ELSEVIER
Annals of Pure and Applied Logic 92 (1998) 283–295

ANNALS OF PURE AND APPLIED LOGIC

Complexity of reals in inner models of set theory

Boban Velickovic [a,*], W. Hugh Woodin [b]

[a] Equipe de Logique, UFR de Math., Universite Paris 7, Tour 45-55, 5e etage, 2 Place Jussieu, 75251 Paris, Cedex 7, France

[b] Department of Mathematics, University of California, Berkeley, CA 94720, USA

Received 6 November 1996; received in revised form 15 January 1998
Communicated by T. Jech

Abstract

We consider the possible complexity of the set of reals belonging to an inner model M of set theory. We show that if this set is analytic then either $\aleph_1^M$ is countable or else all reals are in M. We also show that if an inner model contains a superperfect set of reals as a subset then it contains all reals. On the other hand, it is possible to have an inner model M whose reals are an uncountable F_σ set and which does not have all reals. A similar construction shows that there can be an inner model M which computes correctly $\aleph_1$, contains a perfect set of reals as a subset and yet not all reals are in M. These results were motivated by questions of H. Friedman and K. Prikry.

AMS Classification: primary 04E03; secondary 03E35; 03E45

Keywords: Colorings; Perfect sets; Inner models; Forcing

1. Introduction

The usual definition of the set of constructible reals $\mathbb{R}^L$ is Σ_2^1 (see, for example, [5, Theorem 97, p. 527]). This set can have a simpler definition if, for example, it is countable or if every real is in L. Martin and Solovay [7] have shown that if $\mathrm{MA}_{\aleph_1}$ holds and there is a real r such that $\aleph_1^{L[r]} = \aleph_1$ then every set of reals of size $\aleph_1$ is co-analytic. Thus by a ccc forcing over a universe of $V = L$ we can obtain a universe of set theory in which $\mathbb{R}^L$ is an uncountable co-analytic set yet not every real is in L. The results of this paper were motivated by a question of H. Friedman [2, problem 86], who asked if $\mathbb{R}^L$ can be analytic or even Borel in a nontrivial way, that is both uncountable and not equal to the set of all reals. There is a companion question due to K. Prikry whether $\mathbb{R}^L$ could contain a perfect set and not be equal to the set of all reals. Clearly, a positive answer to the first question would also imply a positive answer to the second one.

* Corresponding author.

0168-0072/98/$19.00
PII S0168-0072(98)00010-4

The main result of this paper is a negative answer to Friedman's question. In fact we prove that if M is an inner model of set theory and the set $\mathbb{R}^M$ of reals in M is analytic then either all reals are in M or else $\aleph_1^M$ is countable. Since the cardinality of $\mathbb{R}^L$ is $\aleph_1^L$ this implies the desired result in the case $M = L$. We also show that in the context of large cardinals this result can be extended to projective sets in place of analytic sets. However, the conclusion of the main theorem cannot be strengthened to say that either all reals are in M or else the continuum of M is countable. We produce a pair of generic extensions W and V of L such that $W \subseteq V$, the reals of W form an uncountable F_σ set in V, and yet not all reals from V are in W.

In relation to Prikry's problem we show that if an inner model M contains a superperfect set of reals then it contains all the reals. The proof is based on a construction of a simply definable continuous coloring of triples of reals into 2^ω such that for any superperfect set P the triples from P obtain all colors. A similar partition was used by Gitik [3], who showed that if V is a universe of set theory and r is a real not in V then the set of countable subsets of ω_2 in $V[r]$ which are not in V form a stationary set in $[\omega_2]^{\aleph_0}$. It was observed by the first author in [12] that this implies that if the semi proper forcing axiom (SPFA) holds and M is an inner model of set theory such that $\aleph_2^M = \aleph_2$ then all reals are in M.

After seeing the first draft of this paper Slaman and Groszek [4] found a negative solution of Prikry's problem. We will show how to derive the same result from the coloring theorem described above. The key observation is that if M is an inner model of set theory which contains a perfect set P of reals and every countable subset of $\mathbb{R} \cap M$ is covered by a countable set from M then one can convert P to a superperfect set by a continuous function coded in M. If M satisfies CH (as L does!) this covering property is necessarily true.

In the positive direction of Prikry's problem we give an example of two generic extensions V and W of L such that $W \subseteq V$, $\aleph_1^W = \aleph_1^V$, and there is a perfect set in V consisting of reals from W. In the construction we use Namba forcing to introduce a countable sequence of ordinals which is not covered by any countable set in L. By the above result this is an essential ingredient of the proof.

The paper is organized as follows. In Section 1 we present the coloring of triples of reals described above. It uses oscillations of reals numbers, a technique commonly used in the construction of examples to negative partition relations (see for example [11]). We then deduce that any inner model M of set theory which contains a superperfect set of reals contains all the reals. We also show that if in addition M satisfies a form of countable covering that any perfect set of reals contained in M can be mapped by a continuous function coded in M to a superperfect set. This gives another proof of the main result of [4] and of the negative answer to Prikry's problem.

In Section 2 we prove a kind of regularity property for $\boldsymbol{\Sigma}_1^1$ sets saying that if an analytic set A contains codes for all countable ordinals then every real is hyperarithmetic in a finite sequence of elements of A. From this our main result follows easily. We then extend this to higher levels of the projective hierarchy under appropriate large cardinal assumptions or projective determinacy.

Section 3 contains examples of pairs of models of set theory which show that the above results are in some sense best possible. We prove that it is possible to have an inner model of set theory W whose reals form an uncountable F_σ set and yet not all reals belong to W. Then necessarily $\aleph_1^W$ is countable. However, it is possible to have $\aleph_1^W = \aleph_1$ if we only require that W contains a perfect set of reals.

Finally, in Section 4 we prove in ZF alone that assuming there is a nonconstructible real there is an inner model M of ZF containing a prewellordering of reals of length $\aleph_1^V$ and such that not all reals belong to M. This is related to the conjecture that assuming AD + $V = L(\mathbb{R})$ then if M is an inner model of ZF containing a Souslin prewellordering of reals of length $\aleph_1^V$ then all reals are in M.

Our notation is fairly standard or self-explanatory. For all undefined notions see [13]. For an index set I we shall let $\mathscr{C}(I)$ denote the usual forcing for adding I Cohen reals. Thus conditions in $\mathscr{C}(I)$ are finite partial functions from $\omega \times I$ to $\{0,1\}$ and the order is inclusion.

2. Coloring triples of reals

We now present the coloring of triples of reals described in the introduction. First, we make some relevant definitions. We identify the set of reals $\mathbb{R}$ with the set $(\omega)^\omega$ of all infinite nondecreasing sequences of natural numbers. We make this nonstandard convention only for the convenience of notation and at any rate this space is naturally homeomorphic to the standard Baire space ω^ω.

We let $\leqslant_*$ denote the ordering of eventual dominance on $(\omega)^\omega$. We also let $(\omega)^{<\omega}$ denote the set of all finite nondecreasing sequences of natural numbers. Then $(\omega)^{<\omega}$ forms a tree under inclusion. Given a subtree T of $(\omega)^{<\omega}$ we say that a node $s \in T$ is *ω-splitting* if the set $\{k : s^\frown k \in T\}$ is infinite. T is called *superperfect* if above every node $s \in T$ there is a node $t \in T$ which is ω-splitting. A subset P of $(\omega)^\omega$ is called *superperfect* if the set T of all finite initial segments of members of P forms a superperfect tree. We can now state the key coloring theorem which will be used in the sequal.

Theorem 1. *There is a partial continous function $o : ((\omega)^\omega)^3 \to \{0,1\}^\omega$ such that for every superperfect subset P of $(\omega)^\omega$ $o[P^3] = \{0,1\}^\omega$. Moreover, o is defined by an arithmetic formula.*

Proof. Given distinct reals $x, y, z \in (\omega)^\omega$ let

$$O(x,y,z) = \{n : z(n-1) \leqslant x(n-1), y(n-1) \text{ and } x(n), y(n) < z(n)\}.$$

$o(x,y,z)$ will be defined iff $O(x,y,z)$ is infinite. If $O(x,y,z)$ is infinite let $\{n_k : k < \omega\}$ be the increasing enumeration of its members. Define $o(x,y,z)$ to be the function $\alpha : \omega \to \{0,1\}$ where for every $k < \omega$,

$$\alpha(k) = 0 \quad \text{iff } x(n_k) \leqslant y(n_k).$$

We show that if P is a superperfect subset of $(\omega)^\omega$ and $\alpha \in \{0,1\}^\omega$ there are $x,y,z \in P$ such that $o(x,y,z)=\alpha$. Thus, fix such a superperfect set P and let T be the tree of all initial segments of elements of P. We define inductively strictly increasing sequences of ω-splitting nodes of T $\{x_k : k<\omega\}$, $\{y_k : k<\omega\}$, $\{z_k : k<\omega\}$ as follows. The lengths of x_k, y_k, and z_k will be l_k, m_k, and n_k, respectively. We will arrange that for every k:
(1) $n_k < \min\{l_k, m_k\}$,
(2) $z_k(n_k-1) \leqslant \min\{x_k(n_k-1), y_k(n_k-1)\}$.
To begin let s by any ω-splitting node of T of length say $n_0>0$, and let $z_0=s$. For definiteness let us assume that $\alpha(0)=0$. First fix an ω-splitting node x_0 extending s of length say $l_0>n_0$. Then, using the fact that s is an ω-splitting node find some $j>x_0(l_0-1)$ such that $s^\frown j \in T$ and an ω-splitting node y_0 extending $s^\frown j$ of length say $m_0 \geqslant l_0$. Finally, find some $i>y_0(m_0-1)$ such that $s^\frown i \in T$ and then fix some ω-splitting node z_1 extending $s^\frown i$ of some length $n_1>l_0, m_0$. If $\alpha(0)=1$ then reverse the construction of x_0 and y_0.

At stage k we use the fact that z_k is an ω-splitting node of T to find an integer $i>x_k(l_k-1), y_k(m_k-1)$ such that $z_k{}^\frown i \in T$. We then let z_{k+1} be any ω-splitting node of T of length $n_{k+1}>l_k, m_k$ extending $z_k{}^\frown i$. Notice that since z_{k+1} is a nondecreasing sequence then for all j with $n_k \leqslant j < l_k$ we have $z_{k+1}(j)>x_k(j)$, similarly, for all $n_k \leqslant j \leqslant m_k$ we have $z_{k+1}(j)>y_k(j)$.

Now look at $\alpha(k+1)$. Let us assume first, as in case $k=0$, that it is equal 0. Since x_k is a ω-splitting node of T we can find an integer $j>z_{k+1}(n_{k+1}-1)$ such that $x_k{}^\frown j \in T$. Now, let x_{k+1} be any ω-splitting node of T extending $x_k{}^\frown j$ of length $l_{k+1}>n_{k+1}$. Finally, find some integer $h>x_{k+1}(l_{k+1}-1)$ such that $y_k{}^\frown h \in T$ and let y_{k+1} be any ω-splitting node of T extending $y_k{}^\frown h$ of length $m_{k+1} \geqslant l_{k+1}$. Note again, that since x_{k+1} is a nondecreasing sequence then for all e such that $l_k \leqslant e \leqslant n_{k+1}$ we have $x_{k+1}(e)>z_{k+1}(e)$. If $\alpha(k+1)=1$ then, as in the case $k=0$, reverse the construction of x_{k+1} and y_{k+1}.

At the end we let $x=\bigcup\{x_k : k<\omega\}$, $y=\bigcup\{y_k : k<\omega\}$, and $z=\bigcup\{z_k : k<\omega\}$. Note now that our construction guarantees that $O(x,y,z)=\{n_k : k<\omega\}$ and that we have $o(x,y,z)=\alpha$. □

Remark. By a little bit more effort we can define a function o as above such that for any three superperfect subsets P,Q,S we have $o[P\times Q\times S]=2^\omega$. It is interesting to note that Spinas [10] has shown that for any Borel coloring of unordered pairs of reals into finitely many colors there is a homogeneous superperfect set. This can also easily be deduced from a partition theorem of Milliken [14]. Therefore, the dimension 3 in the above theorem is optimal.

Theorem 2. *Let V and W be models of set theory such that W is a subuniverse of V. If V contains a superperfect tree T all of whose branches lie in W then V and W have the same reals.*

Following the preliminary version of this paper Groszek and Slaman [4] found a solution of the original Prikry's problem. Their result is more general and concerns pairs of models of set theory as in our Theorem 2 but with a key additional condition. Given two universes $V \subset W$ of set theory let us say that the pair (V, W) satisfies *countable covering for reals* if for every $X \in W$ such that $X \subset \mathbb{R}^V$ and X is countable in W, there is $Y \in V$ such that $X \subset Y$ and Y is countable in V. We now show how the main result of [4] can be derived easily from Theorem 2 above. This gives a different proof from the one in [4].

Theorem 3. *Suppose $V \subset W$ are models of set theory such that (V, W) satisfies countable covering for reals. Assume there is a perfect set of reals P in W such that $P \subset V$. Then V and W have the same reals.*

Proof. Let P be a perfect subset of 2^ω in W such that $P \subset V$. By countable covering for reals there is $D \in V$ which is countable in V, dense in 2^ω and such that $D \cap P$ is dense in P. Then $2^\omega \backslash D$ is homeomorphic to the irrationals which in turn is homeomorphic to $(\omega)^\omega$ (see [6, Section 7.7]). Since $D \in V$ there is such a homeomorphism f which is coded in V. Since $D \cap P$ is a countable dense subset of P we know that $P \backslash D$ is nowhere compact, i.e. contains no compact subset with nonempty interior. Thus $f[P \backslash D]$ is a closed subset of $(\omega)^\omega$ which is nowhere compact and thus is superperfect. Since $P \subset V$ and f is coded in V it follows that $f[P \backslash D] \subset V$. Now, by Theorem 2, we conclude that V and W have the same reals. $\square$

Note that if there is a perfect set of constructible reals then $\aleph_1^L = \aleph_1$ and hence (L, V) satisfies countable covering for reals. Thus as in [4] we have the following.

Corollary 1. *If there is a perfect set consisting only of constructible reals then all reals are constructible.*

3. The main theorem

In this section we prove the main result of this paper. We start with a lemma establishing a kind of regularity property for analytic sets of reals. Given a real x recall that ω_1^x denotes the first ordinal admissible in x, i.e. the least ordinal α such there is no well-ordering of ω of order type α which is recursive in x.

Given a limit ordinal λ let $(\lambda)^{<\omega}$ denote the set of finite nondecreasing sequences of ordinals $<\lambda$ and let $(\lambda)^\omega$ denote the set of all nondecreasing ω-sequences in λ. Say that a subtree T of $(\lambda)^{<\omega}$ is *λ-superperfect* if for every node $s \in T$ there is $t \in T$ extending s such that the set $\{\alpha : t^\frown \alpha \in T\}$ is cofinal in λ. Say that a subset P of $(\lambda)^\omega$ is *λ-superperfect* if the set of finite initial segments of members of P forms a λ-superperfect tree. The same construction as in Theorem 1 gives a coloring $o_\lambda : ((\lambda)^\omega)^3 \to \{0,1\}^\omega$ such that for any λ-superperfect set $P \subseteq (\lambda)^\omega$ $o_\lambda[P^3] = \{0,1\}^\omega$. Moreover, if $x, y, z \in (\lambda)^\omega$ then $o_\lambda(x, y, z) \in L[x, y, z]$.

Lemma 1. *Suppose that A is an analytic set such that $\sup\{\omega_1^x : x \in A\} = \aleph_1$. Then every real is hyperarithmetic in a quadruple of elements of A.*

Proof. Let $T \subset (\omega \times \omega)^{<\omega}$ be a tree such that $A = p[T]$. Note that the statement that $\sup\{\omega_1^x : x \in p[T]\} = \aleph_1$ is $\Pi_2^1(T)$ and thus, by Shoenfield's absoluteness theorem, it is absolute.

For an ordinal α let $Coll(\aleph_0, \alpha)$ be the usual collapse of α to $\aleph_0$ using finite conditions. Let $\mathscr{B}$ denote the regular open algebra of $Coll(\aleph_0, \aleph_1)$ as computed in V. Then we can indentify $Coll(\aleph_0, \aleph_1)$ with a dense subset of $\mathscr{B}$. If G is $\mathscr{B}$-generic over V then by the above remark in $V[G]$ there is $x \in p[T]$ such that $\omega_1^x > \aleph_1^V$. In V fix a $\mathscr{B}$-name $\dot{x}$ for x and a $\mathscr{B}$-name σ for a cofinal ω-sequence in $\aleph_1^V$ such that the maximal condition forces that $\dot{x} \in p[T]$ and $\sigma \in L[\dot{x}]$. Let $\mathscr{B}^*$ denote $\mathscr{B} \setminus \{0\}$. Given a condition $p \in \mathscr{B}^*$ and an integer k let

$$W_k(p) = \{\alpha < \aleph_1^V : \|\sigma(k) = \alpha\| \cdot p > 0\}.$$

Claim 1. *For every $p \in \mathscr{B}^*$ there is $k < \omega$, $s \in (\aleph_1^V)^k$ and $q \in \mathscr{B}^*$ such that $q \leqslant p$, $q \Vdash s \subset \sigma$ and $W_k(q)$ is uncountable in V.*

Proof. Note first that there is k such that $W_k(p)$ is uncountable. For otherwise letting $W = \bigcup_{k<\omega} W_k(p)$ we see that $p \Vdash ran(\sigma) \subseteq W$ contradicting the fact that σ is forced to be cofinal in $\aleph_1^V$. Now, fix the least k such that $W_k(p)$ is uncountable. Then the set Z of all $s \in (\aleph_1^V)^k$ such that $\|s \subset \sigma\| \cdot p > 0$ is countable. Since $W_k(p)$ is the union of the $W_k(\|s \subset \sigma\| \cdot p)$, for $s \in Z$, there is one such s such that $W_k(\|s \subset \sigma\| \cdot p)$ is uncountable. Therefore, the conclusion of the claim is satisfied for k, s, and $q = \|s \subset \sigma\| \cdot p$. □

Let $\mathscr{Q}$ denote $Coll(\aleph_0, 2^{\aleph_1})$ as defined in V. Suppose H is $\mathscr{Q}$-generic over V. Work for a moment in $V[H]$. If G is a $\mathscr{B}$-generic filter over V let σ_G denote the interpretation of σ in $V[G]$. Let S be the set of all σ_G where G ranges over all $\mathscr{B}$-generic filters over V.

Claim 2. *In $V[H]$ S contains an $\aleph_1^V$-superperfect set in $(\aleph_1^V)^\omega$.*

Proof. We work in $V[H]$. Let $\{D_n : n < \omega\}$ be an enumeration of all dense subsets of $\mathscr{B}$ which belong to V. This is possible since in V there are $2^{\aleph_1}$ such dense sets and H collapses $2^{\aleph_1}$ to $\aleph_0$. For each $t \in (\aleph_1^V)^{<\omega}$ we define a condition p_t in $\mathscr{B}$ and $s_t \in (\aleph_1^V)^{<\omega}$ inductively on the length of t such that

(1) $p_t \in D_{lh(t)}$

(2) $p_t \Vdash s_t \subset \sigma$

(3) if $t \leqslant r$ then $p_r \leqslant p_t$ and $s_t \subset s_r$

(4) if t and r are incomparable then s_t and s_r are incomparable

(5) for every t if k_t is the length of s_t then $W_{p_t}(k_t)$ is unbounded in $\aleph_1^V$.

Suppose p_t and s_t have been defined. Using (5) choose in V a 1–1 order preserving function $f_t : \aleph_1^V \to W_{p_t}(k_t)$ for every $\alpha < \aleph_1^V$ a condition $q_{t,\alpha} \leqslant \|s_t \subset \sigma\| \cdot p_t$ and $q_t \in D_{lh(t)+1}$. Now by applying Claim 1 for each $\alpha < \aleph_1^V$ we can find a condition $p \leqslant q_{t,\alpha}$ and $k > k_t + 1$ such that for some $s \in (\aleph_1^V)^k$ $p \Vdash s \subset \sigma$ and $W_k(p)$ is uncountable. Let then $s_{t^\frown\alpha} = s$ and $p_{t^\frown\alpha} = p$. This completes the inductive construction.

Now if $b \in (\aleph_1^V)^\omega$ then $\{p_{b\restriction n} : n < \omega\}$ generates a filter G_b which is $\mathscr{P}$-generic over V. The interpretation of σ under G_b is $s_b = \bigcup_{n<\omega} s_{b\restriction n}$. Since the set $R = \{s_b : b \in (\aleph_1^V)^\omega\}$ is $\aleph_1^V$-superperfect this proves Claim 2. $\square$

Now, using the remark from the beginning of this section, for any real $r \in \{0,1\}^\omega$ we can find $b_1, b_2, b_3 \in (\aleph_1^V)^\omega$ such that $r \in L[s_{b_1}, s_{b_2}, s_{b_3}]$. Let x_i be the interpretation of $\dot{x}$ under G_{b_i}. Then it follows that $x_i \in p[T]$ and $s_{b_i} \in L[x_i]$, for $i = 1,2,3$. Thus $r \in L[x_1, x_2, x_3]$. Pick a countable ordinal δ such that $r \in L_\delta[x_1, x_2, x_3]$. Using the fact that in $V[H]$ $\sup\{\omega_1^x : x \in p[T]\} = \aleph_1$ we can find $y \in p[T]$ such that $\omega_1^y > \delta$. Then we have that r is $\Delta_1^1(x_1, x_2, x_3, y)$. Note that the statement that there are $x_1, x_2, x_3, y \in p[T]$ such that $r \in \Delta_1^1(x_1, x_2, x_3, y)$ is $\Sigma_2^1(r, T)$. Thus for any $r \in V$, by Shoenfield absoluteness again, it must be true in V. This proves Lemma 1. $\square$

We now have as an immediate consequence the following.

Theorem 4. *Suppose M is an inner model of set theory and $\mathbb{R}^M$ is analytic. Then either $\aleph_1^M$ is countable or all reals are in M.*

To extend Lemma 1 and consequently Theorem 4 to higher levels of the projective hierarchy we need the appropriate form of projective absoluteness in place of Shoenfield's theorem. We first do this in the case of $\boldsymbol{\Sigma}_2^1$ sets.

Lemma 2. *Let a be a real such that $a^\#$ exists and assume that A is a $\Sigma_2^1(a)$ set such that $\sup\{\omega_1^x : x \in A\} = \aleph_1$. Then every real is hyperarithmetic in a quadruple of elements of A.*

Proof. Suppose A is defined by a $\Sigma_2^1(a)$ formula $\varphi(x,a)$. Following the proof of Lemma 1 we have to show that if G is $Coll(\aleph_0, \aleph_1)$-generic over V then in $V[G]$ $\sup\{\omega_1^x : \varphi(x,a) \text{ holds}\} > \aleph_1^V$. Let $\alpha < \aleph_1^V$ be indiscernible for $L[a]$. In V pick an $Coll(\aleph_0, \alpha)$-generic filter G_α over $L[a]$. This can be done since $\aleph_1^V$ is inaccessible in $L[a]$. In $L[a, G_\alpha]$ pick a linear ordering R on ω such that (ω, R) is isomorphic to $(\alpha, <)$. The formula which says that there exists x such that $\varphi(x,a)$ holds and such that $\omega_1^x > \alpha$ is $\Sigma_2^1(a, R)$ and is true in V. By Shoenfield's absoluteness theorem it is true in $L[a, G_\alpha]$ as well. Since G_α can be chosen to contain any condition in $Coll(\aleph_0, \alpha)$ it follows that the maximal condition in $Coll(\aleph_0, \alpha)$ forces the above statement. Since both α and $\aleph_1^V$ are indiscernibles over $L[a]$ it follows that the maximal condition in $Coll(\aleph_0, \aleph_1^V)$ forces over $L[a]$ that there is x such that $\varphi(x,a)$ holds and $\omega_1^x > \aleph_1^V$.

As in the proof of Lemma 1 we show that if H is $Coll(\aleph_0, 2^{\aleph_1})$-generic over V then in $V[H]$ for any real r there are reals x_1, x_2, x_3, y all satisfying $\varphi(x,a)$ and such that r is $\Delta_1^1(x_1, x_2, x_3, y)$. The existence of such quadruple is $\Sigma_2^1(a, r)$ so if r is in V it follows, by Shoenfield's theorem again, there such a quadruple exists already in V. $\square$

Theorem 5. *Assume $x^\#$ exists, for every real x. If M is an inner model of set theory such that $\aleph_1^M$ is uncountable and $\mathbb{R}^M$ is $\boldsymbol{\Sigma}_2^1$ then all reals are in M.*

For an integer n and an infinite cardinal κ let us say that a universe V satisfies Σ^1_n-absoluteness for posets of size $<\kappa$ if whenever $\mathscr{P}$ is a forcing notion of size $<\kappa$ and in $V^{\mathscr{P}}\mathscr{Q}$ is a forcing notion of size $<\kappa$ then for any Σ^1_n formula φ with parameters from $V^{\mathscr{P}}$, φ holds in $V^{\mathscr{P}\star\mathscr{Q}}$ if and only if it holds in $V^{\mathscr{P}}$. Woodin has shown that assuming the existence of n Woodin cardinals with a measurable cardinal above then Σ^1_{n+3} absoluteness holds for posets of size less than the first Woodin cardinal. The analogous proof to Lemma 1 goes through for Σ^1_{n+2} sets under this assumption. Therefore, we have the following.

Theorem 6. *Assume the existence of n Woodin cardinals with measurable above. If M is an inner model of set theory such that $\aleph_1^M$ is uncountable and $\mathbb{R}^M$ is a Σ^1_{n+2} set then all reals are in M.*

4. Adding perfect sets of ground model reals

In this section we show that the conclusion of Theorem 4 cannot be strengthened to say that either all reals are in M or the continuum of M is countable. We also show that it is possible to have an inner model of set theory W such that $\aleph_1^W=\aleph_1$, W contains a perfect set of reals, and not all reals are in W. We start with the following.

Theorem 7 (CH). *Suppose there is a club in ω_1 consisting of ordinals of uncountable cofinality in L. Then there is an L-generic filter G for adding ω_1^V many Cohen reals to L such that the reals of $L[G]$ are an F_σ set in V.*

Proof. Let C be a club in ω_1 consisting of ordinals of uncountable cofinality in L. Let P be a perfect subset of 2^ω such that any finite subset of P consists of mutually generic Cohen reals over L. Fix a recursive partition of ω into infinitely many disjoint infinite sets $\{A_i : i<\omega\}$ and for each $i<\omega$ fix a recursive partition $\{A_{i,j} : j<\omega\}$ of A_i into infinitely many disjoint infinite pieces. For each each $d\in 2^\omega$ let d_i be the real obtained by restricting d to A_i and transfering it to 2^ω using the order preserving bijection between A_i and ω. Let $d_{i,j}$ be obtained by restricting d to $A_{i,j}$ and transfering to 2^ω in a similar fashion.

Construct the generic G by constructing an L-generic filter G_α over $\mathscr{C}(\alpha)$ by induction on $\alpha\in C$. The requirements are that for each $\alpha\in C$ there exists a countable subset S_α of P such that

(1) for all $\beta<\alpha$ $G_\alpha(\beta)=d_{i,j}$, for some $d\in S_\alpha$, and some $i,j<\omega$,
(2) for all $d\in S_\alpha$ and for all $i,j<\omega$ there is $\beta<\alpha$ such that $G_\alpha(\beta)=d_{i,j}$,
(3) the set of reals of $L[G_\alpha]$ is the union of the sets of reals in $L[s]$, where s is a finite sequence of members of $\{d_i : d\in S_\alpha$ and $i<\omega\}$.

Since every $\alpha\in C$ has uncountable cofinality in L genericity and these conditions are preserved at a stage δ which is a limit point of C by using $S_\delta=\bigcup\{S_\alpha:\alpha<\delta\}$. We now verify the successor step. Let G_α and S_α be given. By condition (3) any finite subset of $P\setminus S_\alpha$ consists of mutually generic Cohen reals over $L[G_\alpha]$. Let α^* be the next element of C above α. Let $\{X_i : i<\omega\}$ be an increasing sequence of subsets of $[\alpha,\alpha^*)$ such that each $X_i \in L[G_\alpha]$, X_i is countable in $L[G_\alpha]$, and such that if $Y \subset [\alpha,\alpha^*)$ is countable in $L[G_\alpha]$ then $Y \subseteq X_i$, for some $i<\omega$. Moreover arrange that $X_{i+1}\setminus X_i$ is infinite, for each i. Fix any $d \in P\setminus S_\alpha$. It is routine to construct G^* satisfying (1) and (2) for $S_{\alpha^*} = S_\alpha \cup \{d\}$ and such that for all i

$$L[G_\alpha][g_i] = L[G_\alpha][d_i]$$

where $g_i = G_{\alpha^*} \restriction (X_i \setminus X_{i-1})$. Then condition (3) follows.

Assuming CH we can easily arrange that $P = \bigcup\{S_\alpha : \alpha<\omega_1\}$. Thus the set of reals in $L[G]$ is exactly the union of the reals of $L[s]$, where s is a finite sequence of elements of $\{d_i : d \in P \text{ and } i<\omega\}$. Since there are only countably many terms for reals in Cohen extensions which are in L and P is compact, it follows that this set is F_σ. $\square$

To obtain a model satisfying the assumptions of Theorem 7 we can start with a model of $V=L$, collapse $\aleph_1$ to $\aleph_0$ and then shoot a club through the set of ordinals $<\aleph_2^L$ of uncountable cofinality in L. Thus, we have the following:

Theorem 8. *There is a pair V and W of generic extensions of L such that $W \subseteq V$, the reals of W form an uncountable F_σ set in V, and V and W do not have the same reals.*

The following result says that we can have an inner model of set theory for which Prikry's question has a positive answer.

Theorem 9. *Assume ZFC. Then there is a pair (W,V) of generic extensions of L such that $W \subseteq V$, $\aleph_1^W = \aleph_1^V$, and V contains a perfect P set of W-reals which is not in W.*

We will need the following lemma (cf. Theorem 1 from [9]).

Lemma 3. *There is a generic extension V_0 of L such that $\aleph_1^{V_0} = \aleph_1^L$, and V_0 contains a club C in $\aleph_3^L$ consisting of ordinals of uncountable cofinality in L.*

Proof. V_0 will be obtained as a two step forcing extension of L. Let $\mathcal{N}$ be the following version of Namba forcing. Conditions in $\mathcal{N}$ are subtrees T of $\omega_2^{<\omega}$ such that for every $s \in T$ the set $\{t \in T : s \subseteq t\}$ has cardinality $\aleph_2$. The partial ordering is defined in the natural way: $R \leqslant T$ if and only if $R \subseteq T$. For a node $s \in T$ we let $T_s = \{t \in T : t \subseteq s \text{ or } s \subseteq t\}$. Then $\mathcal{N}$ preserves $\aleph_1$, changes the cofinality of $\aleph_2$ to $\aleph_0$, and collapses the cardinality of $\aleph_3$ to $\aleph_1$, (see [5, p. 289]). Define in L the set $S = \{\alpha < \omega_3 : cof(\alpha) = \omega_2\}$. Suppose now that G is $\mathcal{N}$-generic over L.

Claim. S remains stationary in $L[G]$.

Proof. Working in L let a name $\dot{C}$ for a club in ω_3 and a condition $T \in \mathcal{N}$ be given. Fix a sufficiently large regular cardinal θ and take an elementary submodel M of H_θ of cardinality $\aleph_2$ containing $\dot{C}$ and T such that $M \cap \omega_3 = \delta \in S$.

By shrinking if necessary we may assume that every node in T has either 1 or $\aleph_2$ immediate extensions. Fix a strictly increasing sequence $\langle \delta_\xi : \xi < \omega_2 \rangle$ of ordinals converging to δ. We build by a fusion argument a condition $R \leqslant T$ such that $R \Vdash \delta \in \dot{C}$. Set $R_0 = T$. Let s be the stem of T. For each $\xi < \omega_2$ such that $s^\frown \xi \in T$ the condition $T_{s^\frown \xi}$ belongs to $\mathcal{N} \cap M$. By elementarity and the fact that $\dot{C}$ is forced to be unbounded in ω_3 there is a condition $Q_{s,\xi} \leqslant T_{s^\frown \xi}$ such that $Q_{s,\xi} \in \mathcal{N} \cap M$ and for some $\delta_\xi < \gamma < \delta$ $Q_{s,\xi} \Vdash \gamma \in \dot{C}$. Let

$$R_1 = \bigcup \{ Q_{x,\xi} : \xi < \omega_2 \text{ and } s^\frown \xi \in T \}.$$

Now given R_n let L_n be the set of nodes of R_n which are $\aleph_2$-splitting and have exactly n $\aleph_2$-splitting nodes below them. For each $t \in L_n$ we have $(R_n)_t \in M$ so, by a similar argument, for each $\xi < \omega_2$ such that $t^\frown \xi \in R_n$ we can pick $Q_{t,\xi} \leqslant R_{t^\frown \xi}$ with $Q_{t,\xi} \in M$ such that for some $\delta_\xi < \gamma < \delta$ $Q_{t,\xi} \Vdash \gamma \in \dot{C}$. Then we let

$$R_{n+1} = \bigcup \{ Q_{t,\xi} : t \in L_n \text{ and } t^\frown \xi \in R_n \}.$$

Finally, let $R = \bigcap \{ R_n : n < \omega \}$. Then $R \in \mathcal{N}$ and if t is an $\aleph_2$-splitting node of R it follows that for every $\xi < \omega_2$ such that $t^\frown \xi \in R$ there is $\delta_\xi < \gamma < \delta$ such that $R_{t^\frown \xi} \Vdash \gamma \in \dot{C}$. This implies that $R \Vdash \delta \in \dot{C} \cap S$, as required. □

Now if G is $\mathcal{N}$-generic over L let in $L[G]$ $\mathcal{Q}$ be the standard poset for shooting a club through S with countable conditions. Then if C is the generic club it consists of ordinals of L-cofinality $\aleph_2$. □

Proof of Theorem 9. For any index set I let $\mathscr{C}(I)$ denote the standard poset for adding I Cohen reals. Let $\mathscr{P}$ be the poset for adding a perfect set of mutually generic Cohen reals, that is a perfect set P_g of reals such that for any 1–1 sequence $\bar{b}$ of length n of members of P_g $\bar{b}$ is $\mathscr{C}(n)$-generic over V. A condition σ belongs to $\mathscr{P}$ if there is an integer $m = m(\sigma)$ such that σ is an initial segment of $\{0,1\}^{\leqslant m}$ with the property that every $s \in \sigma$ has an extension in σ of height m. Say that $\tau \leqslant \sigma$ iff $\tau \restriction \{0,1\}^{\leqslant m(\sigma)} = \sigma$. Thus, in terms of forcing, $\mathscr{P}$ is equivalent to the standard poset for adding a single Cohen real. If g is $\mathscr{P}$-generic over V then $T_g = \bigcup g$ is a perfect tree. Let $P_g = [T_g]$ denote the set of all infinite branches of T_g as computed in the model $V[g]$.

Let now V_0 be the generic extension of L as in Lemma 3. We shall force over V_0 with the poset $\mathscr{C}(\omega_3^L) \times \mathscr{P}$. Note that this poset is equivalent to $\mathscr{C}(\omega_1)$. Suppose $G \times g$ is V_0-generic for $\mathscr{C}(\omega_3^L) \times \mathscr{P}$. Then we can identify G with an ω_3^L-sequence $\langle G(\xi) : \xi < \omega_3^L \rangle$ of Cohen reals. Let $P = P_g^{V_0[G \times g]}$ denote $[T_g]$ as computed in the model $V_0[G \times g]$. Note that since the forcing notion $\mathscr{P}$ is the same whether defined in V_0 or $V_0[G]$ we conclude that the reals in P are mutually Cohen generic over $V_0[G]$.

In V_0 fix a club C in ω_3^L consisting of ordinals of uncountable cofinality in L. Note that for any $X \in L$ which is countable in L $X \cap C$ is finite. In $V_0[G \times g]$ fix an enumeration $\{r_\alpha : \alpha < \omega_1\}$ of P and an increasing enumeration $\{\gamma_\alpha : \alpha < \omega_1\}$ of C. We now define an ω_3^L-sequence of reals G^* as follows. If $\gamma = \gamma_\alpha$ for some $\alpha < \omega_1$ then let $G^*(\gamma) = r_\alpha$, otherwise let $G^*(\gamma) = G(\gamma)$.

Claim. G^* is $\mathscr{C}(\omega_3^L)$-generic over L.

Proof. Since $\mathscr{C}(I)$ has the ccc for any I it suffices to show that for any $I \subseteq \omega_3^L$ which is countable in L $G^* \restriction I$ is $\mathscr{C}(I)$-generic over L. Fix such I. By the property of the club C it follows that $I \cap C$ is finite. Let $F \subseteq \omega_1$ be finite such that $I \cap C \subseteq \{\gamma_\alpha : \alpha \in F\}$. Now $G^* \restriction (I \backslash F) = G \restriction (I \backslash F)$ and the sequence $\langle r_\alpha : \alpha \in F \rangle$ is $\mathscr{C}(F)$-generic over $L[G]$. It follows that $G^* \restriction I$ is $\mathscr{C}(I)$-generic over L. □

Now, let $W = L[G^*]$ and $V = V_0[G \times g]$. By the definition of G^* we have that $P_g \subseteq W$. We claim that T_g does not belong to W. Otherwise there would be a countable $I \subseteq \omega_3^L$ such that $I \in L$ and $T_g \in L[G^* \restriction I]$. Since T_g is a perfect tree it would have infinitely many branches in $L[G^* \restriction I]$. Since $I \cap C$ is finite there would exist $\alpha \in \omega_1$ such that $\gamma_\alpha \notin I$ and $r_\alpha \in L[G^* \restriction I]$. This contradicts the fact that r_α is Cohen generic over $L[G^* \restriction I]$. □

5. Inner models of a choiceless universe

A possible application of the coding techniques presented in this paper would be to inner models of a universe of set theory satisfying some amount of determinacy. Suppose that an inner model of a model of $\mathrm{AD} + V = L(\mathbb{R})$ contains an uncountable prewellording of reals. Does it necessarily contain all reals? We show that some restrictions in this context are necessary.

Theorem 10 (ZF). *Assume there is a nonconstructible real. Then there is a transitive inner model M of* ZF *in which there is a prewellordering of the reals of length ω_1^V and such that not all reals belong to M.*

Proof. Suppose $\langle c_i : i < \omega \rangle$ is a sequence of mutually generic Cohen reals over L. Let S be the set of reals constructible from finitely many of the c_i's and let T be the set of Turing degrees of the c_i's. Then $L(S, T)$ is a symmetric extension of L. For an ordinal δ in $L(S, T)$ consider the partial ordering $\mathscr{Q}$ for adding a map from T to δ with finite conditions. Thus members of $\mathscr{Q}$ are finite partial functions from T to δ and the ordering is reverse inclusion. We can identify the generic filter G with a prewellordering $\leqslant_G$ of T where $\tau \leqslant_G \sigma$ iff $(\bigcup G)(\tau) \leqslant (\bigcup G)(\sigma)$.

Claim. If $\leqslant$ is *any* prewellordering of T of length δ for which the induced equivalence classes are infinite then $\leqslant$ is $\mathscr{Q}$-generic over $L(S, T)$. Moreover, $L(S, T)[\leqslant]$ and $L(S, T)$ have the same reals.

Proof. Let $\leqslant$ be any prewellordering of T satisfying the requirements of the claim and let H be the corresponding filter in $\mathcal{Q}$. Then $h=(\bigcup H):T\to\delta$ and $h^{-1}(\xi)$ is infinite, for every $\xi<\delta$. Let $D\in L(S,T)$ be a dense subset of $\mathcal{Q}$. We have to show that $D\cap H\neq\emptyset$. There is $n<\omega$ such that D is definable in $L(S,T)$ from parameters $\{c_1,\dots,c_n\}\cup\{S,T\}$. For each i let d_i be the Turing degree of c_i. Let $F=\{d_1,\dots,d_n\}$ and let $p=h\restriction F$. Then $p\in\mathcal{Q}$. Using the density of D find $q\leqslant p$ such that $q\in D$. We may assume, without loss of generality, that for some $m\geqslant n$ $\mathrm{dom}(q)=\{d_1,\dots,d_m\}$. By the property of h we can find a 1–1 function $f:m\backslash n\to\omega\backslash n$ such that for all $i\in[n,m)$ $q(d_i)=h(d_{f(i)})$. Let

$$q^*=h\restriction(F\cup\{d_{f(j)}:n\leqslant j<m\}).$$

We show that $q^*\in D$. To see this fix a recursive permutation φ of ω extending $(id\restriction n)\cup f$. φ induces a permutation of $\{c_i:i<\omega\}$ which in turn induces an automorphism φ^* of $L(S,T)$ which fixes $c_1,\dots,c_n$, and each Turing degree in T. Then $\varphi(D)=D$ and $\varphi^*(q)=q^*$. From this it follows that $q^*\in D$, as required.

To prove that $\mathcal{Q}$ does not add any reals to $L(S,T)$ let H and h be as above and suppose $\dot r$ is a $\mathcal{Q}$-name for a real. Then as before there is n such that $\dot r$ is definable from $\{c_1,\dots,c_n\}\cup\{S,T\}$. Let $F=\{d_1,\dots,d_n\}$ and $p=h\restriction F$. Let $m<\omega$ and suppose a condition $q\leqslant p$ decides the value of $\dot r(m)$. Then as in the previous argument there is a condition $q^*\in H$ such that some automorphism of $L(S,T)$ fixes $\dot r$ and maps q to q^*. Thus q^* forces the same information about $\dot r(m)$ as q. This implies that p forces that $\dot r$ is in $L(S,T)$ as desired. □

To finish the proof of Theorem 10 notice that we may assume that ω_1^L is countable since otherwise we can take $M=L$. Let P be a perfect set of mutually generic Cohen reals over L. Let S be the set of reals constructible from finitely many members of P and let T be the set of Turing degrees of the c_i's. Let $\leqslant$ be any prewellordering of T of length ω_1^V whose induced equivalence classes are infinite. Then $\leqslant$ will be generic over $L(S,T)$. To see this go to a generic extension of the universe in which the continuum of V is countable and apply the claim. Let $M=L(S,T)[\leqslant]$. Then by applying the second part of the claim M and $L(S,T)$ have the same reals and therefore not all reals are in M. Therefore, M satisfies the conclusions of the theorem. □

Conjecture 1. *Assume* AD + $V=L(\mathbb{R})$. *If M is an inner model of* ZF *containing a Souslin prewellordering of reals of length $\aleph_1^V$ then all reals are in M.*

References

[1] K. Devlin, R. Jensen, Marginalia to a theorem of Silver, ISILC Logic Conference, Lecture Notes in Mathematics, vol. 499, Springer, Berlin, 1975, pp. 115–142.
[2] H. Friedman, One hundred and two problems in mathematical logic, J. Symbolic Logic 40 (2) (1975) 113–129.
[3] M. Gitik, Nonsplitting subsets of $\mathcal{P}_\kappa(\kappa^+)$, J. Symbolic Logic 50 (1985) 881–894.
[4] M.J. Groszek, T.A. Slaman, A basis theorem for perfect sets, preprint.

[5] T. Jech, Set theory, Academic Press, London, New York, San Francisco, 1988.
[6] A. Kechris, Classical Descriptive Set Theory, Springer, Berlin, 1995.
[7] D. Martin, R. Solovay, Internal Cohen extension, Ann. Math. Logic 2 (1970) 143–178.
[8] D. Martin, R. Solovay, A basis theorem for Σ^1_3 sets, Ann. Math. 89 (2) (1969) 138–160.
[9] S. Shelah, W.H. Woodin, Forcing the failure of *CH* by adding a real, J. Symbolic Logic 49 (1984) 1185–1189.
[10] O. Spinas, A Ramsey theorem for Polish planes, manuscript.
[11] S. Todorcevic, Oscillations of real numbers, in: F.R. Drake and J.K. Truss (Eds.), Logic Colloquium '86, Studies in Logic and the Foundations of Mathematics, vol. 124, North-Holland, Amsterdam, 1988, pp. 325–331.
[12] B. Velickovic, Forcing axioms and stationary sets, Advances in Mathematics, 94 (2) (1992) 256–284.
[13] K. Kunen, Set Theory, North-Holland, Amsterdam, 1980.
[14] K.R. Milliken, Ramsey's theorem with sums or unions, J. Combinatorial Theory A 18 (1975) 276–290.