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On the Existence of Cartan Subgroups of Finite Groups

SURINDER K. SEHGAL

Department of Mathematics, Ohio State University, Columbus, Ohio 43210

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INTRODUCTION

In 1961 Carter found the existence of a characteristic class of conjugate subgroups of a finite solvable group that are characterized as nilpotent, self-normalizing subgroups of the given group. He also pointed out certain covering and avoidance properties of these subgroups, which relate them strongly with the structure of the whole group. The question was raised by Professor Zassenhaus in the group theory conference held at the University of Michigan and Michigan State University in March 1964 whether there is a significant generalization of the Carter subgroups of finite solvable groups so as to form an equivalent of the Cartan subgroups of algebraic groups. In this paper such a generalization will be defined and explored.

It should be remarked that nilpotent subgroups of finite groups that are their own normalizer, even if they exist (as is the case for the simple group of order 168), do not have the structure theoretical significance corresponding to the Cartan subgroups of semisimple algebraic groups or the Cartan subalgebras of Lie algebras. Therefore another way of defining the Carter subgroups of finite solvable groups, lending itself better to a generalization for nonsolvable finite groups, was needed. After some experimentation it was found that Carter subgroups of finite solvable groups also can be characterized as maximal nilpotent subgroups that remain maximal nilpotent after the application of any group homomorphism [7]. However this new invariance property still needs a skillful modification in the case of nonsolvable groups; for example, it would obviously be inappropriate to describe every maximal nilpotent subgroup of a given finite simple group as a Cartan subgroup. Moreover, the subgroups of known finite simple groups that obviously do correspond to Cartan subalgebras of corresponding Lie algebras are not necessarily

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maximal nilpotent. For example, the subgroup of order 2 of A_5 corresponds to the Cartan subalgebra formed by the diagnal matrices in the Lie algebra of all 2×2 matrices of nonvanishing traces over a field of 5 elements. One cannot expect in general that a Cartan subgroup will be its own normalizer, because it is already known from the theory of semisimple Lie groups that the factor group of the normalizer of a Cartan subgroup over itself is isomorphic to the weyl group of the corresponding semi-simple Lie algebra. However there will occur in the final definition a solvable subgroup L that plays a role analogous to the Borel subgroups of Lie groups.

These results are extracted from my thesis for the Ph.D. degree at the University of Notre Dame, I take this opportunity to express my gratitude to Professor H. Zassenhaus for his help and encouragement in supervising this research.

Throughout this paper, all groups considered are finite.

1. *C*-Subgroups

Definition. Define $\Lambda(G)$ to be the intersection of all maximal normal subgroups of G. For example, $\Lambda(S_5) = A_5$ and $\Lambda(A_5) = \{1\}$, where S_5 is the group of all permutations of five letters and A_5 is the group of all even permutations of five letters.

Definition. Define $V(G)$ to be the group generated by nontrivial minimal normal subgroups of G. Thus $V(S_5) = A_5$ and $(A_5) = A_5$.

Remark 1.1. $\Lambda(G)$ is called the first term of the descending Loewy series of G, and $V(G)$ is known as the first term of the ascending Lowey series of G. (For further results, see [8].)

Theorem 1. *If N is any normal subgroup of G, then $\Lambda(G/N) = \Lambda(G)N/N$ (see* [8]).

Remark 1.2. If G is a finite group satisfying $\Lambda(G) = \{1\}$, then G is the direct product of simple groups. (For proof, see [8].)

Definition. Let G be a finite group. A subgroup U of G is called a *c-subgroup* of G if it staisfies the following conditions.

(a) U is nilpotent.

(b) There exists a solvable subgroup L of G containing U and satisfying $\langle L^{N_G(U)} \rangle . \Lambda(G) = G$ [$\langle L^{N_G(U)} \rangle$ stands for the group generated by L and all its conjugates under $N_G(U)$].

(c) U is maximal with respect to conditions (a) and (b).

(d) Let $\{1\} \subset N_1 \subset N_2 \subset \cdots \subset N_s = G$ be any fixed chief series of G; then UN_i/N_i satisfies conditions (a)–(c) in G/N_i.

Remark 1.3. If G is a simple group, then there exist c-subgroups of G.

Proof. Let $U_1 = \{1\}$. Then U_1 satisfies (a) and (b). Let U be a subgroup of G, $U \supseteq U_1$ and U maximal with respect to conditions (a) and (b). Then this U satisfies (a)–(d). Hence U is a c-subgroup of G.

Remark 1.4. If G is solvable, then c-subgroups of G are the same as Carter subgroups of G.

Proof. If G is solvable, then condition (b) is trivially satisfied with $L = G$. So conditions (a)–(d) are equivalent to

(A) U is a maximal nilpotent subgroup of G.

(B) If $\{1\} \subset N_1 \subset \cdots \subset N_s = G$ is a chief series of $G\{1\} \subset N_1 \subset \cdots \subset N_s = G$, then UN_i/N_i is a maximal nilpotent subgroup of G/N_i. By [7] these conditions are necessary as well as sufficient in order that U be a Carter subgroup of G.

Remark 1.5. (1) c-subgroups of A_5 are all subgroups of order 2 and of order 3. c-subgroups of A_6 are all subgroups of order 4 and of order 3. c-subgroups of S_5 are all cyclic groups of order 6. c-subgroups of S_6 are all groups of order 16 and all cyclic subgroups of order 6.

Remark 1.6. If G is the direct product of simple groups, then c-subgroups of G exist.

Proof. Let $\{1\} \subset N_1 \subset N_2 \subset \cdots \subset N_s = G$ be a chief series of G. We can write $G = F_1 \times F_2 \times \cdots \times F_s$ such that $N_1 = F_1$, $N_2 = F_1 \times F_2, \ldots, N_s = F_1 \times \cdots \times F_s$, where F_i are simple groups (see [8]).

Let U_i be the c-subgroups of F_i. Let L_i be solvable such that $L_i \supseteq U_i$ and $\langle L_i^{N_{F_i}(U_i)} \rangle = F_i$. Then $U = U_1 \times U_2 \times \cdots \times U_s$ is a c-subgroup of G with $L = L_1 \times L_2 \times \cdots \times L_s$ as the solvable subgroup of G. Clearly $\langle L^{N_G(U)} \rangle = G$.

Now U satisfies properties (a) and (b). If U is not maximal with respect to properties (a) and (b), let $V \supset U$. Let σ_i be the projection maps. Then we must have for one i, $V^{\sigma_i} = V_i > U_i$.

Let M be solvable, $M \supseteq V$ and satisfying $\langle M^{N_G(V)} \rangle = G$. Then $\langle M_i^{N_{F_i}(V_i)} \rangle = F_i$, where $M_i = M^{\sigma_i}$. This gives a contradiction to the choice of U_i. Similarly U satisfies property (d).

THEOREM 1.4. *Any finite group has c-subgroups.*

Proof. Let G be a group of smallest order for which the result is not true. Let $\{1\} \subset N_1 \subset N_2 \subset \cdots \subset N_s = G$ be a chief series of G.

Case (i). $N_1 \not\subset \Lambda(G)$. Now $\exists G_1$ maximal normal in G such that $N_1 \not\subset G_1$. Therefore $G = G_1 \times N_1$ and $G/N_1 = G_1$. Let $\{1\} \subset M_2 \subset \cdots \subset M_s = G_1$ be a corresponding chief series of G_1. Since G_1 is not a criminal, there exists U_2 a c-subgroup of G_1 with respect to this series of G_1. Let U_1 be a c-subgroup of N_1 (which exists by Remark 1.3) and $U = U_1 \times U_2$. Then as one can easily see, U is a c-subgroup of G with $L = L_1 \times L_2$ and with respect to the chief series $\{1\} \subset N_1 \subset \cdots \subset N_s = G$, i.e., G is not a criminal. Hence case (i) doesn't arise.

Case (ii). N_1 is solvable. $N_1 \subseteq \Lambda(G)$. Now G/N_1 is not a criminal. Therefore $\exists \bar{U}/N_1$ nilpotent contained in $\bar{L}/N_1$ and staisfying properties (a)–(d) with respect to the chief series $\{1\} \subseteq N_2/N_1 \subset \cdots \subset N_s/N_1 = G/N_1$. That is, we have $\langle \bar{L}^{N_G(\bar{U})} \rangle \cdot \Lambda(G) = G$.

Now $\bar{U}$, $\bar{L}$ are solvable. Let U be a Carter subgroup of $\bar{U}$. Therefore U is contained in a solvable subgroup $\bar{L}$ and satisfies $\langle \bar{L}^{N_G(U)} \rangle \Lambda(G) = \langle \bar{L}^{N_G(U) \cdot N} \rangle \cdot \Lambda(G) \supseteq \langle \bar{L}^{N_G(\bar{U})} \rangle \cdot \Lambda(G) = G$. Thus U satisfies conditions (a) and (b). Suppose U does not satisfy (c), that is, there exists $V \supset U$, V satisfying (a) and (b). Let L' be the solvable group satisfying $\langle L'^{N_G(V)} \rangle \Lambda(G) = G$. Therefore $\langle L'N/N^{N_{G/N}(VN)/N} \rangle \Lambda(G/N) = G/N$. Now $UN/N = \bar{U}/N \subseteq VN/N$, and VN/N satisfies conditions (a) and (b). Therefore $\bar{U}/N = VN/N$ because $\bar{U}/N$ satisfies (c). Now we have $U \subset V \subset \bar{U}$. But U is maximal nilpotent in $\bar{U}$, which gives us a contradiction and U satisfies condition (c). Now $UN_i/N_i = \bar{U}/N_1 \cdot N_i/N_1/N_i/N_1$ satisfies (a)–(c) in G/N_i. Hence U is a c-subgroup of G and G is not a criminal. Therefore we can now assume that $N_1 \subseteq \Lambda(G)$, N_1 minimal normal in G and N_1 is not solvable.

Let S_p be a sylow p-subgroup of N_1 for some prime p dividing the order of N_1. Then we have $G = N_G(S_p) \cdot N_1$. Let $G_1 = N_G(S_p)$. Then $|G_1| < |G|$. Therefore G_1 is not a criminal.

Now $G_1/G_1 \cap N_1 \cong G_1N_1/N_1 = G/N_1$ and $1 \subseteq N_2/N_1 \subseteq \cdots \subseteq N_s/N_1 = G/N_1$ is a chief series of G/N_1. Let $\{1\} \subset M_2 \subset \cdots \subset M_s = G_1/G_1 \cap N_1$ be the corresponding chief series of $G_1/G_1 \cap N_1$. Consider a chief series of G_1 through $1 \subset \cdots \subset G_1 \cap N_1 \subset M_2 \subset \cdots \subset M_s = G_1$. Let U_1 be a c-subgroup of G_1 with respect to the chief sreies of G_1 with the solvable group L_1 satisfying all the conditions. That is, we have $\langle L_1^{N_{G_1}(U_1)} \rangle \Lambda(G_1) = G_1$.

Now $\Lambda(G)/N_1 = \Lambda(G/N_1) = \Lambda(G_1N_1/N_1) = \Lambda(G_1)N_1/N_1$. So

$$\Lambda(G_1) \subseteq \Lambda(G) \quad \text{and} \quad \langle L_1^{N_G(U_1)} \rangle \Lambda(G) \supseteq G_1N_1 = G.$$

Thus U_1 satisfies conditions (a) and (b). Let U be a subgroup of G containing U_1 such that U is maximal with respect to conditions (a) and (b). Then $U_1N_1/N_1 \subseteq UN_1/N_1$ satisfies conditions (a) and (b) in G/N_1. But U_1N_1/N_1 is maximal with respect to conditions (a) and (b) in $G/N_1 = G_1N_1/N_1 \cong G_1/G_1 \cap N_1$. Thus $U_1N_1/N_1 = UN_1/N_1$. Now $UN_i/N_i = U_1N_i/N_i$ satisfies conditions (a)–(c) in G/N_i. Hence U is a c-subgroup of G, which gives us a contradiction.

2. Cartan Subgroups

Definition. Let G be a finite group. A subgroup U of G is called a *Cartan subgroup* of G if it staisfies the following conditions.

(a) U is nilpotent.

(b) There exists a solvable subgroup L of G containing U and satisfying $\langle L^{N_G(U)}\rangle \Lambda(G) = G$, (i.e., L and all its conjugates under the elements of $N_G(U)$ generate a subgroup G_1 satisfying $G_1 \cdot \Lambda(G) = G$).

(c) U is maximal with respect to the properties (a) and (b).

(d) If α is an automorphism of G, then U^α is a conjugate of U under an element of G.

(e) If N is any normal subgroup of G, then UN/N satisfies the properties (a)–(d) in G/N.

Examples. All subgroups of order 2 and 3 are Cartan subgroups of A_5. All subgroups of order 8 and all cyclic subgroups of order 6 are Cartan subgroups of S_5. All subgroups of order 4 and 3 are Cartan subgroups of A_6. All subgroups of order 16 and all cyclic subgroups of order 6 are Cartan subgroups of S_6. If $G = G_{168}$, the simple group of order 168, then all Cartan subgroups are of order 3.

Theorem 2.1. *If G is a finite solvable group, then Cartan subgroups are the same as Carter subgroups.*

Proof. Since G is a solvable group. Conditions (a)–(e) are equivalent to the conditions

(a^1) U is maximal nilpotent in G.

(b^1) If α is any automorphism of G, then U^α is a conjugate of U.

(c^1) UN/N satisfies the conditions (a^1) and (b^1) in G/N.

Now by [7] conditions (a^1) and (c^1) are equivalent to saying that U is a Carter subgroup of G and (b^1) is always satisfied by all Carter subgroups.

Hence Cartan subgroups of solvable groups are the same as Carter subgroups.

We will make the following assumptions on simple groups. Our results hold good for those classes of composite groups for which all simple composition factors have those properties.

POSTULATE I. *Let N be a simple nonabelian group. Suppose G is an automorphism group of N satisfying*

(1) *The inner automorphism group $\bar{N}$ induced by elements of N lies in G.*

(2) *$G/\bar{N}$ is nilpotent.*

(3) *Y is a nilpotent covering of G/N.*

Then there exists a single characteristic class S of conjugate, maximal nilpotent subgroups of N satisfying

(a) *Y normalizes X for some X in S.*

(b) *$N_N(X)$ is solvable.*

(By a singly characteristic class we mean: (A) if $U \in S$, then $U^\alpha \in S$. (B) $U, V \in S$, $U = V^g$.)

POSTULATE II. *If G is an automorphism group of $N = N_1 \times N_2 \times \cdots \times N_s$, where $N_i \cong N_1$, N_1 is simple nonabelian, and G/N is nilpotent, then there exists a Cartan subgroup of G.*

(In this case Cartan subgroups are the same as N projectors. See [8].)

THEOREM 2.2. *If G is a finite group whose compositino factors satisfy Postulates* I *and* II, *then G has Cartan subgroups.*

Proof. Suppose the statement is false. Let G be a group of smallest possible order for which the result is not true. Therefore G is not solvable.

Case (1). $R(G)$ is a nontrivial normal subgroup of G (where $R(G)$ the radical of G stands for the maximal normal solvable subgroup of G). Let N be a minimal normal fully invariant subgroup of G, $N \subseteq R(G)$. As is well known, N is an elementary abelian group. Now G/N is not a criminal. Therefore there exists a Cartan subgroup $\bar{U}/N$ of G/N, i.e., $\bar{U}/N \subseteq \bar{L}/N$ solvable satisfying (a)–(e).

Now $\bar{U} \subseteq \bar{L}$ is a solvable subgroup of G. Let U be a Carter subgroup of $\bar{U}$. Then $UN = \bar{U}$. Consider

$$\langle \bar{L}^{N_G(U)} \rangle \cdot \Lambda(G) = \langle \bar{L}^{N_G(U)N} \rangle \, \Lambda(G) \supseteq \langle \bar{L}^{N_G(\bar{U})} \rangle \cdot \Lambda(G) = G.$$

Therefore U satisfies conditions (a) and (b) in G. Suppose $U \subsetneq V$ and V satisfies conditions (a) and (b). Therefore $UN/N \subsetneq VN/N$ and VN/N satisfies conditions (a) and (b). But $UN/N = \bar{U}/N$ is maximal with respect to conditions (a) and (b). Therefore $VN = \bar{U}$. Now $U \subsetneq V$ nilpotent $\subseteq \bar{U}$. But U is a Carter subgroup of $\bar{U}$ and therefore is maximal nilpotent in $\bar{U}$. Therefore $U = V$ and U satisfies conditions (a)–(c) in G.

Now let α be any automorphism of G. Let $\bar{\alpha}$ be the induced automorphism of G/N. Now $\bar{U}/N$ is a Cartan subgroup of G/N. Therefore we have $[\bar{U}/N]^{\bar{\alpha}} = [\bar{U}/N]^{g}$ for some $g \in G$. Therefore $U^{\alpha g^{-1}} = U^{h}$ or $U^{\alpha} = U^{g'}$ for $g' = hg \in G$. Now let M be any normal subgroup of G. Trivially UM/M satisfies the conditions (a) and (b) in G/M. Suppose $UM/M \subsetneq V/M$ and V/M satisfies the conditions (a) and (b) in G/M. Therefore $\bar{U}M/NM \subsetneq VN/MN$ and VN/MN satisfies conditions (a) and (b) in G/MN. But $\bar{U}M/NM$ satisfies conditions (a) and (b). Thus $\bar{U}M/NM = VN/MN$.

Now U is a Carter subgroup of $\bar{U}$. Therefore UM/M is a Carter subgroup of $\bar{U}M/M = VN/M$, and UM/M is maximal nilpotent in $\bar{U}M/M = VN/M$.

Now $UM/M \subsetneq V/M$ nilpotent $\subsetneq VN/M$, which applies $UM/M = V/M$. Therefore UM/M satisfies conditions (a)–(c) in G/M. Now let us check condition (d) in G/M. Let σ be any automorphism of G/M. σ induces an automorphism $\bar{\sigma}$ of G/NM. But $\bar{U}M/NM$ satisfies condition (d) in G/NM. Therefore $[\bar{U}M/NM]^{\bar{\sigma}} = [\bar{U}M/NM]^{g}$ for some g in G. Therefore $[\bar{U}M/M]^{\sigma g^{-1}} = \bar{U}M/M$.

Now since UM/M is a Carter subgroup of $\bar{U}M/M$, we have $[UM/M]^{\sigma g^{-1}} = [UM/M]^{h}$, and therefore $[UM/M]^{\sigma} = [UM/M]^{g'}$ for $g' = hg \in G$. Hence U is a Cartan subgroup of G, which is a contradiction to our assumptions. So this case can not occur.

Case (2). $R(G) = 1 = \Lambda(G)$. By [8], G is the direct product of simple nonabelian groups, say, $G = N_1 \times N_2 \times \cdots \times N_s$. By Postulate II, N_i has Cartan subgroups, say, U_i. Let L_i be the solvable groups satisfying $\langle L_i^{N_{N_i}(U_i)} \rangle = N_i$ and also satisfying the conditions (d) and (e). We pick U_i in such a way that if $N_i \cong N_j$, then $U_i \cong U_j$. Now let $U = U_1 \times U_2 \times \cdots \times U_s$. We claim that U is a Cartan subgroup of G.

Trivially U satisfies conditions (a) and (b) with $L = L_1 \times L_2 \times \cdots \times L_s$. Suppose U does not satisfy condition (c). Let $U \subsetneq V \subsetneq L'$ solvable and satisfy $\langle L'^{N_G(V)} \rangle = G$. Since V is greater than U, the projection of V on some N_i, say, N_1, is greater than U_1. Let σ_1 be the projection map of G on N_1. Let $L_1 = L'^{\sigma_1}$ and $V_1 = V^{\sigma_1}$. Then we have $\langle L_1^{N_{N_1}(V_1)} \rangle = N_1$. Therefore V_1 satisfies the conditions (a) and (b) in N_1, but U_1 is maximal with respect to conditions (a) and (b) in N_1, which is a contradiction. Thus U satisfies the conditions (a)–(c). Now we assert that U satisfies the condition (d). Let σ be any automorphism of G. Suppose $N_i = N_j^\sigma$. Let $U_i^\sigma = U_j'$. Therefore U_j' is automorhpic to U_j and $U_j' = U_j^{n_j}$. Hence $[U_1 \times U_2 \times \cdots \times U_s]^\sigma = [U_1 \times U_2 \times \cdots \times U_s]^{n_1 n_2 \cdots n_s} = U^g$ for $g \in G$. Now let M be any normal subgroup of G. Since G is completely reducible, M is a direct factor. Consider a Remak decomposition of G through M. Because of the uniqueness of decomposition of G, we get that M is a direct product of some of the simple factors in G. Therefore G/M is a direct product of some of the simple factors in G. Therefore UM/M satisfies conditions (a)–(d) in G/M. Hence U is a Cartan subgroup of G. So this case cannot occur.

Case (3). $\Lambda(G) \neq 1$ and $R(G) = 1$. Let N be a minimal normal fully invariant subgroup of G. $N \subseteq \Lambda(G)$. Again N is the direct product of simple nonabelian isomorphic groups, say, $N = N_1 \times N_2 \times \cdots \times N_s$, where $N_i \cong N_1$.

Now G/N does not violate the assumptions of the theorem. Therefore there exists a Cartan subgroup $\bar{U}/N$ of G/N satisfying (a)–(e) with the solvable subgroup as $\bar{L}/N$.

(*) Let us first suppose that $\bar{U} < G$ and $\Lambda(\bar{U}) \supseteq N$.

Now $\bar{U}$ has a Cartan subgroup, say, U. Since $\bar{U}/N$ is nilpotent, we have $\bar{U} = U \cdot N$. Let $N = N_1 \times N_2 \times \cdots \times N_{i_1} \times N_{i_1+1} \times \cdots \times N_s$, where $N_j = N_1^{u_j}$ if $j = 1, 2,..., i_1$; $N_{i_1+1} \cong N_1 ,...,$ where $u_j \in U$ (i.e., all N_i's conjugate to N_1 are put together, and again all N_i's conjugate to N_{i_1+1} are put together, etc.). Let $A_1 = \{u \in U / N_1^u \subseteq N_1\}$. Then $U = \{A_1, u_2, u_3,..., u_{i_1}\}$. Consider the inner automorphism group $\underline{A_1 N_1}$ induced by the elements of A_1N_1. It is an automorphism group of $\underline{N_1}$ containing $\underline{N_1}$ ($\underline{N_1}$ is the inner automorphism group of N_1). It satisfies the conditions $\underline{A_1N_1} \supseteq \underline{N_1}$, and $\underline{A_1N_1}/\underline{N_1}$ is nilpotent. Therefore by Postulate I there exists a characteristic class of conjugate nilpotent subgroups satisfying A_1 normalizes X_1 for some $X_1 \in S$, and $N_{N_1}(X_1)$ is solvable. Let $X_2 = X_1^{u_2},..., X_{i_1} = X_1^{u_{i_1}}$. Then U normalizes $X_1 \times X_2 \times \cdots \times X_{i_1}$. Similarly consider the second set $N_{i_1+1} \times \cdots \times N_{i_2}$ such that $N_j = N_{i_1+1}^{u_j}$ for $j = i_1 + 2,..., i_2$, where $u_j \in U$. Similarly there exists $X_{i_1+1} ,..., X_{i_2}$ satisfying U normalizes $X_{i_1+1} \times \cdots \times X_{i_2}$. Continuing, by this procedure we get $U \subseteq N_G(X_1 \times X_2 \times \cdots \times X_s)$, where the X_i's belong to the same conjugate class. Therefore $N_G(X_1 \times \cdots \times X_s)N = G$. Call $N_G(X_1 \times X_2 \times \cdots \times X_s) = G_1$. Let $L = \bar{L} \cap G_1$. Now $\bar{L}/N = LN/N$ is solvable. Also $LN/N \cong L/L \cap N$ and $L \cap N = N_G(X_1 \times \cdots \times X_s) \cap N = N_{N_1}(X_1) \times \cdots \times N_{N_s}(X_s)$ is solvable. Therefore L is solvable and $L \supseteq U$. Now

$$\langle \bar{L}^{N_G(U)} \rangle \cdot \Lambda(G) = \langle \bar{L}^{N_G(U)N} \rangle \cdot \Lambda(G) \supseteq \langle \bar{L}^{N_G(\bar{U})} \rangle \cdot \Lambda(G) = G.$$

Suppose $\langle L^{N_G(U)}\rangle \cdot \Lambda(G) = G_2$. Then

$$G_2/N = \langle LN/N^{N_G(U)N/N}\rangle \cdot \Lambda(G/N) = \langle \bar{L}/N^{N_G(U)N/N}\rangle \cdot \Lambda(G/N) = G/N.$$

Therefore $G_2 = G$. Hence U satisfies conditions (a) and (b) in G. We claim that U is maximal nilpotent in $\bar{U}$. Since U is a Cartan subgroup of $\bar{U}$, it satisfies (a)–(e) in $\bar{U}$. Suppose U is not maximal nilpotent in $\bar{U}$. Let $V \supset U$ be nilpotent. Take $L_1 = V$. Then $\langle L_1^{N_{\bar{U}}(V)}\rangle\, \Lambda(\bar{U}) \supseteq U \cdot N = \bar{U}$. Therefore U is not maximal with respcet to conditions (a) and (b) in $\bar{U}$. This gives a contradiction to the fact that U is a Cartan subgroup of $\bar{U}$. Hence the assertion is established. Now suppose U does not satisfy condition (c) in G, say, $U \subset V$ and V satisfies the conditions (a) and (b). Therefore $UN/N = \bar{U}/N \subsetneq VN/N$ satisfies the conditions (a) and (b). But $\bar{U}/N$ is a Cartan subgroup of G/N. Therefore $\bar{U}/N$ is maximal with respect to conditions (a) and (b) in G/N. Therefore $\bar{U}/N = UN/N = VN/N$. Now U is contained in a nilpotent subgroup V, which is continued in $\bar{U}$. But U is maximal nilpotent in $\bar{U}$, which is a contradiction. Hence U satisfies conditions (a)–(c). Let α be any automorphism of G. α induces an automorphism $\bar{\alpha}$ on G/N. $\bar{U}/N$ is a Cartan subgroup of G/N. Therefore $[\bar{U}/N]^{\bar{\alpha}} = [\bar{U}/N]^g$. Therefore $\bar{U}^{\alpha g^{-1}} = \bar{U}$. But U is a Cartan subgroup of $\bar{U}$. Therefore $U^{\alpha g^{-1}} = U^h$. Thus $U^\alpha = U^{g'}$ for $g' = hg \in G$. Therefore U satisfies the conditions (a)–(d) in G. Now let M be any normal subgroup of G. Then trivially UM/M satisfies conditions (a) and (b) in G/M.

Suppose $UM/M \subset V/M$ and V/M satisfies the conditions (a) and (b). Therefore $UMN/MN = \bar{U}M/MN \subsetneq VN/MN$ satisfies the conditions (a) and (b). But $\bar{U}M/MN$ satisfies the condition (c) in G/MN and therefore is maximal with respect to the conditions (a) and (b) in G/MN. Therefore $\bar{U}M = VN$. Now since U is a Cartan subgroup of $\bar{U}$, UM/M is a Cartan subgroup of $\bar{U}M/M$. Also $\Lambda(\bar{U}M/M) \subseteq NM/M$ and $\bar{U}M/M/\Lambda(\bar{U}M/M)$ is nilpotent. Therefore by the same reasoning as before, UM/M is maximal nilpotent in $\bar{U}M/M$, which is a contradiction. Thus UM/M satisfies conditions (a)–(c) in G/M. Now let τ be any automorphism of G/M. τ induces an automorphism $\bar{\tau}$ of G/NM. Therefore $[\bar{U}M/MN]^{\bar{\tau}} = [\bar{U}M/MN]^g$ becuase $\bar{U}M/NM$ is a Cartan subgroup of G/NM. Therefore $[\bar{U}M/M]^{\tau g^{-1}} = \bar{U}M/M$. But UM/M is a Cartan subgroup of $\bar{U}M/M$. Therefore $[UM/M]^{\tau g^{-1}} = [UM/M]^h$, i.e., $[UM/M]^\tau = [UM/M]^{g'}$ for $g' = hg \in G$. Hence U satisfies the conditions (a)–(e). Therefore U is a Cartan subgroup of G, which gives us a contradiction. Hence (*) cannot occur.

(**) Now let us suppose $\bar{U} \subset G$ and $\Lambda(\bar{U}) \not\supseteq N$.

Therefore we can write $\bar{U} = \bar{U}_1 \times N_2$, where N can be split as $N_1 \times N_2$ and $\Lambda(\bar{U}_1) \supseteq N_1$. Now since N is the direct product of simple nonabelian groups, so is N_2. Therefore $\bar{U}_1$ and N_2 are characteristic subgroups of $\bar{U}$ (see [8]). Let $N = M_1 \times M_2 \times \cdots \times M_t$, where $N_1 = M_1 \times M_2 \times \cdots \times M_s$ and $N_2 = M_{s+1} \times M_{s+2} \times \cdots \times M_t$; M_i simple nonabelian and $M_i \cong M_1$. Now let U_1 be a Cartan subgroup of $\bar{U}_1$. As before we can pick characteristic classes S_i of conjugate subgroups satisfying $U_1 \subseteq N_G(X_1 \times \cdots \times X_s)$, where $X_i \in S_i$. Pick $U = U_1 \times X_{s+1} \times \cdots \times X_t$, where $X_{s+1},..., X_t$ are subgroups of $M_{s+1},..., M_t$ belonging to the same class as before. Then $U \subseteq N_G(X_1 \times X_2 \times \cdots \times X_t)$.

Now $\bar{U}_1/N_1$ is nilpotent and $\Lambda(\bar{U}_1) \supseteq N_1$. Therefore U_1 is maximal nilpotent in $\bar{U}_1$. Also by Postulate I, $X_{s+1},..., X_t$ are maximal nilpotent in $N_{s+1},..., N_t$. Therefore U is maximal nilpotent in $\bar{U}$. As in Case $(*)$, U satisfies conditions (a) and (b). Keeping in mind that

$$U^\alpha = U_1^\alpha \times [X_{s+1} \times \cdots \times X_t]^\alpha = U^{l_1} \times [X_{s+1} \times \cdots \times X_t]^{l_2},$$

where $l_1 \in \bar{U}_1$ and $l_2 \in N_2$, we get that $U^\alpha = U^l$ for $l = l_1 l_2 \in \bar{U}$ and α is an automorphism of $\bar{U}$. Suppose U does not satisfy condition (c). Let U be a proper subgroup of V, where V satisfies conditions (a) and (b) in G. Then $UN/N = \bar{U}/N \subseteq VN/N$ satisfies conditions (a) and (b) in G/N. But $\bar{U}/N$ satisfies condition (c) in G/N, i.e., $\bar{U}/N$ is maximal with respect to conditions (a) and (b) in G/N. Therefore $UN = VN = \bar{U}$. Now $U \subset V \subseteq \bar{U}$ and U is maximal nilpotent in $\bar{U}$ implies $U = V$. Hence conditoin (c) is satisfied.

Now let α be an automorphism of G. Then α induces an automorphism $\bar{\alpha}$ of G/N. Therefore $[\bar{U}/N]^{\bar{\alpha}} = [\bar{U}/N]^g$ for some $g \in G$ and $\bar{U}^{\alpha g^{-1}} = \bar{U}$. Now as above αg^{-1} is an automorphism of $\bar{U}$ and $U^{\alpha g^{-1}} = U^h$ for some $h \in \bar{U}$. Therefore $U^\alpha = U^{g'}$ for some $g' \in G$.

As to condition (e), let M be any normal subgroup of G. Then $\bar{U}M/M = \bar{U}_1 M/M \times [X_{s+1} \times \cdots \times X/M]M$. Therefore UM/M satisfies conditions (a)–(c) in G/M.

Now let τ be any automorphism of G/M. It induces an automorphism $\bar{\tau}$ of G/MN. Therefore $[\bar{U}M/MN]^{\bar{\tau}} = [\bar{U}M/MN]^g$ and $[\bar{U}M/M]^{\tau g^{-1}} = \bar{U}M/M$. Therefore $[UM/M]^{\tau g^{-1}} = [UM/M]^h$ for some $h \in G$ and $[UM/M]^\tau = [UM/M]^{g'}$ for some $g' \in G$. Hence U satisfies conditions (a)–(e) in G, i.e., U is a Cartan subgroup of G. Therefore Case $(**)$ cannot arise.

$(***)$ Let us now suppose that $\Lambda(\bar{U}) \supseteq N$, $\bar{U} = G$, and $R(G) = 1$.

Therefore G is an outer automorphism group of N that is a direct product of isomorphic simple nonabelian groups and $G/N = \bar{U}/N$ is nilpotent. Therefore by Postulate II G has a Cartan subgroup, which gives us a contradiction. Hence Case $(***)$ cannot occur. Hence our assumptions were wrong.

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