Inequalities and Bounds for Quasi-symmetric 3-Designs

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Quasi-symmetric 3-designs with block intersection numbers \( x \) and \( y \) \((0 < x < y < k)\) are studied, several inequalities satisfied by the parameters of a quasi-symmetric 3-designs are obtained. Let \( D \) be a quasi-symmetric 3-design with the block size \( k \) and intersection numbers \( x, y; y > x \geq 1 \) and suppose \( D' \) denote the complement of \( D \) with the block size \( k' \) and intersection numbers \( x', y' \). If \( k - 1 < x + y \) then it is proved that \( x' + y' \leq k' \). Using this it is shown that the quasi-symmetric 3-designs corresponding to \( y = x + 1, x + 2 \) are either extensions of symmetric designs or designs corresponding to the Witt-design (or trivial design, i.e., \( v = k + 2 \)) or the complement of above designs.


1. INTRODUCTION

A quasi-symmetric block design \( D \) is a \( t-(v, k, \lambda) \) design in which any two blocks intersect in \( x \) or \( y \) points. If \( x = y \) then \( t = 2 \) and in that case \( b = v \), where \( b \) is the number of blocks. Such designs are called symmetric designs where any two blocks intersect in \( \lambda \) points. It is well known that a 3-design has at least two block intersection numbers. In this paper we study quasi-symmetric 3-designs with intersection numbers \( x, y \) \((0 < x < y < k)\).

Cameron [6] classified quasi-symmetric 3-designs with intersection number \( x = 0 \) (Theorem 2.1). In [12] Sane and Shrikhande made the conjecture: Let \( D \) be a quasi-symmetric 3-design. Then one of the following cases occurs:

(i) \( x = 0 \) and \( D \) is a design in Cameron's family (see Theorem 2.1);

(ii) \( x = 1 \) and \( D \) is the Witt-Lüneburg design on 23 points or its residual;

(iii) \( D \) is the complement of some design in (i) or (ii) above.

In support of the conjecture, the case \( x = 1 \) was settled by Calderbank and Morton [5] and Pawale and Sane [10]. Though the conjecture is still
far from settled, it is hoped that the results of this paper will contribute a step towards its proof. To that end we prove that if the pair \((x, y)\) satisfies \(4xy > (x + y - 1)^2\) then the number of quasi-symmetric 3-designs with block intersection numbers \(x, y\) are finite.

The main purpose of this paper is to give the following bounds for intersection numbers \(x, y\) of quasi-symmetric 3-design \(D\):

\[
\begin{align*}
(i) \quad & \frac{(k-1)^2(v-k+1)}{(v-2)(v-k)} \leq x + y - 1 \leq \frac{2(k-1)(k-2)}{(v-3)}; \\
(ii) \quad & \frac{k(k-1)^2}{(v-2)(v-k)} \leq xy \leq \frac{k(k-1)^2(k-2)}{(v-2)(v-3)}; \\
(iii) \quad & \frac{k}{v-k+1} \leq \frac{xy}{x+y-1} \leq \frac{k(k-1)}{2(v-2)}; \\
(iv) \quad & \frac{(v-1)(2k-v)+(k-1)(k-2)}{(v-2)} \leq x + y - 1; \\
(v) \quad & \frac{k(k-1)(2k-v)}{(v-2)} \leq xy; \\
(vi) \quad & \frac{k(k-1)(2k-v)}{(v-1)(2k-v)+(k-1)(k-2)} \leq \frac{xy}{x+y-1}.
\end{align*}
\]

The upper bounds are attained in (i), (ii), and (iii) if and only if \(D\) is the Witt 4-(23, 7, 1) design or its complement. Equality holds in (iv), (v), and (vi) if and only if \(D\) is the complement of a design in Cameron’s family. We add that inequality (i) was first obtained by Calderbank [4], using linear programming techniques. From inequality (iii) it is clear that \(v-2 \leq k(k-1)/2\). We characterise the cases \(v-2 = k(k-1)/2\) and \(v-1 = k(k-1)/2\) in terms of the Witt 4-(23, 7, 1) design and its residual.

Let \(D'\) denote the complement of \(D\) with block size \(k'\) and intersection numbers \(x'\) and \(y'\) using inequality (i) we show that, if \(k-1 \leq x + y\) then \(x' + y' \leq k'\). This result is used to determine quasi-symmetric 3-designs with intersection numbers \(x\) and \(y = x + 1, x + 2\).

Section 2 contains preliminary results. For basic definitions and results we refer to [2, 7].

2. PRELIMINARIES

Throughout this paper \(D\) will denote a quasi-symmetric 3-design with standard parameter set \((v, b, r, k, \lambda; x, y)\), where \(x, y\) are two block inter-
section numbers with $0 \leq x < y < k$. For $x = 0$ such designs were classified by Cameron [6] in the following theorem.

**Theorem 2.1.** $D$ is a quasi-symmetric 3-design with an intersection number 0 if and only if $D$ is an extension of a symmetric 2-design. In that case the parameters of $D$ are one of the following four types:

1. $D$ is a Hadamard 3-design;
2. $v = (\lambda + 2)(\lambda^2 + 4\lambda + 2) + 1$, $k - \lambda^2 + 3\lambda + 2$, and $\lambda = 1, 2, ...$;
3. $D$ is the extension of a projective plane of order 10;
4. $v = 496$, $k = 40$, and $\lambda = 3$.

Quasi-symmetric 3-designs for $x = 1$ were classified by Calderbank and Morton [5] and Pawale and Sane [10] in the following theorem.

**Theorem 2.2.** Let $D$ be a quasi-symmetric 3-design with the smaller intersection number $x = 1$. Then $D$ is either the unique Witt 4-(23, 7, 1) design or its residual the unique 3-(22, 7, 4) design (we consider the 3-(5, 3, 1) design to be trivial).

The following recursive relation will be used throughout this paper.

**Lemma 2.3.** In $t$-$(v, k, \lambda)$ design, let $\lambda_i$ be the number of blocks containing given $i$-tuple, $i = 0, 1, ..., t$, with $\lambda_t = \lambda$, $\lambda_0 = b$, and $\lambda_1 = r$. Then

$$\lambda_i = \frac{(v - i)}{(k - i)} \lambda_{i+1}, \quad i = 0, 1, ..., t - 1. \quad (1)$$

**Lemma 2.4** [12, Lemma 2.5]. The following relation holds for any proper quasi-symmetric design:

$$k(r - 1)(y + x - 1) - xy(b - 1) = k(k - 1)(\lambda - 1). \quad (2)$$

**Lemma 2.5** [4, 9, 12]. Parameters of $D$ satisfy the following equation:

$$xy(v - 2)^2 + [xy - k(k - 1)(x + y - 1)](v - 2) + k(k - 1)^2 (k - 2) = 0. \quad (3)$$

**Proof.** We divide the proof into two parts. If the residual $E$ of $D$ is a proper quasi-symmetric design then Eq. (2) for $E$ is

$$k(r - \lambda_2 - 1)(y + x - 1) - xy(b - r - 1) = k(k - 1)(\lambda_2 - \lambda_3 - 1). \quad (4)$$

Subtracting (4) from (2) and using relation (1) we obtain (3). Also if $x = 0$ then by Theorem 2.1, (3) holds. Let $D$ have non-zero intersection numbers
and suppose the residual of \( D \) is a symmetric design. In this case \( b - r = v - 1 \), \( r - \lambda_2 = k \), \((\lambda_2 - \lambda_3)(v - 2) = k(k - 1)\), and the derived design of \( D \) is a proper quasi-symmetric design. Equation (2) for the derived design of \( D \) is

\[
(k - 1)(\lambda_2 - 1)(y + x - 3) - (x - 1)(y - 1)(r - 1) = (k - 1)(k - 2)(\lambda_3 - 1).
\]

Now subtract (2) from (5) and use above relations to obtain (3).

**Remark 2.6.** Neumaier [9] first obtained an inequality for a quasi-symmetric 2-design in which equality holds for a quasi-symmetric 3-design resulting in Eq. (3). Calderbank [4] also obtained the same inequality using Hahn polynomials. However, their proof is quite involved and as far as quasi-symmetric 3-designs are concerned above, the proof of (3) is quite elementary and short.

**Corollary 2.7.** Let \( D \) be a quasi-symmetric 3-design then \( v - 2 \) divides \( k(k - 1)^2(k - 2) \).

**Proof.** Clear from (3).

**Proposition 2.8.** Let \( D \) be a quasi-symmetric 3-design with \( x \geq 1 \). If \( 4xy - (x + y - 1)^2 > 0 \), then

\[
k < \frac{8xy}{[4xy - (x + y - 1)^2]}.
\]

**Proof.** The discriminant \( \Delta \) of quadratic (3) is given by

\[
\Delta = (xy)^2 - 2xy(x + y - 1)k(k - 1) + k(k - 1)^2 \\
\times \{ -k[4xy - (x + y - 1)^2] + 8xy \}.
\]

Since \( \Delta \geq 0 \), \(-k[4xy - (x + y - 1)^2] + 8xy > 0\), which gives the above inequality.

**Remark 2.9.** It is clear by Proposition 2.8 that for fixed integers \( x, y \), \( 1 \leq x < y \) such that \( 4xy - (x + y - 1)^2 > 0 \), \( k \) takes finitely many values. Hence by [12, Theorem 2.6], there exist finitely many quasi-symmetric 3-designs with intersection numbers \( x, y \).

**Proposition 2.10.** Let \( D \) be a quasi-symmetric 3-design with \( x \geq 1 \), then

\[
x + y - 1 = \frac{(k - 1)^2 [2(v - 1)\lambda_2 - k(v + k - 3)]}{(v - 2)[(v - 1)\lambda_2 - k(k - 1)]}; \tag{6}
\]

\[
xy = \frac{k^2(k - 1)^2 [\lambda_2 - (k - 1)]}{(v - 2)[(v - 1)\lambda_2 - k(k - 1)]}. \tag{7}
\]
**Proof.** Equation (3) can be written as

\[ k(k-1)(v-2)(x+y-1)-(v-1)(v-2)xy = k(k-1)^2 (k-2). \]  

(8)

Now consider (2) and (8) as simultaneous linear equations in unknowns \( x+y-1 \) and \( xy \). It is clear from Crammer's rule that if \( A \neq 0 \) then

\[ x+y-1 = \frac{B}{A} \quad \text{and} \quad xy = \frac{C}{A}, \]

where

\[ A = \frac{(v-2)(v-k)[k(k-1)-(v-1)\lambda_2]}{(k-1)}, \]

\[ B = (k-1)(v-k)[k(v+k-3)-2(v-1)\lambda_2]; \]

\[ C = k^2(k-1)(v-k)[(k-1)-\lambda_2]. \]

Since \( D \) is not a symmetric design, we have \( r > k \). Using (1) we obtain \((v-1)\lambda_2 > k(k-1)\). This implies that \( A < 0 \) and completes the proof.

**Corollary 2.11.** Let \( D \) be a quasi-symmetric 3-design with \( x \geq 1 \). Then \( \lambda_2 \geq k \).

**Proof.** Since \( xy > 0 \), \( \lambda_2 - (k-1) > 0 \) i.e. \( \lambda_2 \geq k \).

3. **Main Results**

**Theorem 3.1.** Let \( D \) be a quasi-symmetric 3-design with the smaller block intersection number \( x \geq 1 \). Then

\[ \frac{(k-1)^2(v-k+1)}{(v-2)(v-k)} \leq x+y-1 \leq \frac{2(k-1)(k-2)}{(v-3)}; \]  

(9)

\[ \frac{k(k-1)^2}{(v-2)(v-k)} \leq xy \leq \frac{k(k-1)^2 (k-2)}{(v-2)(v-3)}; \]  

(10)

\[ \frac{k}{v-k+1} \leq \frac{xy}{x+y-1} \leq \frac{k(k-1)}{2(v-2)}; \]  

(11)

\[ \frac{(v-1)(2k-v)+(k-1)(k-2)}{(v-2)} \leq x+y-1; \]

\[ \frac{k(k-1)(2k-v)}{(v-2)} \leq xy; \]

\[ \frac{k(k-1)(2k-v)}{(v-1)(2k-v)+(k-1)(k-2)} \leq \frac{xy}{x+y-1}. \]

Upper bounds are attained in (i), (ii), and (iii) if and only if \( D \)
is 4-design. Equality holds in (iv), (v), and (vi) if and only if \( D \) is the complement of a design in Cameron's family.

(vii) \( v - 2 \leq k(k - 1)/2 \), Equality holds if and only if \( D \) is the 4-(23, 7, 1) design or the trivial 3-(5, 3, 1) design.

(viii) \( v - 1 = k(k - 1)/2 \) if and only if \( D \) is the 3-(22, 7, 4) design.

**Proof.** (i), (ii), and (iii). By [7] the inequality \( b \leq v(v - 1)/2 \) holds for any quasi-symmetric design with equality if and only if \( D \) is a 4-design. By (1) we obtain \( \lambda_2 \leq k(k - 1)/2 \), with equality if and only if \( D \) is a 4-design. Hence by Corollary 2.11, \( k \leq \lambda_2 \leq k(k - 1)/2 \). Let

\[
f_1(\theta) = \frac{(k - 1)^2 [2(v - 1)\theta - k(v + k - 3)]}{(v - 2)[(v - 1)\theta - k(k - 1)]};
\]

\[
f_2(\theta) = \frac{k^2(k - 1)^2 [\theta - (k - 1)]}{(v - 2)[(v - 1)\theta - k(k - 1)]};
\]

\[
f_3(\theta) = \frac{k^2[\theta - (k - 1)]}{[2(v - 1)\theta - k(v + k - 3)]}.
\]

Then \( f_i \)'s for \( i = 1, 2, 3 \) are differentiable real-valued functions on the interval \([k, k(k - 1)/2]\) and the derivatives are

\[
\frac{\delta f_1(\theta)}{\delta \theta} = \frac{k(k - 1)^2 (v - 1)(v - k - 1)}{(v - 2)[(v - 1)\theta - k(k - 1)]^2};
\]

\[
\frac{\delta f_2(\theta)}{\delta \theta} = \frac{k^2(k - 1)^3 (v - k - 1)}{(v - 2)[(v - 1)\theta - k(k - 1)]^2};
\]

\[
\frac{\delta f_3(\theta)}{\delta \theta} = \frac{k^2(k - 2)(v - k - 1)}{[2(v - 1)\theta - k(v + k - 3)]^2}.
\]

It is clear that all the above derivatives are non-negative. Hence \( f_i \)'s for \( i = 1, 2, 3 \) are increasing functions of \( \theta \), therefore \( f_i(k) \leq f_i(\lambda_2) \leq f_i(k(k - 1)/2) \) for \( i = 1, 2, 3 \). Now use (6) and (7) to complete the proof.

(iv), (v), (vi). The residual of \( D \) is a 2-design with the parameters \((v, k, \lambda_2 - \lambda_3)\). By Fisher's inequality we obtain \( r - \lambda_2 \geq k \), with equality if and only if the residual of \( D \) is a symmetric design. It is easy to see that the residual of \( D \) is a symmetric design if and only if \( D \) is the complement of a quasi-symmetric 3-design which is an extension of a symmetric design. That is, by Theorem 2.1, \( D \) is the complement of a design in Cameron's family.

Now \( r - \lambda_2 \geq k \) implies \( \lambda_2 \geq k(k - 1)/(v - k) \) which implies \( f_i(k(k - 1)/(v - k)) \leq f_i(k(k - 1)/(v - k)) \) for \( i = 1, 2, 3 \) with equality if and only if \( D \) is the comple-
ment of a design in Cameron's family. Again use (6) and (7) to complete the proof.

(vii) Since $x \geq 1$, $xy/x + y - 1 \geq 1$. Hence by (11), $k(k-1)/2(v-2) \geq 1$, implies $(v-2) \leq k(k-1)/2$. If $v-2 = k(k-1)/2$, then by (11) $xy/x + y - 1 \leq 1$ implies $(x-1)(y-1) \leq 0$; hence $x = 1$. By Theorem 2.2 $D$ is the $4$-$(23, 7, 1)$ design or the trivial $3$-$(5, 3, 1)$ design.

(viii) Let $v-1 = k(k-1)/2$. In this case $xy - (x+y-1) = [2(k-1)/(v-2)] \{1 - [k(k - 3)/2(\lambda_2 - 2)]\} < 2$. Therefore, $xy - (x+y-1) = (x-1)(y-1) \leq 1$; hence $x = 1$. By Theorem 2.2, $D$ is the $3$-$(22, 7, 4)$ design.

**Corollary 3.2.** Let $D$ be a non-trivial quasi-symmetric $3$-design. Then:

(i) $x + y - 1 = 2(k-1)(k-2)/(v-3)$ if and only if $D$ is the Witt $4$-design or its complement;

(ii) $xy = k(k-1)^2(k-2)/(v-2)(v-3)$ if and only if $D$ is the Witt $4$-design or its complement.

**Proof.** By Theorem 3.1 in both the cases $D$ is a $4$-design. By [3, 8], $D$ is the Witt $4$-design or its complement.

**Remark 3.3.** Inequality (i) of Theorem 3.1, as also Corollary 3.2(i) were first obtained by Calderbank [4] using liner programming techniques.

**Theorem 3.4.** Let $D$ be a quasi-symmetric $3$-design with the block intersection numbers $x$ and $y$, $y > x \geq 1$, and $D'$ denote the complement of $D$ with block size $k'$ and intersection numbers $x'$ and $y'$. If $k-1 \leq x+y$ then $x'+y' \leq k'$.

**Proof.** It is clear that

$$x' + y' = k' + (v - 3k + x + y). \quad (12)$$

Using (9) we obtain

$$v - 3k + x + y \leq \frac{[(k-1)-(x+y)][2(k-1)-(x+y-1)]}{x+y-1}.$$

If $k-1 \leq xy$, then $v - 3k + x + y \leq 0$; using (12) we obtain $x'+y' \leq k'$.

**Remark 3.5.** While dealing with complementation problem it is always preferable to start with assumption such as $v < 2k$. In this context inequalities (iv), (v), (vi) of Theorem 3.1 and Theorem 3.4 are important. Now observe that $y' - x' = y - x$; i.e., the difference between the block intersection numbers is the same for both $D$ and $D'$. We use Theorem 3.4 to characterise $y - x = 1$ and 2.
Now we will investigate quasi-symmetric 3-design with the intersection numbers $x$ and $y = x + 1$, $x + 2$. It is enough to consider $x + y \leq k$, since designs obtained by considering $x + y > k$ are complements of designs obtained in the previous case.

**Theorem 3.6.** Let $D$ be a quasi-symmetric 3-design with intersection numbers $x$ and $y = x + 1$. Then $D$ is a trivial design (i.e., $v = k + 2$ and $b = v(v - 1)/2$).

**Proof.** Let, if possible, $D$ be a non-trivial quasi-symmetric 3-design, i.e., $v > k + 2$. If $x = 0$ then by Theorem 2.1 a non-trivial quasi-symmetric 3-design with $y = 1$ does not exist. Now consider $x \geq 1$, $y = x + 1$, and $x + y \leq k$, which may be assumed by Remark 3.5. By Proposition 2.8 we obtain $k < 2x + 2$; hence $k = 2x + 1$. In this case the discriminant $A$ of the quadratic (3) is

$$A = -16x^3 + 11x^2 + 10x + 1.$$  

It is clear that $A < 0$ for all $x \geq 2$. Therefore $x = 1$, by the Theorem 2.2 $D$ is the $3-(5, 3, 1)$ design, a contradiction. Hence $v = k + 2$; in this case using Eq. (3) we obtain $k = x + 2$. Now by (6) or (7) and (1), we obtain $b = v(v - 1)/2$, which implies $D$ is a trivial design. This complete our proof.

**Theorem 3.7.** Let $D$ be a quasi-symmetric 3-design with intersection numbers $x$ and $y = x + 2$. Then $D$ is either the 3-(22, 6, 1) or the 3-(22, 7, 4) design or the 4-(23, 7, 1) design or the complement of one of these three designs.

**Proof.** If $x = 0$ then $y = 2$ and by Theorem 2.1 $D$ is the 3-(22, 6, 1) design. Let $x \geq 1$, $y = x + 2$; then by Proposition 2.8 $k \leq 2x + 5$. Now assume by an earlier remark that $x + y \leq k$; therefore $2x + 2 \leq k \leq 2x + 5$. Compute the discriminant $A$ of quadratic (3) in the four possible cases. It is easily seen that the condition $A \geq 0$ and $A$ a perfect square forces $x = 1$. By Theorem 2.2 $D$ is the 4-(23, 7, 1) or the 3-(2, 7, 4) design.

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**References**

3. A. Bremner, A diophantine equation arising from tight 4-designs, Osaka J. Math. 16 (1979), 353-356.
12. S. S. Sane and M. S. Shrikhande, Quasi-symmetric 2, 3, 4-designs, Combinatorica 7, No. 3 (1987), 291-301.