On the cohomology and deformations of differential graded algebras

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Abstract

In this paper we work out the deformation theory for differential graded algebras (dga's) and for differential graded Hopf algebras (dgha's). The constructions generalize the theory of deformations of algebras developed in late sixties by Gerstenhaber and of Hopf algebras, introduced more recently by Gerstenhaber and Schack and the authors. Namely, we introduce a cohomology theory for dga's and for dgha's, "controlling" their deformations. Our main example of a dga will be the de Rham algebra $\Omega$ of a smooth algebraic variety. We prove that $H^*(\Omega, M) = H^*(M)$ for any symmetric dg module $M$ over $\Omega$. From this result we deduce that the deformation cohomology of the de Rham algebra of a Lie group coincides with cohomology of its classifying space. We introduce the notion of a Poisson–de Rham Lie group – this is just a usual Poisson Lie group with a graded Poisson bracket on its de Rham algebra extending the Poisson bracket on functions. We prove that for any simple Lie group $G$ the standard Poisson structure cannot be extended to a Poisson–de Rham structure. Hence, there are no deformations of the de Rham algebra of $G$ extending the Drinfeld–Jimbo deformation.

1. Introduction

In this paper we construct the deformation complex for differential graded algebras (dga's) and for differential graded Hopf algebras (dgha's). The first one will be a bicomplex and the second one will be a tricomplex. These constructions are natural generalizations of the ones elaborated in [3,4,6]. As usual, the first cohomology group corresponds to the derivations of the algebra (or Hopf algebra), which commute with an intrinsic differential, the second one corresponds to the infinitesimal deformations and the obstructions will belong to the third cohomology group. Observe that to

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obtain the actual deformation complex for dga’s or dgha’s we need to make some truncations in the natural complexes.

If we work with dga’s we of course can define the complex for calculating cohomology with coefficients in an arbitrary dg bimodule over our dga (the module of coefficients in the deformation complex is the dga itself).

Our main example of a dga will be the de Rham algebra $\Omega$ of a smooth algebraic manifold. We prove that $H^*(\Omega, M) = H^*(M)$ for any symmetric dg module $M$ over $\Omega$. From this result one can easily deduce that the deformation cohomology (as a dgha) of the de Rham algebra of a Lie group $G$ coincides with cohomology of its classifying space $BG$. We are concerned with the untruncated cohomology in both these cases.

For the de Rham algebra $\Omega$ of the Lie group there is a natural map from the exterior algebra of its Lie algebra to $\Omega$, which is compatible with grading. In the semisimple case, this map determines the imbedding of infinitesimal deformations of the algebra of functions into those of the de Rham algebra. Nevertheless, one can prove that there are no deformations of the de Rham algebra of any simple Lie algebra, which extend the standard Drinfeld–Jimbo deformation. The problem is that the corresponding infinitesimal deformation does not extend to a Poisson structure on the de Rham algebra, i.e. the primary obstruction to the deformation is not zero. On the other hand if we start with infinitesimal deformation determined by a unitary solution of the Yang–Baxter equation there are no such difficulties, because the corresponding Poisson structure on the algebra of functions does determine the Poisson structure on the de Rham algebra.

2.

Let $A$ be a differential graded algebra (with nonnegative degrees) over the field $k$. Consider the double complex $C_{ij}(A, A) = \text{Hom}_k(A^\otimes i, A)$. The horizontal differential is the usual Hochschild differential (below it will be denoted as $d_1$ or $du$) and the vertical differential is induced by the intrinsic differential of $A$ (it will be denoted by $d_2$ or $d_{DR}$ if $A$ is the de Rham algebra of the algebraic manifold). If we restrict to the case $i \geq 0$, $j > 0$ we obtain a new complex which is denoted by $F_*$. The corresponding cohomology is denoted as $\widetilde{H}^*$. The corresponding cohomology is denoted as $\widetilde{H}^*$.

**Definition 2.1.** A deformation of $A$ is a dga $A_h$ over the ring $k[[h]]$ with an isomorphism of $k[[h]]$-modules: $A[[h]] \cong A_h$ and

\[
m_h(a, b) - ab + h\xi(a, b) + h^2\xi_2(a, b) + \cdots,
\]

\[
d_h(a) = d_0(a) + hd_1(a) + h^2d_2(a) + \cdots.
\]

Here $a, b \in A$ and $m_h(\cdot, \cdot)$, $d_h(\cdot)$ are the (deformed) multiplication and differential. The pair $(\xi, d_1)$ is called an infinitesimal deformation of $A$.

We omit an evident definition of the equivalence of the deformations.
Proposition 2.1. The equivalence classes of infinitesimal deformations of $A$ are in one-to-one correspondence with $H^2(A, A)$. Here the infinitesimal deformations of multiplication are represented by a $\xi$ element in $C^{0,2} = \text{Hom}^0(A^2, A)$ and the infinitesimal deformation of the differential by the element $d_1$ in $C^{1,1} = \text{Hom}^1(A, A)$.

Proof. Easy computation. \hfill \Box

We also omit the proofs and formulations of the standard theorems from the deformation theory which can be found in [3] and transferred to the case of the differential graded algebras without any difficulties.

Before passing to the calculation of the cohomology of our main example of dga's, i.e. the de Rham algebra on a smooth manifold, we give the definition of the cohomology of a dgha and of the deformation of a dgha. We will also omit proofs and even formulations of the standard theorems. We make, however, one remark. Unlike the case of the differential graded algebras, we cannot define the cohomology of a dgha when the module of coefficients is not the dgha itself. So, there is no naive interpretation of this cohomology in terms of derived functors.

Let $H$ be a dgha. Consider the tricomplex $C^{***}$, where $C^{ijk} = \text{Hom}^i(H \otimes^j, H \otimes^k)$. Here the vertical differential $d_2 : i \rightarrow i + 1$ is induced by the de Rham differential, and horizontal ones $d_1 : j \rightarrow j + 1$ and $d_3 : k \rightarrow k + 1$ are nothing but Hochschild and coHochschild differentials. We omit the evident, but rather nasty explicit formulas. The cohomology of this tricomplex will be denoted as $H^*(H, H)$. If we restrict to the case $i \geq 0, j, k > 0$ we obtain a new complex which is denoted $C^{**}$. The corresponding cohomology is denoted $\tilde{H}^*$.

Definition 2.2. A deformation of $H$ is a dgha $H_h$ over the ring $k[[h]]$ such that the isomorphism of $k[[h]]$-modules exists: $H[[h]] \cong A_h$ and

$$m_h(a, b) = ab + h\xi_1(a, b) + h^2\xi_2(a, b) + \cdots,$$
$$d_h(a) = d_0(a) + hd_1(a) + h^2d_2(a) + \cdots,$$
$$\Delta_h(a) = \Delta(a) + h\eta_1(a) + h^2\eta_2(a) + \cdots.$$

Here $a, b \in H$ and $m_h(\cdot, \cdot), d_h(\cdot), \Delta_h(\cdot)$ are the deformed multiplication, differential and comultiplication. The triple $(\xi, d_1, \eta)$ is called the infinitesimal deformation of $H$.

Again, we omit the obvious definition of equivalence of deformation.

Proposition 2.2. The equivalence classes of infinitesimal deformation of $H$ are in one-to-one correspondence with $H^2(H, H)$. Here an infinitesimal deformation of multiplication is represented by an element in $C^{0,2,1} = \text{Hom}^0(A \otimes^2, A)$ the infinitesimal deformation of the differential by an element in $C^{1,1,1} = \text{Hom}^1(A, A)$ and the infinitesimal deformation of comultiplication by an element in $C^{0,1,2} = \text{Hom}^0(A, A \otimes^2)$.

Proof. Easy computation. \hfill \Box
**Theorem 2.1.** Let $M$ be a symmetric dg bimodule over the de Rham algebra $\Omega$ of a smooth manifold $N$. Then the dga Hochschild cohomology $H^\bullet(\Omega, M)$ is equal to the cohomology of $M$, considered as an abstract complex.

**Proof.** The plan of the proof is the following. At the beginning we calculate $H^1(\Omega, \Omega)$ regarding $\Omega$ just as a graded algebra (without differential). Then we compute all higher cohomology $H^i(\Omega, \Omega)$ using the analog of the Kostant–Hochschild–Rosenberg theorem which asserts that this higher cohomology is equal to the exterior algebra of $H^1(\Omega, \Omega)$ over $\Omega$. Notice that $\Omega$ may be considered as the coordinate ring on a smooth supermanifold. By the same theorem $H^\bullet(\Omega, M) = H^\bullet(\Omega, \Omega) \otimes_\Omega M$. Recall that here cohomology is considered only with respect to the Hochschild differential. In other words we have described the $E_1$ term of the spectral sequence associated with our bicomplex. The differential $d_1$ is induced by the intrinsic differentials in $\Omega$ and $M$. To obtain the ultimate result, it is sufficient to prove that $H^\bullet(H^1(\Omega, M))$ is equal to zero when $i > 0$.

Now our aim is to compute $H^1(\Omega, \Omega)$. As we mentioned above, we will consider $\Omega$ as a ring of functions on the supermanifold. Then $H^1(\Omega, \Omega)$ consists of derivations of this ring (in a super sense). In local coordinate we can write down every derivation in the form

$$\lambda = \sum f_i \frac{\partial}{\partial x_i} + g_i \frac{\partial}{\partial dx_i}.$$

Here $f_i, g_i$ are differential forms. The term $g_i \partial/\partial x_i$ is a $\Omega^0$-linear derivation of $\Omega^\bullet$. To determine the action of such a derivation, it is sufficient to know the values of this on forms of first degree. Hence the space of $\Omega^0$-linear derivations coincides with the space of global sections of $T(N) \otimes_{\Omega^0} \Omega^\bullet(N)$, where $T(N)$ is the module of vector fields on $N$. The following lemma holds.

**Lemma 2.1.** The Lie algebra of derivations of $\Omega^\bullet(N)$ is generated as a Lie algebra by the $\Omega^0(N)$-linear derivatives and the de Rham differential $d$.

**Proof.** For an arbitrary derivation $\eta$ of $\Omega^\bullet(N)$, define $\delta(\eta) = [d, \eta]$. It is easy to verify the following operator identity:

$$(-1)^{\deg \omega} \omega L(\xi) = \partial(\omega \xi) - (\omega)\xi.$$

Here $\xi$ denotes the operator of inner product with the vector field $\xi$ and $L(\xi)$ the operator of Lie derivative along $\xi$. The lemma is an easy consequence of this identity.

So the derivations are of two sorts: $\Omega^0(N)$-linear and (super)commutators of those with $d$. Lemma 2.1 tells us that the space of derivations is generated by the $\Omega^0(N)$-linear derivations and their $\partial$-images.

**Lemma 2.2.** If $\eta$ is an $\Omega^0$-linear derivation, and it commutes with $d$, then $\eta = 0$. 
**Proof.** The hypotheses imply that \( \eta \) equals zero on \( \Omega^0 \) and on the space of all exact 1-forms. But 0-forms and exact 1-forms generate \( \Omega^*(N) \). Hence \( \eta \) equals zero identically. \( \square \)

These two lemmas with the obvious observation that the derivations of the form \([d, \eta]\) for some \( \eta \) commute with \( d \) give us the following result.

**Lemma 2.3.** The space of all derivations can be decomposed into the direct sum of the \( \Omega^0 \)-linear derivations and their \( \partial \)-images. Moreover \( \partial \) determines the isomorphism between these two spaces.

Further we need to know the \( \Omega^* \)-module structure on \( \text{Der}(\Omega^*) \). Any element \( \lambda \) in \( \text{Der}(\Omega^*) \) can be represented as a pair \( \lambda = (a, b) = \partial a + b \) where \( a, b \) are in \( \Omega^* \otimes_{\Omega^0} T \).

Then

\[
\begin{align*}
\omega(a, b) &= \omega \partial a + \omega b = \omega(ad + (-1)^i da) + \omega b \\
&= \omega ad + (-1)^{i+\omega} d\omega a - (-1)^\omega d\omega a + \omega b \\
&= (\omega a, \omega b - (-1)^\omega (d\omega)a). \quad \square
\end{align*}
\]

**Remark.** \( \text{Der}(\Omega^*) \) is an extension of the \( \Omega^* \)-module \( \Omega^* \otimes_{\Omega^0} T \) by itself. When does a splitting exist? A splitting is a map \( \tau: \Omega^* \otimes_{\Omega^0} T \rightarrow \Omega^* \otimes_{\Omega^0} T \) such that

\[
(\omega a, \tau(\omega a)) = (\omega a, \omega \tau(a) - (-1)^\omega d\omega a),
\]

i.e.

\[
\tau(\omega a) = \omega \tau(a) - (-1)^\omega d\omega a.
\]

Thus, \( \tau \) gives us a connection on the tangent bundle on \( M \).

For our purposes it is suitable to change notations slightly. Recall that we identified the space of \( \Omega \)-linear derivations with \( \Omega^* \otimes_{\Omega^0} T \). Define \( \partial T = \tilde{T} \). Then one has the isomorphism of linear spaces:

\[
\text{Der} \Omega^* = (\Omega^* \otimes_{\Omega^0} T) \oplus (\Omega^* \otimes_{\Omega^0} \tilde{T}) = \Omega^* \otimes_{\Omega^0} (T \oplus \tilde{T}).
\]

One can consider this identity as an identity between sheaves. Further we will use such notations as \( \Omega^*(U), T(U), \) etc. for an open set \( U \) in \( N \).

Notice that the de Rham differential induces a differential on the graded module \( \Lambda^i(\text{Der} \Omega^*) \). Here \( \Lambda^* \) is considered in the super sense since \( \text{Der} \Omega^* \) is a super Lie algebra, where elements in \( T \) and \( \tilde{T} \) have degree \(-1\) and \( 0 \), respectively.

**Lemma 2.4.** The complex \( \Lambda^i(\text{Der} \Omega^*) \) with the differential described above is acyclic if \( i > 0 \).
Proof. We will prove it locally. Let $U$ be a small contractible disk in $N$. Choose a basis $e_i$ in the $\Omega^0(U)$-module $T(U)$ of vector fields on $U$. Denote $\tilde{e}_i = \partial e_i$, and let $E$ and $\tilde{E}$ be the vector spaces spanned by the $e_i$s and $\tilde{e}_i$s, respectively. Then $e_i$, $\tilde{e}_i$ form a free basis of $\Omega^*(U)$-module $\text{Der } \Omega^*(U)$. One has

$$A_{d\Omega^*(U)}(\text{Der } \Omega^*(U)) = A_{d\Omega^*(U)}(\Omega^*(U) \otimes (E \oplus \tilde{E})) = \Omega^*(U) \otimes A^i(E \oplus \tilde{E}).$$

The last complex, being the tensor product of $\Omega^*(U)$ and the acyclic complex $A^i(E \oplus \tilde{E})$, is acyclic. \(\square\)

Observe that all sheaves in the complex $A_{d\Omega^*(U)}(\text{Der } \Omega^*(U))$ are fine and hence the complex of global sections of this complex is exact.

In the special case when $M' = A'$, the theorem is a consequence of Lemma 2.4.

Now we pass to the general case. Associate with the d.g. $\Omega^*$-module $M^*$ its complex of sheaves. We also denote it $M^*$. Then due to the Hochschild–Kostant–Rosenberg theorem, the $E_1$-term of the corresponding spectral sequence in the disk $U$ has the form

$$A_{d\Omega^*(U)}(\text{Der } \Omega^*(U)) \otimes_{\Omega^*(U)} M^*(U) = \Omega^*(U) \otimes A^i(E \oplus \tilde{E}) \otimes_{\Omega^*(U)} M^*(U).$$

For $i > 0$ this complex is acyclic and the above arguments show that our spectral sequence collapses at the level of global sections. The theorem is proved. \(\square\)

Next we compute the cohomology of the truncated complex $\tilde{C}^*(\Omega^*, \Omega^*)$. It is easy to see that the zero cohomology of this complex is just zeroth de Rham cohomology of the underlying manifold $N$ and the first cohomology is the module of the vector fields on $M$. These vector fields act on the space of differential forms on $M$ as Lie derivatives. The usual cup-product supplies the space $\tilde{H}^*(\Omega^*, \Omega^*)$ with the structure of the differential graded (skew-commutative) algebra. It so happens that this algebra is generated by its first component, i.e. by the module of vector fields. Denote $\tilde{H}^i(\Omega^*, \Omega^*) = \bigoplus_{i > 0} \tilde{H}^i(\Omega^*, \Omega^*)$, so that $\tilde{H}^*(\Omega^*, \Omega^*) = H^*_{DR}(M) \oplus \tilde{H}^i(\Omega^*, \Omega^*)$.

Proposition 2.3. Any element in $\tilde{H}^i(\Omega^*, \Omega^*)$ can be represented as a product of elements in $\tilde{H}^1(\Omega^*, \Omega^*)$.

Proof. Recall that we use the notation $L(\xi)$ for the operation of the Lie derivative along the vector field $\xi$ and the notation $\xi$ for the operator of the inner product with $\xi$. The preceding theorem tells us that the space $\tilde{H}^1(\Omega^*, \Omega^*)$ is equal to the image of the $1$-st row of the $E_1$-term of our spectral sequence under the action of the differential $d_1^{-1}$. As above, we will denote this differential as $\partial$. Every element in $E_1^{i-1} = A^{i-1}_\omega(\text{Der } \Omega^*)^{-1}$ can be represented in the form $\Sigma L(\xi_1) \ldots \xi_i$, where $\omega$ is an $(i - 1)$-form. Since the elements $L(\xi)$ are the $\partial$-cocycles, it is sufficient to prove that the element $\partial(\omega \xi_1, \ldots, \xi_i)$ can be represented as a sum of the monomials $L(\tau_1), \ldots, L(\tau_k)$. 

Notice that for any number of functions $f_i$ and vector fields $\xi_i$ on the manifold $M$, one has the equalities:

\[ \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} L(f_{\sigma(1)}\xi_1) \cdots L(f_{\sigma(n)}\xi_n) \]

\[ = \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} \sum_{i} df_{\sigma(i)}\xi_i \cdots f_{\sigma(i)}L(\xi_i) \cdots df_{\sigma(n)}\xi_n \]

\[ = n! (-1)^{n(n-1)/2} df_1 \cdots \Lambda df_n\xi_1 \cdots \xi_n \]

\[ \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} L(f_{\sigma(1)}\xi_1) \cdots L(f_{\sigma(n)}\xi_n) \]

\[ = \partial((-1)^{n(n-1)/2}(n-1)!\Sigma(-1)^{i}df_1 \cdots f_i \cdots df_n\xi_1 \cdots \xi_n). \]

These equalities can be verified by direct computation. To complete the proof one needs to show that the space of all $(n-1)$-forms is generated by expressions of the form $\Sigma_i(-1)^i df_1 \cdots f_i \cdots df_n$. Denote the last expression by $\lambda(f_1, \ldots, f_n)$. Then

\[ \lambda(f_1, \ldots, f_n) - \lambda(\lambda(f_1, f_2, 1, f_3, \ldots, f_n) + 2\lambda(f_1, f_2, f_2f_3, 1, f_4, \ldots, f_n) \]

\[ \cdots + (-1)^{n+1}(n-1)\lambda(f_1, \ldots, f_{n-1}f_n, 1) = (-1)^{n+1}df_1, \ldots, df_n f_n. \]

Expressions of the form $df_1, \ldots, df_{n-1} f_n$ generate the space of all $n - 1$-forms. The proposition is proved.

Remark. Notice that the space $\mathcal{H}'(\Omega^*, \Omega^*) \oplus H^0_{\text{DR}}(M)$ does not coincide with the exterior algebra of vector fields over the ground field; there is an easily verified relation

\[ fgL(\xi)\Lambda g L(\eta) + L(\xi)\Lambda fg L(\eta) = fL(\xi)\Lambda g L(\eta) + g L(\xi)\Lambda f L(\eta), \]

where $\xi, \eta$ are vector fields and $f, g$ are functions on $M$.

Now suppose that $G$ is a Lie group, $\Omega^*(G)$ is the de Rham algebra on $G$. One has

**Theorem 2.2.** The dgla cohomology of $\Omega^*(G)$ coincides with the cohomology of $BG$, the classifying space of $G$.

**Proof.** We have the spectral sequence $E^{\ast \ast}$, converging to $H_{\text{dgha}}(\Omega^*(G))$ and $E_{1}^{ij} = H^i_{\text{dga}}(\Omega^*(G), \Omega^*(G))^\otimes_j$. The last expression is the cohomology of $\Omega^*(G)$ as a dga with coefficients in $\Omega^*(G)^\otimes$. The previous theorem gives us that $E_{1}^{ij} = E_{1}^{0,j} = (H^0_{\text{DR}}(G))^\otimes_j$. The $E_1$-term of the spectral sequence coincides with the bar-construction of the algebra $H_{\text{DR}}^*(G)$. Hence, according to the Eilenberg–Moore theorem it converges to $H^*(BG)$. The theorem is proved.
3.

Now we are going to investigate the structure of infinitesimal deformations of $\Omega^*(G)$ in more detail. To do this, we need to generalize the notion of the Poisson–Lie group to our situation.

**Definition 3.1.** A group $G$ is called a Poisson–de Rham–Lie group if there is a Poisson bracket on the de Rham algebra $\Omega^*(G)$ which is compatible with the group structure, i.e. the multiplication may $G \times G \to G$ preserves this bracket.

The infinitesimal analog of a Poisson–de Rham–Lie group is what we call the de Rham–Lie bialgebra. It is a Lie superalgebra $\tilde{g}$ with the underlying vector space $g^0 \oplus g^{-1}$, $g^0$ and $g^{-1}$ are two copies of the vector space of some ordinary Lie algebra $g$ with degrees 0 and $-1$, respectively. We will denote the elements in $g^0$ and $g^{-1}$ which correspond to the element $a \in g$ as $a^0$ and $a^{-1}$. Then

\[
[a^0, b^0] = [a, b]^0, \quad [a^0, b^{-1}] = [a, b]^{-1}, \quad [a^{-1}, b^{-1}] = 0.
\]

The natural map $g^{-1} \to g^0 : a^{-1} \to a^0$ will be called the de Rham differential $d$. We require the existence of the Lie cobracket $\lambda : \tilde{g} \to \tilde{g} \otimes \tilde{g}$ which preserves the grading and commutes with the differential $d$.

From now on assume $g$ is a semisimple Lie algebra. There are two sorts of 1-cocycles of $\tilde{g}$ with coefficients in $A^2(\tilde{g})$ of degree zero: the coboundaries which are determined by the elements in $A^2(\tilde{g})$ and the elements representing the nonzero cohomology classes. The latter can be identified with the module $\text{Hom}_{\mathfrak{g}^0}(g^{-1}, g^{-1} \otimes g^0)$, where the expression $\text{Hom}_{\mathfrak{g}^0}$ means homomorphisms of the $g^0$-representations. This can be easily deduced from the Hochschild–Serre spectral sequence connected with the subalgebra $g^0$ in $\tilde{g}$. Given a cocycle of $\tilde{g}$ with coefficients in $A^2(\tilde{g})$ there is a unique representation as a sum of cocycles of these two sorts.

Consider the compatibility condition of the cocycle with the de Rham differential $d$. Let $\lambda \in Z^1(\tilde{g}, A^2(\tilde{g}))$ and $\lambda(x) = [r, x] + \phi(x)$, where $r \in A^2(\tilde{g})$ and $\phi \in \text{Hom}_{\mathfrak{g}^0}(g^{-1}, g^{-1} \otimes g^0)$. Remark that there and later on we will use, slightly abusing the language, the notation $[x, y]$ for the Schouten bracket of elements $x$ and $y$ belonging to the exterior algebra of the Lie algebra. One can check that $\lambda$ is compatible with $d$ if and only if there exist $\mu \in \text{Hom}_{\mathfrak{g}^0}(g^{-1}, S^2(g^{-1}))$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
g^{-1} & \xrightarrow{\mu} & S^2(g^{-1}) \\
\phi & \downarrow & \\
g^{-1} \otimes g^0 & \xrightarrow{\mu} & S^2(g^{-1})
\end{array}
\]

We obtain the following.
Proposition 3.1. For a semisimple Lie algebra \( g \), any element \( \Lambda \in Z^1(\tilde{g}, \Lambda^2(\tilde{g})) \), compatible with the de Rham differential \( d \) on \( \tilde{g} \), is of the following form:

\[
\lambda(x) = [x, r] + d\mu(x).
\]

Here \( x \in g \), \( r \in \Lambda^2(\tilde{g}) \), \( \mu \in \text{Hom}_{\text{gr}}(g^{-1}, S^2(g^{-1})) \) and \( d : S^2(g^{-1}) \to g^0 \otimes g^{-1} \) — the de Rham differential.

Let \( v \) be any map from any vector space \( V \) into \( \Lambda^2(V) \). For \( x \in V \) denote

\[
v \circ v(x) = (v \otimes \text{id})(v)(x) + \text{cyclic permutations of } (v \otimes \text{id})(v)(x)
\]

Proposition 3.2. The coJacobi identity for the cobracket \( \lambda(x) = [x, r] + \phi(x) \) is equivalent to the following equation:

\[
\phi \circ \phi(x) + \frac{1}{2} [[r, r], x] = 0
\]

Proof. Direct calculation. \( \square \)

Proposition 3.3. Suppose \( g \) is a simple Lie algebra.

(i) If \([r, r] = 0\), then there exists a representation \( v \) of the abelian Lie algebra with the underlying vector space \( g^0 \) (we will denote it \( \tilde{g}^0 \)) on \( g^{-1} \) such that \([\text{ad}(x), \mu(y)] = \mu([x, y])\).

(ii) If \([r, r] \neq 0 \) and \([r, r] \) is the \( g \)-invariant element in \( \Lambda^3(\tilde{g}) \) (in this case \( r \) is called a solution of the modified Yang–Baxter equation), then there exist two commuting representations of the Lie algebra \( g^0 \) on \( g^{-1} \).

Proof. (i) According to the previous proposition \([r, r] = 0\) implies \( \phi \circ \phi(x) = 0 \). The Killing form enables us to associate the map \( v \circ g^0 \to \text{End}(g^1) \) with the tensor \( \phi \) (lowering the upper index). Then the equality \( \phi \circ \phi(x) = 0 \) is equivalent to \( v \) being a representation of \( \tilde{g}^0 \).

(ii) If \([r, r] \neq 0 \) then \([r, r] \) is a \( g^0 \)-invariant tensor in \( \Lambda^3(g^0) \). Such a tensor is unique up to a constant factor and equals \( \alpha \epsilon \epsilon \epsilon^{-\beta} \lambda^i \), where \( \lambda^i \) is the structure tensor of the Lie algebra \( g \), \( \epsilon \) is the inverse matrix of the Killing form and \( \alpha \) is a number. Using this fact, it is not difficult to verify that the equality \( \phi \circ \phi(x) + \frac{1}{2} [[r, r], x] = 0 \) can be rewritten in the form \( [[\mu(x), \mu(y)] - \beta \text{ad}(x, y)] = 0 \), where \( \beta \) is a number which we may regard as equal to one. Then consider the two maps \( \rho_1 : \frac{1}{2} (\text{ad}(x) + \mu(x)) \) and \( \rho_2 : \frac{1}{2} (\text{ad}(x) - \mu(x)) \) from \( g^0 \) to \( \text{End}(g^{-1}) \). A direct verification shows that these are our desired representations of \( g^0 \) on \( g^{-1} \).

Remark 1. The representations \( \rho_1 \) and \( \rho_2 \) give us a representation of \( g \oplus g \) on \( g^{-1} \) such that the restriction of this representation to the diagonally embedded Lie algebra \( g \) coincides with the adjoint representation of \( g \).
Remark 2. Both $\rho_1$ and $\rho_2$ are not equal to zero. Indeed, assume that $\rho_1 = 0$, i.e. $ad(x) = -\mu(x)$. Then the tensor $\phi_i^j$ must be skew-symmetric in the upper indices, which is impossible as we saw. Similarly, $\rho_2 \neq 0$.

Proposition 3.4. Suppose $g$ is a simple Lie algebra, $r$ is a solution of the modified Yang–Baxter equation, which determined the structure of a Lie bialgebra on $g$. Then there is no de Rham–Lie bialgebra $\tilde{g}$ having the bialgebra $g$ as the zero component, i.e. such that $\tilde{g} = g^0 \oplus g^{-1}$ and $g = g^0$.

Proof. Suppose such a de Rham–Lie algebra does exist. Then there exists, as we saw, a representation of the Lie algebra $g \oplus g$ on the vector space of the Lie algebra $g$ and the restriction of this representation to the diagonal $g \rightarrow g \oplus g$ is equal to the adjoint representation of $g$. Since the representation of $g \oplus g$ is irreducible, it splits into the tensor product of representation of the subalgebras $(g, 0)$ and $(0, g)$. Hence $ad(g) = L \otimes M$, where $L, M$ are two irreducible representations of $g$. If $\lambda_1$ and $\lambda_2$ are the highest weights of $L$ and $M$ then $ad(g)$ has to have highest weight $\lambda_1 + \lambda_2$. This is possible only in the cases $A_1$ (the highest weight equals $\pi_1 + \pi_2$), $C_1$ (the highest weight equals $2\pi_1$) and $D_3$ (the highest weight equals $\pi_2 + \pi_3$). Consider each of these cases. Below $\rho(\pi_i)$ will denote the representation which corresponds to the dominant weight $\pi_i$.

1) $A_1$ case: $\rho(\pi_1) \otimes \rho(\pi_1) = ad + id \neq ad$.

2) $C_1$ case: Since $\dim \rho(\pi_1) = 2l$ the representation $\rho(\pi_1) \otimes \rho(\pi_1)$ has a dimension $4l^2$ and cannot be equal to the representation $ad$ which has dimension $2l^2 + l$.

3) $D_3$ case: Here are similar dimensional considerations. The dimension of $ad$ equals to the dimension of $\rho(\pi_2)$ and equals 15, but the dimension of $\rho(\pi_2)$ is not equal to 1 whence the equality $ad = \rho(\pi_2) \otimes \rho(\pi_2)$ cannot hold. □

Remark. Let us point out that if $r$ is a solution of (unmodified) Yang–Baxter equation for $g$ then obviously $r$ can be also considered as a solution of the Yang–Baxter equation for the Lie algebra $\tilde{g}$ and hence $g$ makes $\tilde{g}$ into a de Rham–Lie bialgebra.

Proposition 3.5. Suppose now that $g = \bigoplus_{i=1}^k g^i$ is a semisimple Lie algebra, $g^i$ its simple components, $r \in \Lambda^2(g)$ is the solution of the modified Yang–Baxter equation of a generic form, which means that $r$ determines the solutions of the modified Yang–Baxter equation for any $g^i$. Then there is no de Rham–Lie bialgebra $\tilde{g}$ having the bialgebra $g$ as the zero component, i.e. such that $\tilde{g} = g^0 \oplus g^{-1}$ and $g = g^0$.

Proof. Suppose such a de Rham–Lie algebra does exist. Repeating the previous arguments, we obtain a representation of $g \oplus g$ on $g$ of the form $\bigoplus_{j=1}^r L_i \otimes M_j$ where $L_j$ is an irreducible representation of the subalgebra $g \oplus 0$ and $M_j$ is that of the subalgebra $0 \oplus g$. Restriction of our representation to the diagonally embedded algebra $g$ determines the adjoint representation. It is easy to see that for any simple subalgebra $g^i$ there is a term $L_j \otimes M_j$ on which the action of $g^i$ is equivalent to the
adjoint representation of $g^i$. Thus, our problem is reduced to the previous proposition. □

References