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NOTE

ADDENDUM TO "ODD AND EVEN HAMMING SPHERES ALSO HAVE MINIMUM BOUNDARY"

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In the paper "Odd and even Hamming spheres also have minimum boundary (Discrete Math. 51 (1984) 147–165), the authors presented two proofs to Theorem 2, an important result in the development of the paper. One of the proofs (in Section 3) uses a Frankl–Furedi type approach, the other (in Appendix A) relies on several technical lemmas. Here, we present another proof which is simpler than both. We follow the same notations as those used in the mentioned paper.

By Lemma 2 and algebraic manipulations, we have $\varphi^*(m, n) = \varphi^{**}(m, n) = G(m, n-1)$. Hence we can restate Theorem 2 as follows:

Theorem 2. *For every* $m \leqslant 2^{n-1}$, *we have*

$$\varphi(m, n) = G(m, n-1).$$

Proof. The easier direction $\varphi(m, n) \leqslant G(m, n-1)$ is established in the same way as in the original proof in Section 3.

Let $\mathbf{B} \subset \mathbf{X}^n$ be a set satisfying $|\mathbf{B}| = m$ and $|\Gamma\mathbf{B} - \mathbf{B}| = \varphi(m, n)$. By Lemma 2 of the original paper, $\mathbf{B}$ is a pure-parity set. Without loss of generality, assume $\mathbf{B}$ is an even-parity set.

Let $\mathbf{B}_\epsilon = \{\mathbf{x} \in \mathbf{X}^{n-1} \mid (\epsilon, \mathbf{x}) \in \mathbf{B}\}$, $\epsilon = 0, 1$, where $(\epsilon, \mathbf{x})$ is the binary n-tuple obtained by prefixing $\mathbf{x}$ with ϵ. Note that $\mathbf{B}_0$ is an even-parity set and $\mathbf{B}_1$ is an odd-parity set, hence $\mathbf{B}_0 \cap \mathbf{B}_1 = \emptyset$. Also note $|\mathbf{B}_0| + |\mathbf{B}_1| = |\mathbf{B}| = m$.

We claim $|\Gamma\mathbf{B} - \mathbf{B}| \geqslant |\Gamma'(\mathbf{B}_0 \cup \mathbf{B}_1)|$, where Γ' operates in $\mathbf{X}^{n-1}$. Then by Theorem HK of the original paper, we would have $|\Gamma'(\mathbf{B}_0 \cup \mathbf{B}_1)| \geqslant G(m, n-1)$.

We will prove our claim by exhibiting a one-to-one function f from $\Gamma'(\mathbf{B}_0 \cup \mathbf{B}_1)$ to $\Gamma\mathbf{B} - \mathbf{B}$. For an arbitrary $\mathbf{x}$ in $\Gamma'(\mathbf{B}_0 \cup \mathbf{B}_1)$, let

$$f(\mathbf{x}) = \begin{cases} (0, \mathbf{x}), & \text{if } \mathbf{x} \text{ has an odd number of 1's;} \\ (1, \mathbf{x}), & \text{otherwise.} \end{cases}$$

When $\mathbf{x}$ has an odd number of 1's, then $\mathbf{x} \in \mathbf{B}_1$ or $\mathbf{x} \in \Gamma'\mathbf{B}_0 - \mathbf{B}_0$, therefore $(0, \mathbf{x}) \in \Gamma\mathbf{B}$. Since $(0, \mathbf{x})$ has an odd number of 1's, $(0, \mathbf{x}) \notin \mathbf{B}$ so $f(\mathbf{x}) \in \Gamma\mathbf{B} - \mathbf{B}$. The case when $\mathbf{x}$ has an even number of 1's is similar. It is easy to see that f is one-to-one. Hence $|\Gamma'(\mathbf{B}_0 \cup \mathbf{B}_1)| \leqslant |\Gamma\mathbf{B} - \mathbf{B})$. □

This proof is simpler, and does not rely on Lemma 3 as the original proof in Section 3 does.