. Let F be an infinite closed subset of X. Then F contains a countably infinite subset, say E. There is a $\lambda_0 < \Omega$ such that $E \subseteq \bigcup \{A_\lambda : \lambda < \lambda_0\}$, and hence E has continuously many accumulation points in A_{λ_0}. Thus, card $c_X(E) \geqslant c$, but $c_X(E) \subseteq F$, and so card $F \geqslant c$. On the other hand, it is clear that card $F \leqslant c$.

The Pennsylvania State University

BIBLIOGRAPHY

1. Novák, J., On the cartesian product of two compact spaces, Fund. Math. **40**, 106–112 (1953).
2. Mrówka, S., On the potency of compact spaces and the first axiom of countability, Bull. Pol. Acad. Sci. **6**, 7–9 (1958).