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INFINITE HAMILTONIAN PATHS IN CAYLEY DIGRAPHS

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Let Cay($S:H$) be the Cayley digraph of the generators S in the group H. A one-way infinite Hamiltonian path in the digraph G is a listing of all the vertices $[v_i: 1 \leqslant i < \infty]$, such that there is an arc from v_i to v_{i+1}. A two-way infinite Hamiltonian path is similarly defined, with i ranging from $-\infty$ to ∞. In this paper, we give conditions on S and H for the existence of one- and two-way infinite Hamiltonian paths in Cay($S:H$). Two of our results can be summarized as follows. First, if S is countably infinite and H is abelian, then Cay($S:H$) has one- and two-way Hamiltonian paths if and only if it is strongly connected (except for one infinite family). We also give necessary and sufficient conditions on S for Cay($S:H$) to be strongly connected for a large class of Cayley digraphs. Second, we show that any Cayley digraph of a countable locally finite group has both one- and two-way infinite Hamiltonian paths. As a lemma, we give a relation between the strong connectivity and the outer valence of finite vertex-transitive digraphs.

1. Introduction

Let S generate the group H. We define the Cayley digraph of the generating set S in the group H, denoted Cay($S:H$), to be the directed graph with vertex set H and arc set $\{(v, vs): v \in H, s \in S\}$. When we write Cay($S:H$) we will assume implicitly that S generates H. We will sometimes use Cay(S) to mean Cay($S:\langle S\rangle$). A one-way infinite Hamiltonian path in an infinite directed graph, G, is a sequencing of the vertices of G, $[v_i: 1 \leqslant i < \infty]$ such that for each i there is an arc joining v_i to v_{i+1}. Similarly, we define a two-way infinite Hamiltonian path to be a sequencing $[v_i: -\infty < i < \infty]$ with the same property. For simplicity we will frequently write one-(two-)way path instead of one-(two-)way infinite Hamiltonian path.

Witte [3] has shown that every Cayley digraph of a countably infinite abelian torsion group has both one- and two-way paths. In this paper we will investigate the existence of infinite paths in other Cayley Digraphs.

Witte and Gallian [4], Alspach [1], Witte [3], and Witte, Letzter and Gallian [5] survey recent work on finite and infinite Hamiltonian paths and circuits in Cayley graphs and digraphs.

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2. Notation and preliminaries

An *arc* is a directed edge. The *inner* (*outer*) *valence* of a vertex is the number of arcs entering (leaving) the vertex. If G is a digraph, $V(G)$ is the vertex set of G, hence, if $G = \mathrm{Cay}(S:H)$, then $V(G) = H$. If C is a subset of $V(G)$ then by $G - C$ we mean the digraph whose vertex set is $V(G) - C$ and whose arc set consists of those arcs of G both of whose endpoints lie in $V(G) - C$.

A path in G can be denoted either by specifying the vertices, $[v_i: 1 \leqslant i \leqslant n]$, which we will do using square brackets, or by specifying the arcs, $(a_i: 1 \leqslant i \leqslant n-1)$ which we will do using parentheses. In Cayley digraphs we will frequently write $(s_i: 1 \leqslant i \leqslant n-1)$ where for each i, s_i is the generator corresponding to the arc a_i. It is then necessary to specify the initial point, which we will usually do by saying that $(s_i: 1 \leqslant i \leqslant n-1)$ is a path from v_1 to v_n. When speaking of a one-arc path (s) we will frequently leave out the parentheses and write s. If $(p_i: 1 \leqslant i \leqslant n)$ is a set of paths then by $(p_i: 1 \leqslant i \leqslant n)$ we will mean the path whose generating sequence is the concatenation of the sequence of generators corresponding to the paths p_i. Note that the terminal vertex of p_i need not be the initial vertex of p_{i+1}. A *Hamiltonian path* in G is a path which includes each vertex of G exactly once.

A digraph G is *vertex transitive* if for every pair of vertices a and b of G, there is an automorphism of G which takes a to b. Intuitively, a vertex transitive digraph is one that 'looks the same' from every vertex. In $\mathrm{Cay}(S:H)$ the map $x \mapsto ba^{-1}x$ is a digraph automorphism taking a to b; hence every Cayley digraph is vertex transitive. In a vertex transitive digraph, G, the outer (inner) valences of the vertices of G are independent of the choice of vertex, so we may speak of the outer (inner) valence of G. If G is also finite then the inner and outer valences are equal.

If G is a digraph and for every pair of vertices a and b of G there is a path from a to b or a path from b to a, then G is said to be *unilaterally connected*. If for every pair of vertices a and b there is a path from a to b, then G is *strongly connected*. Every vertex, a, of a digraph, G, is contained in a unique, maximal, strongly connected subdigraph of G called the *strongly connected component of a*.

Since every Cayley digraph $\mathrm{Cay}(S:H)$ is vertex transitive, $\mathrm{Cay}(S:H)$ is strongly connected if and only if for each element a of H, there is a path from e to a. (Here e is the identity of H.) Let $(s_1, s_2, \ldots, s_n)$ be such a path. Then the product $s_1 s_2 \cdots s_n = a$. Hence a Cayley digraph $\mathrm{Cay}(S:H)$ is strongly connected if and only if every element of H is a product of elements of S. However, by definition of generating set, every element of H is a product of elements of S and their inverses. Hence it is sufficient for $\mathrm{Cay}(S:H)$ to be strongly connected that for each s in S, s^{-1} is a product of elements of S. Conversely, if $\mathrm{Cay}(S:H)$ is strongly connected, then for each s in S there is a path from e to s^{-1}. We have proven the following.

Theorem. *$\mathrm{Cay}(S:H)$ is strongly connected if and only if, for each s in S, there is a sequence, $(s_i: 1 \leqslant i \leqslant n)$, of elements of S with $s^{-1} = s_1 s_2 \cdots s_n$.*

Most of the Cayley digraphs considered in this paper will be in abelian groups and for these we will write the group operation additively. We will use $\mathbb{Z}$ to denote the integers, $\mathbb{Q}$ the rationals, and $\mathbb{R}$ the real numbers.

A *torsion group* is a group each of whose elements has finite order. The *torsion subgroup* of an abelian group is the set of all elements of finite order. A *torsion-free* group is a group no nonzero element of which has finite order. It is known that every torsion-free abelian group is a subgroup of a vector space over $\mathbb{Q}$. The *rank* of an abelian group is the maximum number of elements that are linearly independent over $\mathbb{Z}$. Every rank n torsion-free abelian group is isomorphic to a subgroup of $\mathbb{Q}^n$.

3. Some necessary conditions

It follows immediately from the definition that in order for a directed graph to have either a one- or two-way infinite Hamiltonian path it must be countable. We can impose further necessary conditions on the graph by considering various forms of connectedness.

A graph G is *k-divisible* if there is a finite set F of vertices such that $G-F$ has at least k infinite connected components. A graph is *k-indivisible* if it is not k-divisible. Every graph with a one-way path is 2-indivisible and every graph with a two-way path is 3-indivisible. Notice that $\text{Cay}(S:Z)$ is 2-divisible for any finite set S, so it cannot have a one-way path and also that $\text{Cay}(\{a, b\}:H)$, where H is the free group on a and b, is k-divisible for every k and so has neither one- nor two-way paths.

Another property of directed graphs which is closely related to the existence of infinite paths is strong connectivity.

Theorem 3.1. *If* $\text{Cay}(S:H)$ *has a one-way infinite Hamiltonian path then it is strongly connected. If* $\text{Cay}(S:H)$ *has a two-way infinite Hamiltonian path then it is unilaterally connected.*

Proof. Choose a and b in H. Suppose $\text{Cay}(S:H)$ has a one-way path. By the vertex transitivity of Cayley digraphs, we lose no generality in assuming that the path starts at a. But then since every vertex, including b, is included somewhere in the path, some initial segment of the one-way path is a path from a to b.

Now suppose instead that $\text{Cay}(S:H)$ has a two-way path. Then both a and b appear in it so some segment of the two-way path joints the earlier one to the later one. Hence the digraph is unilaterally connected. □

In general, it is not true that a Cayley digraph with a two-way path is strongly connected; however, Witte [private communication] has classified all Cayley digraphs with two-way paths which are not strongly connected. We present his result next.

Theorem 3.2 [Witte]. *Suppose* Cay$(S:H)$ *is infinite and not strongly connected. Then* Cay$(S:H)$ *has a two-way infinite Hamiltonian path if and only if there is a subset T of S and an element s in S such that $\langle T\rangle$ is finite and normal in H, $T\cup\{s\}$ generates H, and* Cay(T) *has a Hamiltonian path.*

Proof. First suppose Cay$(S:H)$ has a two-way path. Define $\leqslant$ on H by $g\leqslant h$ if there is a path from g to h. Since Cay$(S:H)$ is unilaterally connected, $g\leqslant h$ or $h\leqslant g$. Let $g\sim h$ if $g\leqslant h$ and $h\leqslant g$. Then $\sim$ is an equivalence relation, and the equivalence classes of $\sim$ (denoted by square brackets) are the strongly connected components of Cay$(S:H)$.

Since paths can be translated by an element of the group, $\leqslant$ is left H-invariant; i.e., $a\leqslant b$ if and only if $xa\leqslant xb$. Thus $a\sim b$ if and only if $e\sim a^{-1}b$. Hence $I=[e]$ is a subgroup and the strongly connected component of a in Cay$(S:H)$ is aI. Let $T=I\cap S$. Suppose $a\in I$, say $a=s_1s_2\cdots s_k$, $a^{-1}=s_{k+1}\cdots s_n$, $s_i\in S$. Then s_1 is an arc from e to s_1 and $(s_2,\ldots,s_n)$ is a path from s_1 to e. Hence $s_1\in I$. Inductively, we can show $s_i\in I$ for each i. Hence $I=\langle T\rangle$.

Let $P=[v_i: -\infty<i<\infty]$ be a two-way path in Cay$(S:G)$. Then $v_i\leqslant v_j$ for $i<j$ and $[h]$ must be a sequence of consecutive vertices of P for each h in H. Since Cay$(S:H)$ is vertex transitive, its strongly connected components are isomorphic, as digraphs. Hence there are three possibilities. Either all components are finite, or there is one infinite component, or there are two infinite components, $C=\{v_i: i<r\}$ and $C'=\{v_i: i\geqslant r\}$ for some r. The second case contradicts our assumption that Cay$(S:H)$ is not strongly connected. In the third case, every vertex a in C has the property that $a\leqslant b$ for all b in H while the vertices in C' do not have this property. This contradicts the vertex transitivity of Cay$(S:H)$. Hence I is finite.

Let N_i, $-\infty<i<\infty$, be the distinct connected components of Cay$(S:H)$, where $N_0=I$ and N_{i+1} is the successor of N_i in P. We have $[b]$ is the successor of $[a]$ if and only if $[a]\neq[b]$ and $a\leqslant c\leqslant b$ if and only if $[a]=[c]$ or $[b]=[c]$. This condition is left H-invariant, so $[b]$ is the successor of $[a]$ if and only if $[xb]$ is the successor of $[xa]$. Let v_i be the last vertex of P in N_0 and $s=v_i^{-1}v_{i+1}$. Since $[v_{i+1}]$ is the successor of $[v_i]$, $[xs]$ is the successor of $[x]$ for all x in H. Therefore $N_i=[s^i]=s^iI$, and $H=\langle s, I\rangle$. For g in I, $[gs]$ is the successor of $[g]=N_0$, so $gs\in sI$. Therefore I is normal in $\langle I\cup\{s\}\rangle=H$. Of course the portion of P in I is a Hamiltonian path in Cay(T).

Conversely, suppose $\langle T\rangle$ is finite and $T\cup\{s_0\}$ generates H, and $(s_1,s_2,\ldots,s_n)$ is a Hamiltonian path in Cay(T). Let $v_0=0$, and define v_i, $-\infty<i<\infty$, such that $v_{i+1}=v_i+s_k$ where $k=i \pmod{n+1}$. Then $[v_i: -\infty<i<\infty]$ is a two-way infinite Hamiltonian path in Cay$(S:H)$. □

4. A sufficient condition

Let G be an infinite digraph. Suppose that for any finite set F of vertices of G, $G-F$ is strongly connected. Then we say that G is *infinitely strongly connected.*

Theorem 4.1. *Let G be a countable, infinitely strongly connected digraph. Then G has both one- and two-way infinite Hamiltonian paths.*

The result is implicit in the proof of Witte [3, Theorem 7.1] but we present it here for completeness.

Proof. Let $G=\{g_1, g_2, \ldots\}$. We construct a one-way path inductively. Let $n(0)=0$, and $x_0=g_1$. Given the sequence $[x_i: 0 \leqslant i \leqslant n(k)]$ we let g be the element of G with least index which does not occur in $[x_i: 0 \leqslant i \leqslant n(k)]$. Let $[x_i: n(k) \leqslant i \leqslant n(k+1)]$ be a path from $x_{n(k)}$ to g in $G-\{x_i: 0 \leqslant i \leqslant n(k)-1\}$. Then $[x_i: 0 \leqslant i \leqslant \infty]$ is a one-way infinite Hamiltonian path in G.

We can construct a two-way path similarly by adding to alternate ends of the sequence. □

The following theorem is a partial converse to Theorem 4.1.

Theorem 4.2. *If G is a digraph with a one-way infinite Hamiltonian path, and if each vertex of G has infinite inner and outer valence, then G is infinitely strongly connected.*

Proof. Let $[g_0, g_1, \ldots]$ be a one-way infinite Hamiltonian path in G. Let F be a finite subset of $V(G)$ and let x and y be two elements of $V(G)-F$. Let n be the largest index of an element of $\{x, y\} \cup F$. Since x has infinite outer valence, there is an integer $k>n$ such that (x, g_k) is an arc of G. Since y has infinite inner valence, there is an integer $t>k$ such that (g_t, y) is an arc of G. Then $[x, g_k, g_{k+1}, \ldots, g_t, y]$ is a path from x to y in $G-F$. Since x and y were arbitrary, $G-F$ is strongly connected and, hence, G is infinitely strongly connected. □

5. Digraphs of infinitely generated groups

In this section, we give necessary and sufficient conditions for Cayley digraphs of many infinitely generated groups to have Hamiltonian paths. (By infinitely generated groups we mean groups which have no finite generating set.)

Theorem 5.1. *Suppose H is infinitely generated and for every finite subset M of H there is a finite subset S' of S such that:*

(1) $M \subset \langle S' \rangle$;

(2) $\mathrm{Cay}(S')$ *is strongly connected*;

(3) $\langle S' \rangle \neq N_H(\langle S' \rangle)$.

Then $\mathrm{Cay}(S:H)$ *is infinitely strongly connected.*

Proof. Given F a finite subset of H and x and y not in F, we wish to find a path from x to y in $H \setminus F$. Take M in the conditions of the theorem to be $F \cup \{x, y\}$.

Let $H' = \langle S' \rangle$ and $N = N_H(H')$. Pick $v \in N \backslash H'$. Take a path in Cay$(S:H)$ of minimum length from H' to v. (We know one exists since the conditions of the theorem imply that Cay(S) is strongly connected.) Follow this with a path of minimum length from v to H'. This gives us a path through v with only its endpoints in H'. We can translate this by an element of H' so that the initial point of the path is 1, giving us a path $[1, v_1, v_2, \ldots, v_m]$ in Cay$(S:H)$ which intersects H' only at its endpoints and which passes through an element of $N \backslash H'$, say v_n.

Let $h = v_n^{-1} x^{-1} y v_m^{-1} v_n$. By the normality of H' in N, $h \in H'$. Since Cay$(S':H')$ is strongly connected it has a path $[1, h_1, \ldots, h_a = h]$. The path

$$[x, xv_1, xv_2, \ldots, xv_n, xv_n h_1, xv_n h_2, \ldots, xv_n h, xv_n h v_n^{-1} v_{n+1},$$
$$xv_n h v_n^{-1} v_{n+2}, \ldots, xv_n h v_n^{-1} v_m = y]$$

does not intersect H' except at endpoints because $[v_1, \ldots, v_{m-1}]$ does not intersect H', and because x, h, $v_n h v_n^{-1}$, and h_i are in H' for all n and i. This gives us the desired path. □

Lemma 5.2. *Assume* Cay$(S:H)$ *is strongly connected. Then for each finite subset R of H, there is a finite subset S' of S such that $R \subset \langle S' \rangle$ and* Cay(S') *is strongly connected.*

Proof. Let R be a finite subset of H. We know there is a finite subset of S which generates all the elements of R, say $\{t_1, t_2, \ldots, t_n\} = T$. Since Cay$(S:H)$ is strongly connected, for each i we can express t_i^{-1} as a product of elements of S, say

$$t_i^{-1} = s_{i,1} s_{i,2} \cdots s_{i,r(i)}.$$

Let $S' = T \cup \{s_{i,j} : 1 \leq i \leq n,\ 1 \leq j < r(i)\}$. Since $T \subset S'$, we have $R \subset \langle S' \rangle$ so all we need to show is that Cay(S') is strongly connected, or equivalently, that for each $s \in S'$, s^{-1} is a product of elements of S'. This is clear if $s \in T$. Suppose then that $s = s_{i,j}$. Then

$$s^{-1} = s_{i,j}^{-1} = s_{i,j+1} \cdots s_{i,r(i)} t_i s_{i,1} \cdots s_{i,j-1}.$$

This completes the proof of the lemma. □

Theorem 5.3. *Suppose H is infinitely generated and there is a finitely generated normal subgroup K of H such that H/K is nilpotent. Then the following are equivalent*:

(i) Cay$(S:H)$ *is strongly connected*;
(ii) Cay$(S:H)$ *has a one-way infinite Hamiltonian path*;
(iii) Cay$(S:H)$ *has a two-way infinite Hamiltonian path.*

Proof. By Theorems 3.1 and 3.2, (ii) and (iii) each imply (i). By Theorem 4.1, it suffices to show that strong connectivity implies infinite strong connectivity. For this we use Theorem 5.1. Assume Cay$(S:H)$ is strongly connected. Given a finite subset M of H we must construct S' to satisfy properties (1), (2) and (3). By Lemma 5.2 there is a finite subset S' of S such that $M \cup K \subset \langle S' \rangle$ and $\text{Cay}(S')$ is strongly connected. Also since H/K is locally nilpotent and $K \subset \langle S' \rangle$, $\langle S' \rangle \neq N_H(\langle S' \rangle)$. This completes the proof. □

Remark. Many groups satisfy the conditions of Theorem 5.3, for example any infinitely generated group H such that either H' (the commutator subgroup of H) is finitely generated or H is nilpotent.

6. Abelian groups

In this section we will give necessary and sufficient conditions on an infinite generating set of an abelian group for the corresponding Cayley digraph to have a one-way path. We will also give conditions for the existence of two-way paths which are always sufficient and usually necessary.

Lemma 6.1. *Suppose F is a finite subgroup of the abelian group H and* $\text{Cay}(S:H)$ *is strongly connected. Then* $\text{Cay}(S:H)$ *is infinitely strongly connected if and only if* $\text{Cay}(S/F:H/F)$ *is infinitely strongly connected.*

Proof. Suppose first that $\text{Cay}(S)$ is infinitely strongly connected. Let R/F be a finite subset of H/F and pick x/F and y/F not in R/F. Since there is a path from x to y in $\text{Cay}(S)$ which avoids $R+F$, the image of this path in $\text{Cay}(S/F)$ is the desired path.

Now suppose $\text{Cay}(S/F)$ is infinitely strongly connected. This implies that S is infinite. For each ordered pair (a, b) of elements in F choose a path in $\text{Cay}(S)$ from a to b. Let P be the union of these paths. Choose any finite subset R of H and x and y disjoint from R. Let Q be the union of all cosets of F which contain the difference of an element of R and an element of P.

Since Q is finite we can find s and t in S such that $x+s$ and $y-t$ are not in Q. Let $(s_1/F, \ldots, s_n/F)$, $s_i \in S$, be a path in $\text{Cay}(S/F)$ from $(x+s)/F$ to $(y-t)/F$ which does not intersect Q/F. Then $(s, s_1, \ldots, s_n)$ is a path in $\text{Cay}(S)$ from x to $y-t-f$ for some f in F. Let $(s_{n+1}, \ldots, s_m)$ be the path in P from 0 to f. Then $(s, s_1, \ldots, s_m, t)$ is a path from x to y which does not intersect R. □

Lemma 6.2. *Suppose* $\text{Cay}(S:\mathbb{Z}^k)$ *is strongly connected, S is infinite, and $k>1$. Then* $\text{Cay}(S:\mathbb{Z}^k)$ *is infinitely strongly connected.*

Proof. Let R be a finite subset of $\mathbb{Z}^k$ and x and y be disjoint from R. Let $S' = \{e_i : -k \leq i \leq k, i \neq 0\}$ where e_i denotes the element $(0, \ldots, 0, 1, 0, \ldots, 0)$ of $\mathbb{Z}^k$, the 1 occurring in the ith position, and $e_{-i} = -e_i$. Let p_i be a path from 0 to e_i in $\text{Cay}(S)$ and let N be the maximum distance from any point in these paths to the origin. Let M be the maximum distance of any point in R from the origin. Choose

s and t in S such that $x+s$ and $y-t$ are at least a distance of $N+M$ from the origin.

Since $k>1$ there is a path from $x+s$ to $y-t$ in Cay($S':\mathbb{Z}^k$) which never goes within $N+M$ of the origin, say $(e_{i(n)}: 1 \leq n \leq m)$. Then $(p_{i(n)}: 1 \leq n \leq m)$ is a path in Cay(S) from $x+s$ to $y-t$ which does not go within M of the origin. The desired path is $(s, p_{i(1)}, \ldots, p_{i(m)}, t)$. □

Theorem 6.3. *If* $\langle S \rangle = H$ *is a countable abelian group with no cyclic subgroup of finite index and S is infinite, then the following three statements are equivalent*:

(i) Cay($S:H$) *is strongly connected*;
(ii) Cay($S:H$) *has a one-way infinite Hamiltonian path*;
(iii) Cay($S:H$) *has a two-way infinite Hamiltonian path.*

Proof. If H is infinitely generated this is a special case of Theorem 5.3, so we may assume H is finitely generated. As in the proof of Theorem 5.3, we need only show that strong connectivity implies infinite strong connectivity. Let T be the torsion subgroup of H. By Lemma 6.1 it suffices to show that Cay($S/T:H/T$) is infinitely strongly connected. But H/T is isomorphic to $\mathbb{Z}^k$ for some $k>1$, so by Lemma 6.2 we are done. □

Theorem 6.4. Cay($S:\mathbb{Z}$) *is infinitely strongly connected if and only if S is unbounded above and below.*

Remark. Combined with Lemma 6.1, Theorem 6.4 gives us a test for infinite strong connectivity of Cayley digraphs in any abelian group with a cyclic subgroup of finite index. From Theorems 4.1 and 4.2 we know that this test is necessary and sufficient for the existence of one-way paths though only sufficient for the existence of two-way paths in Cay(S) with S infinite.

Proof. First suppose S is bounded above. Let n be the maximum element of S. Then there is clearly no path from 0 to $n+1$ in Cay(S) which does not intersect $\{1, 2, \ldots, n\}$, so Cay(S) is not infinitely strongly connected. The proof is similar for S bounded below.

Now suppose that S is unbounded above and below. By the test given in the next section, Cay(S) is strongly connected. Let p be a path from 0 to 1. Since we can change the order of generators to have all negative generators follow all positive generators, we may assume no vertex in the path p is negative.

Pick R finite and x and y disjoint from R. Let r be the maximum element of R. By the unboundedness conditions on S, there exist s and t in S such that $r<x+s<y-t$. The path $(s, p, p, \ldots, p, t)$ where p is repeated $y-t-x-s$ times is the desired path. □

Remark. Let $S=\{-1, 1, 2, 3, \ldots\}$. Then Cay($S:\mathbb{Z}$) is countable, two indivisible, and strongly connected, but Theorem 4.2 shows that it has no one-way infinite Hamiltonian path. This counterexample shows that the conjunction of the necessary conditions given in Section 3 for the existence of a one-way path, is not sufficient.

We conjecture that if H is a countable abelian group and Cay($S:H$) is strongly connected, then Cay($S:H$) has a two-way infinite Hamiltonian path, and furthermore, if H has rank greater than 1, Cay($S:H$) has a one-way infinite Hamiltonian path.

7. A test for strong connectivity

In this section we provide a test for strong connectivity for a large class of Cayley digraphs.

Lemma 7.1. *Suppose T is a torsion subgroup of the center of H. Then* Cay($S:H$) *is strongly connected if and only if* Cay($S/T:H/T$) *is strongly connected.*

Proof. If $s^{-1}=s_1s_2\cdots s_n$ then $(s/T)^{-1}=(s_1/T)\cdots(s_n/T)$ so Cay(S/T) is strongly connected if Cay(S) is.

Now suppose Cay(S/T) is strongly connected. Choose $s\in S$. We must express s^{-1} as a product of elements in S. We know there exist $s_1,\ldots,s_n\in S$ such that $(s/T)^{-1}=(s_1/T)\cdots(s_n/T)$. Hence there is a t in T such that $s^{-1}=s_1s_2\cdots s_nt$. Since T is torsion, t has finite order, say m. Then $s^{-1}=s^{-m}s^{m-1}=(s_1\cdots s_n)^ms^{m-1}$, and we are done. □

Corollary 7.2. *Suppose H is abelian and T is the torsion subgroup of H. Then* Cay($S:H$) *is strongly connected if and only if* Cay($S/T:H/T$) *is strongly connected.*

We next give a test for strong connectivity of Cay($S:H$) for H a torsion free abelian group of finite rank. Combined with Corollary 7.2, this gives a test for strong connectivity of any finite rank abelian group.

Theorem 7.3. *Suppose $H\subset\mathbb{Q}^n$ has rank n. Then* Cay($S:H$) *is strongly connected if and only if the projection of S to every one-dimensional subspace of $\mathbb{R}^n$ has both positive and negative elements.*

Proof. First suppose there is a projection P to a one-dimensional subspace such that $P(s)\geqslant 0$ for all s in S. $P(s)$ can not be 0 for all s since H has rank n, so there is some s with $P(s)>0$. Then $P(-s)<0$, and clearly $-s$ cannot be a sum of elements of S. The proof is similar for $P(s)\leqslant 0$.

Now suppose the projection of S to every one-dimensional subspace of $\mathbb{R}^n$ has both positive and negative elements, or equivalently, that S contains points on each side of every $(n-1)$-dimensional hyperplane containing 0. Let $U=\{r_1s_1+\cdots+r_ks_k: s_i \in S, r_i$ positive real, and $k \geq 0\}$. Clearly U is convex, and since $\operatorname{rank}(H)=n$, U is contained in no $(n-1)$-dimensional hyperplane. It follows that U is the union of a convex open set V and a subset of the boundary of V. If 0 were not in V, there would be an $(n-1)$-dimensional hyperplane containing 0 with V entirely on one side of it, hence with U containing no point on the other side, a contradiction.

Pick $s \in S$. Since $0 \in V$, there is a positive rational multiple of $-s$ in V. Hence there are positive reals $r_1, \ldots, r_k$ and $s_1, \ldots, s_k \in S$ with $-s=r_1s_1+\cdots+r_ks_k$. Since $S \subset \mathbb{Q}^n$, there are also positive rationals $r_1', \ldots, r_k'$ which satisfy the same equation. Multiplying through by the least common denominator, m, we get positive integers $a_1, \ldots, a_k$ with $-ms=a_1s_1+\cdots+a_ks_k$.

Finally, $-s=(m-1)s+a_1s_1+\cdots+a_ks_k$. □

8. Locally finite groups

In this section we prove that every Cayley digraph of a countable locally finite group has one- and two-way paths. (A locally finite group is one in which every finite subset generates a finite subgroup.) We will first need a result relating the strong connectivity of a finite vertex transitive digraph to its outer valence. By *strong connectivity* we mean the minimum number of vertices which can be removed from a digraph such that the remaining digraph is either not strongly connected or is trivial.

Theorem 8.1. *Let G be a finite, connected, vertex transitive digraph with outer valence r and strong connectivity k. Then $k>(1/2)r$ and $1/2$ is the best possible constant in this inequality.*

Proof. Watkins [2] has a similar result for undirected graphs; we will adapt his argument to digraphs. Let G, k, and r be as in the hypotheses of the theorem. Let $n=|V(G)|$. We say a subset C of $V(G)$ is a cut set if $G-C$ is not strongly connected or is trivial, and let $C(G)$ denote the collection of minimum cut sets, i.e. those of cardinality k. If $G-C$ is trivial for some $C \in C(G)$, then $k=n-1$ and the theorem is certainly true so we may assume that $G-C$ has at least two vertices. If A and B are two subsets of $V(G)$ we say that A *points to* B if there are vertices v_1 of A and v_2 of B such that (v_1, v_2) is an arc of G; in this case we will also say that B *is pointed to by* A. If A points exclusively to itself and to B, i.e., A does not point to $V(G)-A-B$, then we say that A is a *pointer* to B. Similarly, if $V(G)-A-B$ does not point to B, then we say that B is an *antipointer* of A. Clearly $V(G)-A$ is a pointer to A, and we will consider this case as trivial.

Lemma 8.2. *Let C be a subset of $V(G)$. Then C is a non-trivial cut set of G if and only if C has a non-trivial pointer.*

Proof. Suppose C has a non-trivial pointer, A. Let $v \in A$ and $w \in V(G)-C-A$. Then there are no arcs from A to $V(G)-C-A$ so there can be no path from v to w in $G-C$. Thus C is a cut set.

Conversely, suppose C is a non-trivial cut set. Then $G-C$ is a union of at least two disjoint strong components with vertex sets $A_1, A_2, \ldots, A_r$. For any s, either A_s is a pointer to C or there is some s' different from s such that A_s points to $A_{s'}$. Suppose there are no pointers to C. Then there is an infinite sequence $A_{i(1)}, A_{i(2)}, \ldots$ such that each strong component points to the next. Since G is a finite digraph, there must be repetitions. However if each set in the sequence $A_{j(1)}, A_{j(2)}, \ldots, A_{j(1)}$ points to the next then the subdigraph on the union of $A_{j(1)}, A_{j(2)}, \ldots$ is strongly connected, contradicting the fact that each A_i is a strong component. Hence at least one of the components must be a pointer to C. $\square$

In an analagous manner we can show that C is a cut set if and only if C has a non-trivial antipointer.

Let $p(G)=\min\{|V(A)|: A \text{ is a pointer to } C \text{ and } C \in C(G)\}$. A pointer, A, is called an *atomic pointer* if $|A|=p(G)$. We analogously define $q(G)$ to be the smallest size of antipointers of minimum cut sets, and an *atomic antipointer* to be an antipointer of this size.

If we reverse the directions of all of the arcs of G, we get a new vertex transitive digraph, G', with the same r and k but $q(G)=p(G')$ and $p(G)=q(G')$ since pointers become antipointers and vice versa. We therefore lose no generality in assuming that $q(G) \geqslant p(G)$. With this assumption we can prove the following lemma.

Lemma 8.3. *Distinct atomic pointers of G are disjoint.*

Proof. Assume the contrary. Then there are two distinct atomic pointers A_1 and A_2 with non-empty intersection. Let the minimum cut sets to which these point be C_1 and C_2, respectively. Let $R_1=V(G)-(A_1 \cup C_1)$ and $R_2=V(G)-(A_2 \cup C_2)$. Let $D_1=(C_1 \cap R_2) \cup (C_1 \cap C_2) \cup (C_2 \cap R_1)$ and $D_2=(C_1 \cap A_2) \cup (C_1 \cap C_2) \cup (C_2 \cap A_1)$. By the definition of pointer, A_1 does not point to $V(G)-A_1-C_1$ and A_2 does not point to $V(G)-A_2-C_2$ so $A_1 \cap A_2$ does not point to $(V(G)-A_1-C_1) \cup (V(G)-A_2-C_2)=V(G)-A_1 \cap A_2-D_2$. Hence $A_1 \cap A_2$ is a pointer to D_2. Clearly $A_1 \cup A_2$ is a pointer to D_1. Also we have

$$|D_1|+|D_2|=|D_1 \cup D_2|+|D_1 \cap D_2|=|C_1 \cup C_2|+|C_1 \cap C_2|=|C_1|+|C_2|=2k.$$

Since $A_1 \cap A_2$ is a non-trivial pointer to D_2, D_2 is a cut set and since $|A_1 \cap A_2|<|A_1|=p(G)$, D_2 is not a minimum cut set; hence $|D_2|>k$. Therefore,

$|D_1|<k$. This makes it is impossible that D_1 is a cut set, so $A_1 \cup A_2$ is a trivial pointer, i.e., $A_1 \cup A_2 \cup D_1 = V(G)$. But A_1 does not point to R_1 so R_1 is an antipointer of C_1.

Thus $|R_1| \geq q(G) \geq p(G) = |A_2|$, and hence

$$2p(G)+k \leq |A_1|+|R_1|+|C_1| = |V(G)| = |A_1 \cup A_2 \cup D_1| \leq |A_1|+|A_2|+|D_1| < 2p(G)+k.$$

This contradiction completes the proof of the lemma. □

Since there is some minimum cut set with an atomic pointer, by vertex transitivity, every vertex must be contained in at least one such cut set. It follows that there are at least $V(G)/k$ minimum cut sets with atomic pointers. Two distinct minimum cut sets cannot share a single pointer since their intersection would also have that pointer and would therefore be a smaller cut set. There are thus at least $V(G)/k$ distinct atomic pointers. These are disjoint so we have $V(G) \geq p(G)(V(G)/k)$ or $p(G) \leq k$. But each vertex of a pointer can point only to other vertices of the pointer and to those of the cut set, so $r \leq p(G)-1+k<2k$.

To show that 1/2 is the best possible constant we exhibit a sequence of digraphs for which k/r approaches 1/2.

Let A be the complete graph on n vertices. Let G be formed by taking three copies of A and adding arcs from each vertex of the first copy to each vertex of the second copy, each of the second to each of the third, and each of the third to each of the first. Then it is easily seen that G has strong connectivity n and outer valence $2n-1$ so $k/r = n/(2n-1)$ which approaches 1/2 as n grows without bound. □

Remark. Theorem 8.1 is considerably stronger than what we will need to prove Theorem 8.4. For our purposes it would have sufficed to prove that k goes to infinity as r does. However we consider the above result to be elegant enough in its own right to present it in full.

Theorem 8.4. *If H is a countable locally finite group, and S is any generating set for H, then* $\mathrm{Cay}(S:H)$ *has one- and two-way infinite Hamiltonian paths.*

Proof. By Theorem 4.1 it suffices to prove that $\mathrm{Cay}(S:H)$ is infinitely strongly connected. Let R be a finite subset of H and x and y elements not in R. We wish to show that there is a path from x to y in $H-R$. Let S' be a finite subset of S with at least $2|R|$ elements such that $R \cup \{x, y\} \subset \langle S' \rangle$. Then $\mathrm{Cay}(S')$ is a finite connected vertex transitive digraph with outer valence at least $2|R|$ so it has strong connectivity at least $|R|+1$. Thus $\mathrm{Cay}(S')-R$ is strongly connected and contains a path from x to y. Since $\mathrm{Cay}(S')$ is a subdigraph of $\mathrm{Cay}(S:H)$, this path also exists in $\mathrm{Cay}(S:H)$. □

9. Groups which always have paths

In Section 8, we showed that every Cayley digraph on a countable locally finite group has both one- and two-way infinite Hamiltonian paths. The following example due to Witte shows that locally finite groups are not the only ones with this property.

Let H be the semidirect product of $\mathbb{Q}$ and $\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathbb{Q}$ by inversion. Then it is not difficult to see that $\text{Cay}(S:H)$ is infinitely strongly connected for any generating set S but it is not locally finite (not even torsion).

In this section we prove that among abelian groups, only torsion groups always have paths. In order to do so we prove the following necessary condition on a group H for every Cayley digraph on H to have a one-way path.

Theorem 9.1. *If there is a non-trivial homomorphism $f: H \mapsto \mathbb{R}$, then there is a generating set S of H for which* $\text{Cay}(S:H)$ *has no one-way infinite Hamiltonian path.*

Proof. Let $S = \{h \in H: f(h) \geqslant 0\}$. Since for each $h \in H$, either h or its inverse is in S, S generates H. Clearly $\text{Cay}(S:H)$ is not strongly connected, hence $\text{Cay}(S:H)$ has no one-way path. □

We can use this to prove the theorem:

Theorem 9.2. *If H is an abelian group and for every generating set S of H,* $\text{Cay}(S:H)$ *has a one-way infinite Hamiltonian path, then H is torsion.*

Proof. Suppose the torsion subgroup, T, of H is a proper subgroup. Then the canonical homomorphism $f: H \mapsto H/T$ is non-trivial. H/T is torsion-free and, hence, is a subgroup of a vector space V over $\mathbb{Q}$. Choose a basis, $\{b_i: i \in I\}$ for V. For each $j \in I$ there is a homomorphism, g_j, of V into $\mathbb{Q}$ defined by $g_j(b_i) = 1$ if $i = j$ and 0 otherwise. For each j, $g_j \circ f$ is a homomorphism of H into $\mathbb{Q}$ and since f is non-trivial, there is some j for which $g_j \circ f$ is non-trivial. By Theorem 9.1, there is a generating set S of H such that $\text{Cay}(S:H)$ has no one-way path. □

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