Journal of Computational and Applied Mathematics 224 (2009) 688–695

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Common fixed points of new iterations for two asymptotically nonexpansive nonself-mappings in a Banach space

Sornsak Thianwan *

School of Science and Technology, Naresuan University at Phayao, Phayao, 56000, Thailand

ARTICLE INFO

Article history:
Received 7 January 2008
Received in revised form 29 May 2008

MSC:
47H10
47H09
46B20

Keywords:
Asymptotically nonexpansive nonself-mapping
Weak and strong convergence
Completely continuous
Opial's condition
Common fixed points

ABSTRACT

In this paper, we introduce a new two-step iterative scheme for two asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space. Weak and strong convergence theorems are established for the new two-step iterative scheme in a uniformly convex Banach space.

1. Introduction

Let C be a nonempty closed convex subset of real normed linear space X. A self-mapping $T : C \to C$ is said to be nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$. A self-mapping $T : C \to C$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n(x) - T^n(y)\| \leq k_n\|x - y\| \tag{1.1}$$

for all $x, y \in C$ and $n \geq 1$. A mapping $T : C \to C$ is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n(x) - T^n(y)\| \leq L\|x - y\| \tag{1.2}$$

for all $x, y \in C$ and $n \geq 1$.

It is easy to see that if T is an asymptotically nonexpansive, then it is uniformly L-Lipschitzian with the uniform Lipschitz constant $L = \sup\{k_n : n \geq 1\}$.

Fixed-point iteration process for nonexpansive self-mappings including Mann and Ishikawa iteration processes have been studied extensively by various authors [1,4,6,12,13,16]. For nonexpansive nonself-mappings, some authors [5,9,18,20,23] have studied the strong and weak convergence theorems in Hilbert space or uniformly convex Banach space. In 1972, Goebel and Kirk [3] introduced the class of asymptotically nonexpansive self-mappings, who proved that if C is a nonempty

* Tel.: +66 89 5688318; fax: +66 54 466664.
E-mail address: sornsakt@nu.ac.th.

0377-0427/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2008.05.051

closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping on C, then T has a fixed point.

In 1991, Schu [17] introduced a modified Mann iteration process to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert space. More precisely, he proved the following theorem.

Theorem 1.1 ([17]). *Let H be a Hilbert space, C a nonempty closed convex and bounded subset of H. Let $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ for all $n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the condition $0 < a \leq \alpha_n \leq b < 1$, $n \geq 1$, for some constant a, b. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in C$ using*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1, \tag{1.3}$$

converges strongly to some fixed point of T.

Since then, Schu's iteration process has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert space or Banach spaces [12,14,15,17,21].

In 2000, Noor [10] introduced a three-step iterative sequence and studied the approximate solutions of variational inclusion in Hilbert spaces. In 2005, Suantai [19] defined a new three-step iteration, which is an extension of Noor iterations, and gave some weak and strong convergence theorems of such iterations for asymptotically nonexpansive mappings in uniformly convex Banach spaces.

The concept of asymptotically nonexpansive nonself-mappings was introduced in [2] in 2003 as the generalization of asymptotically nonexpansive self-mappings. The asymptotically nonexpansive nonself-mapping is defined as follows:

Definition 1.1 ([2]). Let C be a nonempty subset of a real normed linear space X. Let $P : X \to C$ be a nonexpansive retraction of X onto C. A nonself-mapping $T : C \to X$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\| \tag{1.4}$$

for all $x, y \in C$ and $n \geq 1$. T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\| \tag{1.5}$$

for all $x, y \in C$ and $n \geq 1$.

By studying the following iteration process:

$$x_1 \in C, \quad x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \tag{1.6}$$

Chidume, Ofoedu and Zegeye [2] got the following strong and weak convergence theorems for asymptotically nonexpansive nonself-mapping.

Theorem 1.2 ([2]). *Let X be a real uniformly convex Banach space and C a nonempty closed convex subset of X. Let $T : C \to X$ be a completely continuous and asymptotically nonexpansive map with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\} \subset (0, 1)$ be such that $\epsilon \leq 1 - \alpha_n \leq 1 - \epsilon$, $\forall n \geq 1$ and some $\epsilon > 0$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.6). Then $\{x_n\}$ converges strongly to some fixed point of T.*

Theorem 1.3 ([2]). *Let X be a real uniformly convex Banach space which has a Fréchet differentiable norm and C a nonempty closed convex subset of X. Let $T : C \to X$ be an asymptotically nonexpansive map with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\} \subset (0, 1)$ be such that $\epsilon \leq 1 - \alpha_n \leq 1 - \epsilon$, $\forall n \geq 1$ and some $\epsilon > 0$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.6). Then $\{x_n\}$ converges weakly to some fixed point of T.*

If T is a self-mapping, then P becomes the identity mapping so that (1.4) and (1.5) reduce to (1.1) and (1.2), respectively. (1.6) reduces to (1.3).

Recently, Wang [22] generalized the iteration process (1.6) as follows: $x_1 \in C$,

$$\begin{aligned} y_n &= P((1 - \beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), \\ x_{n+1} &= P((1 - \alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n), \quad n \geq 1, \end{aligned} \tag{1.7}$$

where $T_1, T_2 : C \to X$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0, 1)$. He studied the strong and weak convergence of the iterative scheme (1.7) under proper conditions. Meanwhile, the results of [22] generalized the results of [2].

Inspired and motivated by these facts, we introduce and study a new class of iterative schemes in this paper. The scheme is defined as follows.

Let X be a normed space, C a nonempty convex subset of X, $P : X \to C$ a nonexpansive retraction of X onto C and $T_1, T_2 : C \to X$ given mappings. Then for an arbitrary $x_1 \in C$, the following iteration scheme is studied:

$$y_n = P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n),$$
$$x_{n+1} = P((1-\alpha_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n), \quad n \geq 1, \tag{1.8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate real sequences in $[0, 1)$.

The iterative scheme (1.8) is called the projection type Ishikawa iteration for two asymptotically nonexpansive nonself-mappings. If $T_1 = T_2$ and $\beta_n = 0$ for all $n \geq 1$, then (1.8) reduces to (1.6).

The purpose of this paper is to construct an iteration scheme for approximating common fixed points of two asymptotically nonexpansive nonself-mappings and to prove some strong and weak convergence theorems for such mappings in a uniformly convex Banach space.

Now, we recall some well known concepts and results.

Let X be a Banach space with dimension $X \geq 2$. The modulus of X is the function $\delta_X : (0, 2] \to [0, 1]$ defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \|x\| = 1, \|y\| = 1, \epsilon = \|x-y\|\right\}.$$

Banach space X is uniformly convex if and only if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

A subset C of X is said to be a retract if there exists a continuous mapping $P : X \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : X \to X$ is said to be a retraction if $P^2 = P$. It follows that if a mapping P is a retraction, then $Pz = z$ for every $z \in R(P)$, the range of P.

Recall that a Banach space X is said to satisfy *Opial's condition* [11] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ implying that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

A mapping $T : C \to X$ is said to be semi-compact if, for any sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x^* \in C$. Two mappings $S, T : C \to C$, where C is a subset of a normed space X, are said to satisfy condition A' [7] if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that either

$$\|x - Sx\| \geq f(d(x, F)) \quad \text{or} \quad \|x - Tx\| \geq f(d(x, F))$$

for all $x \in C$, where $d(x, F) = \inf\{\|x - q\| : q \in F = F(S) \cap F(T)\}$.

Note that condition A' reduces to condition (A) [21] when $S = T$. Maiti and Ghosh [8] and Tan and Xu [21] have approximated fixed points of a nonexpansive mapping T by Ishikawa iterates under the condition (A).

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1.4 ([21]). *Let $\{a_n\}$ and $\{t_n\}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + t_n \quad \text{for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 1.5 ([17]). *Let X be a real uniformly convex Banach space and $0 \leq p \leq t_n \leq q < 1$ for all positive integer $n \geq 1$. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$; then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 1.6 ([2]). *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X, and let $T : C \to X$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed at zero, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed points of T.*

Lemma 1.7 ([19]). *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

2. Main results

In this section, we prove theorems of strong and weak convergence of the iterative scheme given in (1.8) to a common fixed point for two asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space. In order to prove our main results, the following lemmas are needed.

Lemma 2.1. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $k_n \to 1$, $l_n \to 1$ as $n \to \infty$, respectively and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1)$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ using (1.8). If $q \in F(T_1) \cap F(T_2)$, then $\lim_{n\to\infty} \|x_n - q\|$ exists.*

Proof. Let $q \in F(T_1) \cap F(T_2)$. Setting $k_n = 1 + u_n$, $l_n = 1 + v_n$. Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, so $\sum_{n=1}^{\infty} u_n < \infty$, $\sum_{n=1}^{\infty} v_n < \infty$. Using (1.8), we have

$$\begin{aligned}
\|y_n - q\| &= \|P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n) - P(q)\| \\
&\leq \|(1-\beta_n)(x_n - q) + \beta_n(T_2(PT_2)^{n-1}x_n - q)\| \\
&\leq (1-\beta_n)\|x_n - q\| + \beta_n\|T_2(PT_2)^{n-1}x_n - q\| \\
&\leq (1-\beta_n)\|x_n - q\| + \beta_n(1+v_n)\|x_n - q\| \\
&= (1-\beta_n)\|x_n - q\| + (\beta_n + \beta_n v_n)\|x_n - q\| \\
&\leq (1+v_n)\|x_n - q\|,
\end{aligned}$$

and so

$$\begin{aligned}
\|x_{n+1} - q\| &= \|P((1-\alpha_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n) - P(q)\| \\
&\leq \|(1-\alpha_n)(y_n - q) + \alpha_n(T_1(PT_1)^{n-1}y_n - q)\| \\
&\leq (1-\alpha_n)\|y_n - q\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - q\| \\
&\leq (1-\alpha_n)\|y_n - q\| + \alpha_n(1+u_n)\|y_n - q\| \\
&\leq (1+u_n)\|y_n - q\| \\
&\leq (1+u_n)(1+v_n)\|x_n - q\| \\
&= (1 + v_n + u_n + u_n v_n)\|x_n - q\| \\
&< e^{\sum_{n=1}^{\infty}(v_n + u_n + u_n v_n)}\|x_1 - q\|.
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(v_n + u_n + u_n v_n) < \infty$, then $\{x_n\}$ is bounded. It implies that there exists a constant $M > 0$ such that $\|x_n - q\| \leq M$ for all $n \geq 1$. So,

$$\|x_{n+1} - q\| \leq \|x_n - q\| + (v_n + u_n + u_n v_n)M.$$

It follows from Lemma 1.4 that $\lim_{n\to\infty} \|x_n - q\|$ exists. This completes the proof. □

Lemma 2.2. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $k_n \to 1$, $l_n \to 1$ as $n \to \infty$, respectively and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.8). Then $\lim_{n\to\infty} \|x_n - T_1 x_n\| = \lim_{n\to\infty} \|x_n - T_2 x_n\| = 0$.*

Proof. Let $q \in F(T_1) \cap F(T_2)$. Set $k_n = 1 + u_n$, $l_n = 1 + v_n$. By Lemma 2.1, we see that $\lim_{n\to\infty} \|x_n - q\|$ exists. Assume that $\lim_{n\to\infty} \|x_n - q\| = c$. Using (1.8), we have

$$\|y_n - q\| \leq (1 + v_n)\|x_n - q\|. \tag{2.1}$$

Taking the lim sup on both sides in the inequality (2.1), we have

$$\limsup_{n\to\infty} \|y_n - q\| \leq c. \tag{2.2}$$

In addition, $\|T_1(PT_1)^{n-1}y_n - q\| \leq k_n\|y_n - q\|$, taking the lim sup on both sides in this inequality, we have

$$\limsup_{n\to\infty} \|T_1(PT_1)^{n-1}y_n - q\| \leq c. \tag{2.3}$$

From (1.8), we have

$$\begin{aligned}
\|x_{n+1} - q\| &\leq \|(1-\alpha_n)(y_n - q) + \alpha_n(T_1(PT_1)^{n-1}y_n - q)\| \\
&\leq (1 + v_n + u_n + u_n v_n)\|x_n - q\|.
\end{aligned} \tag{2.4}$$

Since $\sum_{n=1}^{\infty}(v_n + u_n + u_n v_n) < \infty$ and $\lim_{n\to\infty} \|x_{n+1} - q\| = c$, letting $n \to \infty$ in the inequality (2.4), we have

$$\lim_{n\to\infty} \|(1-\alpha_n)(y_n - q) + \alpha_n(T_1(PT_1)^{n-1}y_n - q)\| = c. \tag{2.5}$$

By using (2.2), (2.3) and (2.5) and Lemma 1.5, we have

$$\lim_{n\to\infty} \|T_1(PT_1)^{n-1}y_n - y_n\| = 0. \tag{2.6}$$

In addition, $\|T_2(PT_2)^{n-1}x_n - q\| \le l_n\|x_n - q\|$, and taking the lim sup on both sides in this inequality, we have

$$\limsup_{n\to\infty} \|T_2(PT_2)^{n-1}x_n - q\| \le c. \tag{2.7}$$

Using (1.8), we have

$$\begin{aligned}
\|x_{n+1} - q\| &\le (1-\alpha_n)\|y_n - q\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - q\| \\
&= (1-\alpha_n)\|y_n - q\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - y_n + y_n - q\| \\
&\le (1-\alpha_n)\|y_n - q\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - y_n\| + \alpha_n\|y_n - q\| \\
&\le \|y_n - q\| + \|T_1(PT_1)^{n-1}y_n - y_n\|.
\end{aligned} \tag{2.8}$$

Taking the lim inf on both sides in the inequality (2.8), by (2.6) and $\lim_{n\to\infty}\|x_{n+1} - q\| = c$, we have

$$\liminf_{n\to\infty} \|y_n - q\| \ge c. \tag{2.9}$$

It follows from (2.2) and (2.9) that $\lim_{n\to\infty}\|y_n - q\| = c$. This implies that

$$c = \lim_{n\to\infty}\|y_n - q\| \le \lim_{n\to\infty}\|(1-\beta_n)(x_n - q) + \beta_n(T_2(PT_2)^{n-1}x_n - q)\| \le \lim_{n\to\infty}\|x_n - q\| = c,$$

and so

$$\lim_{n\to\infty}\|(1-\beta_n)(x_n - q) + \beta_n(T_2(PT_2)^{n-1}x_n - q)\| = c.$$

Using (2.7) and Lemma 1.5, we obtain

$$\lim_{n\to\infty}\|T_2(PT_2)^{n-1}x_n - x_n\| = 0. \tag{2.10}$$

From $y_n = P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n)$ and (2.10), we have

$$\begin{aligned}
\|y_n - x_n\| &= \|P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n) - x_n\| \\
&\le \|(1-\beta_n)(x_n - x_n) + \beta_n(T_2(PT_2)^{n-1}x_n - x_n)\| \\
&\le (1-\beta_n)\|x_n - x_n\| + \beta_n\|T_2(PT_2)^{n-1}x_n - x_n\| \\
&\le \|T_2(PT_2)^{n-1}x_n - x_n\| \\
&\to 0 \quad (\text{as } n\to\infty).
\end{aligned} \tag{2.11}$$

In addition,

$$\begin{aligned}
\|T_1(PT_1)^{n-1}x_n - x_n\| &= \|T_1(PT_1)^{n-1}x_n - y_n + y_n - x_n\| \\
&\le \|T_1(PT_1)^{n-1}x_n - y_n\| + \|y_n - x_n\| \\
&= \|T_1(PT_1)^{n-1}x_n - T_1(PT_1)^{n-1}y_n + T_1(PT_1)^{n-1}y_n - y_n\| + \|y_n - x_n\| \\
&\le \|T_1(PT_1)^{n-1}x_n - T_1(PT_1)^{n-1}y_n\| + \|T_1(PT_1)^{n-1}y_n - y_n\| + \|y_n - x_n\| \\
&\le k_n\|x_n - y_n\| + \|T_1(PT_1)^{n-1}y_n - y_n\| + \|y_n - x_n\|.
\end{aligned}$$

Thus, it follows from (2.6) and (2.11) that

$$\lim_{n\to\infty}\|T_1(PT_1)^{n-1}x_n - x_n\| = 0. \tag{2.12}$$

By using (1.8), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le (1-\alpha_n)\|y_n - x_n\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - x_n\| \\
&\le (1-\alpha_n)\|y_n - x_n\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - y_n + y_n - x_n\| \\
&\le (1-\alpha_n)\|y_n - x_n\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - y_n\| + \alpha_n\|y_n - x_n\| \\
&\le \|y_n - x_n\| + \|T_1(PT_1)^{n-1}y_n - y_n\|.
\end{aligned}$$

It follows from (2.6) and (2.11) that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.13}$$

Using (2.12) and (2.13), we have

$$\begin{aligned}\|x_{n+1}-T_1(PT_1)^{n-1}x_{n+1}\| &= \|x_{n+1}-x_n+x_n-T_1(PT_1)^{n-1}x_n+T_1(PT_1)^{n-1}x_n-T_1(PT_1)^{n-1}x_{n+1}\|\\ &\le \|x_{n+1}-x_n\|+\|T_1(PT_1)^{n-1}x_{n+1}-T_1(PT_1)^{n-1}x_n\|+\|T_1(PT_1)^{n-1}x_n-x_n\|\\ &\le \|x_{n+1}-x_n\|+k_n\|x_{n+1}-x_n\|+\|T_1(PT_1)^{n-1}x_n-x_n\|,\\ &\to 0 \quad (\text{as } n\to\infty).\end{aligned} \tag{2.14}$$

In addition,

$$\begin{aligned}\|x_{n+1}-T_1(PT_1)^{n-2}x_{n+1}\| &= \|x_{n+1}-x_n+x_n-T_1(PT_1)^{n-2}x_n+T_1(PT_1)^{n-2}x_n-T_1(PT_1)^{n-2}x_{n+1}\|\\ &\le \|x_{n+1}-x_n\|+\|T_1(PT_1)^{n-2}x_n-x_n\|+\|T_1(PT_1)^{n-2}x_{n+1}-T_1(PT_1)^{n-2}x_n\|\\ &\le \|x_{n+1}-x_n\|+\|T_1(PT_1)^{n-2}x_n-x_n\|+L\|x_{n+1}-x_n\|,\end{aligned}$$

where $L=\sup\{k_n : n\ge 1\}$. It follows from (2.13) and (2.14) that

$$\lim_{n\to\infty}\|x_{n+1}-T_1(PT_1)^{n-2}x_{n+1}\|=0. \tag{2.15}$$

We denote as $(PT_1)^{1-1}$ the identity maps from C onto itself. Thus by the inequality (2.14) and (2.15), we have

$$\begin{aligned}\|x_{n+1}-T_1x_{n+1}\| &= \|x_{n+1}-T_1(PT_1)^{n-1}x_{n+1}+T_1(PT_1)^{n-1}x_{n+1}-T_1x_{n+1}\|\\ &\le \|x_{n+1}-T_1(PT_1)^{n-1}x_{n+1}\|+\|T_1(PT_1)^{n-1}x_{n+1}-T_1x_{n+1}\|\\ &= \|x_{n+1}-T_1(PT_1)^{n-1}x_{n+1}\|+\|T_1(PT_1)^{1-1}(PT_1)^{n-1}x_{n+1}-T_1(PT_1)^{1-1}x_{n+1}\|\\ &\le \|x_{n+1}-T_1(PT_1)^{n-1}x_{n+1}\|+L\|(PT_1)^{n-1}x_{n+1}-x_{n+1}\|\\ &= \|x_{n+1}-T_1(PT_1)^{n-1}x_{n+1}\|+L\|(PT_1)(PT_1)^{n-2}x_{n+1}-P(x_{n+1})\|\\ &\le \|x_{n+1}-T_1(PT_1)^{n-1}x_{n+1}\|+L\|T_1(PT_1)^{n-2}x_{n+1}-x_{n+1}\|\\ &\to 0 \quad (\text{as } n\to\infty),\end{aligned}$$

which implies that $\lim_{n\to\infty}\|x_n-T_1x_n\|=0$. Similarly, we may show that $\lim_{n\to\infty}\|x_n-T_2x_n\|=0$. The proof is completed. □

Theorem 2.3. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C\to X$ be two asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}$, $\{l_n\}\subset[1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1)<\infty$, $\sum_{n=1}^{\infty}(l_n-1)<\infty$, $k_n\to 1$, $l_n\to 1$ as $n\to\infty$, respectively and $F(T_1)\cap F(T_2)\ne\emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon\in(0,1)$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by (1.8). If one of T_1 and T_2 is completely continuous, then $\{x_n\}$ and $\{y_n\}$ converge strongly to a common fixed point of T_1 and T_2.*

Proof. By Lemma 2.1, $\{x_n\}$ is bounded. In addition, by Lemma 2.2, $\lim_{n\to\infty}\|x_n-T_1x_n\|=0$ and $\lim_{n\to\infty}\|x_n-T_2x_n\|=0$, and then $\{T_1x_n\}$ and $\{T_2x_n\}$ are also bounded. If T_1 is completely continuous, there exists a subsequence $\{T_1x_{n_j}\}$ of $\{T_1x_n\}$ such that $T_1x_{n_j}\to q$ as $j\to\infty$. It follows from Lemma 2.2, that $\lim_{j\to\infty}\|x_{n_j}-T_1x_{n_j}\|=\lim_{j\to\infty}\|x_{n_j}-T_2x_{n_j}\|=0$. So by the continuity of T_1 and Lemma 1.6, we have $\lim_{j\to\infty}\|x_{n_j}-q\|=0$ and $q\in F(T_1)\cap F(T_2)$. Furthermore, by Lemma 2.1, we get that $\lim_{n\to\infty}\|x_n-q\|$ exists. Thus $\lim_{n\to\infty}\|x_n-q\|=0$. From (2.11), we have $\lim_{n\to\infty}\|y_n-x_n\|=0$, and it follows that $\lim_{n\to\infty}\|y_n-q\|=0$. The proof is completed. □

Theorem 2.4. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C\to X$ be two asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}$, $\{l_n\}\subset[1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1)<\infty$, $\sum_{n=1}^{\infty}(l_n-1)<\infty$, $k_n\to 1$, $l_n\to 1$ as $n\to\infty$, respectively, and $F(T_1)\cap F(T_2)\ne\emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon\in(0,1)$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by (1.8). If one of T_1 and T_2 is semi-compact, then $\{x_n\}$ and $\{y_n\}$ converge strongly to a common fixed point of T_1 and T_2.*

Proof. Since one of T_1 and T_2 is semi-compact, $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n-T_1x_n\|=\lim_{n\to\infty}\|x_n-T_2x_n\|=0$, and then there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges strongly to q. It follows from Lemma 1.6 that $q\in F(T_1)\cap F(T_2)$. Thus $\lim_{n\to\infty}\|x_n-q\|$ exists by Lemma 2.1. Since the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to q, then $\{x_n\}$ converges strongly to the common fixed point $q\in F(T_1)\cap F(T_2)$. From (2.11), we have

$$\lim_{n\to\infty}\|y_n-x_n\|=0,$$

and it follows that $\lim_{n\to\infty}\|y_n-q\|=0$. The proof is completed. □

In the next result, we prove the strong convergence of the scheme (1.8) under condition A' which is weaker than the compactness of the domain of the mappings.

Theorem 2.5. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two asymptotically nonexpansive nonself-mappings of C satisfying condition A' with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $k_n \to 1$, $l_n \to 1$ as $n \to \infty$, respectively, and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative scheme (1.8) converge strongly to a common fixed point of T_1 and T_2.*

Proof. By Lemma 2.2, we have $\lim_{n\to\infty} \|x_n - T_1x_n\| = \lim_{n\to\infty} \|x_n - T_2x_n\| = 0$. It follows from condition A' that

$$\lim_{n\to\infty} f(d(x_n, F)) \le \lim_{n\to\infty} \|x_n - T_1x_n\| = 0 \quad \text{or} \quad \lim_{n\to\infty} f(d(x_n, F)) \le \lim_{n\to\infty} \|x_n - T_2x_n\| = 0.$$

In the both case, $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$, we obtain that $\lim_{n\to\infty} d(x_n, F) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty}(v_n + u_n + u_nv_n) < \infty$, given $\epsilon > 0$, there exists a natural number n_0 such that $d(x_n, F) < \frac{\epsilon}{4}$ and $\sum_{k=n_0}^{\infty}(v_k + u_k + u_kv_k)M < \frac{\epsilon}{2}$ for all $n \ge n_0$. So, we can find $y^* \in F$ such that $\|x_{n_0} - y^*\| < \frac{\epsilon}{4}$. For $n \ge n_0$ and $m \ge 1$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\le \|x_{n+m} - y^*\| + \|x_n - y^*\| \\
&\le \|x_{n_0} - y^*\| + \|x_{n_0} - y^*\| + \sum_{k=n_0}^{n+m-1}(v_k + u_k + u_kv_k)M \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since X is complete. Let $\lim_{n\to\infty} x_n = u$. Now $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(u, F) = 0$. F is closed; therefore $u \in F$. From (2.11), we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0,$$

and it follows that $\lim_{n\to\infty} \|y_n - u\| = 0$. This completes the proof. □

Finally, we prove the weak convergence of the iterative scheme (1.8) for two asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 2.6. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $k_n \to 1$, $l_n \to 1$ as $n \to \infty$, respectively, and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by (1.8). Then $\{x_n\}$ and $\{y_n\}$ converge weakly to a common fixed point of T_1 and T_2.*

Proof. It follows from Lemma 2.2 that $\lim_{n\to\infty} \|x_n - T_1x_n\| = \lim_{n\to\infty} \|x_n - T_2x_n\| = 0$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 1.6, we have $u \in F(T_1) \cap F(T_2)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 1.6, $u, v \in F(T_1) \cap F(T_2)$. By Lemma 2.1, $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 1.7 that $u = v$. Therefore $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2. Moreover, $\lim_{n\to\infty} \|y_n - x_n\| = 0$ as proved in Lemma 2.2 and $x_n \to u$ weakly as $n \to \infty$, and therefore $y_n \to u$ weakly as $n \to \infty$. This completes the proof of the theorem. □

Acknowledgments

The author would like to thank the Thailand Research Fund, The Commission on Higher Education (MRG5180036), and Naresuan University at Phayao, Phayao, Thailand, for financial support during the preparation of this paper.

References

[1] S.S. Chang, Y.J. Cho, H. Zhou, Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings, J. Korean Math. Soc. 38 (2001) 1245–1260.
[2] C.E. Chidume, E.U. Ofoedu, H. Zegeye, Strong and weak convergence theorems for asymptotically nonexpansive mappings, J. Math. Anal. Appl. 280 (2003) 364–374.
[3] K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 35 (1972) 171–174.
[4] S. Ishikawa, Fixed points and iteration of nonexpansive mappings of in a Banach spaces, Proc. Amer. Math. Soc. 73 (1976) 61–71.
[5] J.S. Jung, S.S. Kim, Strong convergence theorems for nonexpansive nonself-mappings in Banach spaces, Nonlinear Anal. 33 (1998) 321–329.
[6] S.H. Khan, H. Fukhar-ud-din, Weak and strong convergence of a scheme with errors for two nonexpansive mappings, Nonlinear Anal. 61 (2005) 1295–1301.
[7] H. Fukhar-ud-din, S.H. Khan, Convergence of iterates with errors of asymptotically quasi-nonexpansive mappings and applications, J. Math. Anal. Appl. 328 (2007) 821–829.
[8] M. Maiti, M.K. Ghosh, Approximating fixed points by Ishikawa iterates, Bull. Austral. Math. Soc. 40 (1989) 113–117.

[9] S.Y. Matsushita, D. Kuroiwa, Strong convergence of averaging iteration of nonexpansive nonself-mappings, J. Math. Anal. Appl. 294 (2004) 206–214.
[10] M. Aslam Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251 (2000) 217–229.
[11] Z. Opial, Weak convergence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591–597.
[12] M.O. Osilike, A. Udomene, Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings, Math. Comput. Modelling 32 (2000) 1181–1191.
[13] M.O. Osilike, A. Udomene, Demiclosedness principle and convergence theorems for strictly pseudocontractive mappings of Browder–Petryshyn type, J. Math. Anal. Appl. 256 (2001) 431–445.
[14] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 67 (1979) 274–276.
[15] B.E. Rhoades, Fixed point iterations for certain nonlinear mappings, J. Math. Anal. Appl. 183 (1994) 118–120.
[16] J. Schu, Iterative construction of a fixed points of asymptotically nonexpansive mappings, J. Math. Anal. Appl. 158 (1991) 407–413.
[17] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Austral. Math. Soc. 43 (1991) 153–159.
[18] N. Shahzad, Approximating fixed points of non-self nonexpansive mappings in Banach spaces, Nonlinear Anal. 61 (2005) 1031–1039.
[19] S. Suantai, Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings, J. Math. Anal. Appl. 311 (2005) 506–517.
[20] W. Takahashi, G.E. Kim, Strong convergence of approximants to fixed points of nonexpansive nonself-mappings, Nonlinear Anal. 32 (1998) 447–454.
[21] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mapping by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993) 301–308.
[22] L. Wang, Strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings, J. Math. Anal. Appl. 323 (2006) 550–557.
[23] H.K. Xu, X.M. Yin, Strong convergence theorems for nonexpansive nonself-mappings, Nonlinear Anal. 242 (1995) 23–228.