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# A unified interpretation of several combinatorial dualities

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Dedicated to the memory of George Dantzig

**Abstract**

Several combinatorial structures exhibit a duality relation that yields interesting theorems, and, sometimes, useful explanations or interpretations of results that do not concern duality explicitly. We present a common characterization of the duality relations associated with matroids, clutters (Sperner families), oriented matroids, and weakly oriented matroids. The same conditions characterize the orthogonality relation on certain families of vector spaces. This leads to a notion of abstract duality.


*Keywords:* Duality; Minors; Matroid; Clutter

## 1. Introduction

We will examine some combinatorial structures (clutters, matroids, oriented matroids, and weakly oriented matroids) and some algebraic structures (vector spaces coordinatized over such fields as $\mathbb{R}$, $\mathbb{Q}$, or $GF(p^n)$, $n$ odd) in which there are interesting duality or orthogonality relations. Although there are known resemblances among the duality relations in these different settings, a much stronger connection can be made. Specifically, we give a brief set of conditions that characterize each of the duality relations within its domain.

Each of the structures under consideration can be put in the following form. Let $\mathcal{F}$ be a family in which each $F \in \mathcal{F}$ is associated with a finite set $E(F)$. Assume further that there are operations / (contraction) and \ (deletion) that take each $F \in \mathcal{F}$ and $e \in E(F)$ to $F/e \in \mathcal{F}$ and $F\backslash e \in \mathcal{F}$, respectively, having $E(F/e) = E(F\backslash e) = E(F) - \{e\}$. We are concerned with relations $D : \mathcal{F} \to \mathcal{F}$ having such properties as:

$$E(D(F)) = E(F) \quad (\forall F \in \mathcal{F}) \tag{1.1}$$

$$D(D(F)) = F \quad (\forall F \in \mathcal{F}) \tag{1.2}$$

$$D(F/e) = D(F)\backslash e \quad \text{and} \quad D(F\backslash e) = D(F)/e \quad (\forall F \in \mathcal{F}, e \in E(F)). \tag{1.3}$$

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It is not difficult to construct trivial examples of this type. Given $\mathcal{F}$ and, say, the contraction operation, one could take deletion to be the same as contraction and $D$ to be the identity. However, there are several interesting and well-known structures in which there are established contraction and deletion operations. We show that under these operations, conditions (1.1)–(1.3) characterize the established duality relation. Moreover, the established contraction and deletion operations in these examples take the same form when viewed geometrically. The following linear algebraic example will lead us in that direction.

Let $K$ denote a field. For a given finite set $E$ let $K^E$ denote the vector space of all maps from $E$ to $K$. We denote by $\mathcal{F}_K$ the family of all pairs $F = (E(F), \mathcal{V}(F))$, where $E(F)$ is a finite set and $\mathcal{V}(F)$ is a subspace of $K^E$. We say that such a vector (sub)space is *coordinatized over* $K$. The operations of contraction (/) and deletion (\) of a coordinate $e^* \in E(F)$ correspond to projection of $F$ onto, and intersection of $F$ with, the hyperplane $x(e^*) = 0$. Specifically,

$$\begin{aligned} &E(F/e^*) = E(F) - \{e^*\} \quad \text{and} \\ &\mathcal{V}(F/e^*) = \{x : E(F/e^*) \to K \mid \exists\, x' \in \mathcal{V}(F) \text{ s.t. } x'(e) = x(e), \forall e \in E(F/e^*)\} \end{aligned} \tag{1.4}$$

$$\begin{aligned} &E(F\backslash e^*) = E(F) - \{e^*\} \quad \text{and} \\ &\mathcal{V}(F\backslash e^*) = \{x : E(F\backslash e^*) \to K \mid \exists x' \in \mathcal{V}(F) \text{ s.t. } x'(e^*) = 0 \text{ and } x'(e) = x(e), \forall e \in E(F\backslash e^*)\}. \end{aligned} \tag{1.5}$$

The orthogonality relation $D_K$ having $D_K(F) = (E(F), \{y : E(F) \to K \mid y \cdot x = 0, \forall x \in \mathcal{V}(F)\})$ for every $F \in \mathcal{F}_K$, satisfies (1.1)–(1.3). We denote by $\mathcal{F}_{GF(2)}$ and $D_{GF(2)}$ what had been denoted by $\mathcal{F}_K$ and $D_K$ in the case of $K = GF(2)$.

**Theorem 1.1.** *For $K = GF(2)$, the orthogonality relation $D_K$ is the unique $D : \mathcal{F}_K \to \mathcal{F}_K$ satisfying (1.1)–(1.3).*

In [4] we examine $\mathcal{F}_K$ for other fields $K$.

There is a natural bijection between $\mathcal{F}_{GF(2)}$ and $\mathcal{F}_b$, the family of finite binary matroids. Under this bijection, / and \ correspond to the usual matroid contraction and deletion operations (see [23,26]), and orthogonality corresponds to matroid duality. This gives another interpretation of Theorem 1.1. Let $D_b$ denote the matroid duality relation restricted to $\mathcal{F}_b$.

**Corollary 1.2.** *For $\mathcal{F} = \mathcal{F}_b$, the relation $D_b$ is the unique $D : \mathcal{F} \to \mathcal{F}$ satisfying (1.1)–(1.3).*

In the next example we expand the viewpoint from $\mathcal{F}_b$ to $\mathcal{F}_M$, the family of all matroids $F$ on a finite set of elements $E(F)$. We take / and \, respectively, to be the usual matroid contraction and deletion operations.

**Theorem 1.3.** *For $\mathcal{F} = \mathcal{F}_M$, the matroid duality relation $D_M$ is the unique $D : \mathcal{F} \to \mathcal{F}$ satisfying conditions (1.1)–(1.3).*

Another example comes from *Clutters* (also called *Sperner families*). A clutter $F$ can be described by a finite set $E(F)$ and a set $\mathcal{P}(F)$ of noncomparable subsets of $E(F)$; we will call the members of $\mathcal{P}(F)$ the *minimal dependent sets* of $F$. For discussion of the blocking duality and the operations of contraction and deletion on clutters see Edmonds and Fulkerson [14] and Seymour [24]. These concepts are rooted in work of Lehman (see the 1965 preprint of his paper [21]). Let $\mathcal{F}_S$ be the family of all clutters $F$ on a finite set $E(F)$, take / and \ respectively, to be the usual contraction and deletion operations in this setting, and let $D_S$ be the blocking duality relation on $\mathcal{F}_S$. We have the following analog of Theorem 1.1:

**Theorem 1.4.** *For $\mathcal{F} = \mathcal{F}_S$, $D_S$ is the unique $D : \mathcal{F} \to \mathcal{F}$ satisfying conditions (1.1)–(1.3).*

G. Kalai pointed out to us that the matroidal result, Theorem 1.3, is a strengthening of a result from Kung [20]. Kung proved the version of Theorem 1.3 in which one imposes the additional restriction that $D$ preserves isomorphisms,

$$F_1 = \psi\,(F_2) \Rightarrow D(F_1) = \psi(D(F_2)) \quad \forall F_1, F_2 \in \mathcal{F} \text{ and isomorphisms } \psi \text{ from } F_2 \text{ to } F_1. \tag{1.6}$$

Moreover, (1.1)–(1.3) together with (1.6) characterize the standard duality relations $D_O$ on $\mathcal{F}_O$, the family of all oriented matroids $F$ on a finite set $E(F)$, and $D_W$ on $\mathcal{F}_W$, the family of all weakly oriented matroids $F$ on a finite set $E(F)$. Here we again take / and \ to be the usual contraction and deletion operations in these settings. See [6,1,5], respectively, for basic results on oriented matroids and weakly oriented matroids.

**Theorem 1.5.** (a) *For $\mathcal{F} = \mathcal{F}_O$, $D_O$ is the unique $D : \mathcal{F} \to \mathcal{F}$ satisfying* (1.1)–(1.3) *and* (1.6).
(b) *For $\mathcal{F} = \mathcal{F}_W$, $D_W$ is the unique $D : \mathcal{F} \to \mathcal{F}$ satisfying* (1.1)–(1.3) *and* (1.6).

The proof of Theorem 1.3 in Section 4 reveals that for any subfamily $\mathcal{F}_{SM}$ of $\mathcal{F}_M$ that is closed under matroid duality, contraction, and deletion, and includes all matroids on two or fewer elements, the restriction of matroid duality to $\mathcal{F}_{SM}$ is the unique map $D : \mathcal{F}_{SM} \to \mathcal{F}_{SM}$ satisfying conditions (1.1)–(1.3). So, for example, the family $\mathcal{F}_{PM}$ of planar graphic matroids has this property. Within this subfamily of $\mathcal{F}_M$ matroid duality behaves like Whitney's combinatorial duality on planar graphs [27]. This example suggests why the isomorphism-preserving property (1.6), which is not necessary for the matroid duality uniqueness result, Theorem 1.3, is needed in the oriented case, Theorem 1.5. Planar graph duality extends easily to directed graphs, but in this realm, corresponding to planar graphic oriented matroids, other things can happen.

Roughly, each planar graphic matroid $F$ corresponds to a planar graph $G(F)$ whose edge set is $E(F)$; $G(F)$ is not unique. The matroid duality function $D_M$ maps $F$ to $F' = D_M(F)$, which corresponds to a planar graph $G(F')$. Furthermore the graphs $G(F)$ and $G(F')$ are combinatorially dual, i.e., when restricted to planar graphic matroids, matroid duality behaves like Whitney's planar graph duality [27], and it is the unique map on planar graphic matroids that satisfies conditions (1.1)–(1.3).

Suppose now we orient the planar graphs and the associated matroids. Let $\mathcal{F}_{PO}$ be the family of planar graphic oriented matroids. Each $F \in \mathcal{F}_{PO}$ corresponds to a directed planar graph $G(F)$ whose edge set is $E(F)$. The oriented matroid duality function $D_O$ maps $F$ to $F' = D_O(F)$, which corresponds to a directed planar graph $G(F')$. Furthermore the directed graphs $G(F)$ and $G(F')$ are combinatorially dual, i.e., when restricted to directed planar graphic matroids, oriented matroid duality behaves like the natural oriented extension of Whitney's planar graph duality. But so does the map $\hat{D}$ that arises in the following way. Choose a singleton $\{e\}$; $e \in E(F)$ for some, but not all of the planar graphic oriented matroids $F$. Let $F_1 \in \mathcal{F}_{PO}$ with $e \in E(F_1)$. Closely related to $F_1$ and to $G(F_1)$ is another planar graphic oriented matroid $F_2 \in \mathcal{F}_{OM}$ with $E(F_1) = E(F_2)$ and such that the associated directed planar graphs $G(F_1)$ and $G(F_2)$ differ only in the orientation of edge $e$. Now, we could set $\hat{D}(F_1) = D(F_2)$ and $\hat{D}(F_2) = D(F_1)$. If we carry out this dual swapping for every planar graphic oriented matroid $F$ having $e \in E(F)$, but otherwise let $\hat{D}(F) = D(F)\ \forall F \in \mathcal{F}_{PO}, e \notin E(F)$, then the resulting $\hat{D}$ also satisfies conditions (1.1)–(1.3). However, this $\hat{D}$ will violate condition (1.6). On the directed planar graphs themselves this variation corresponds to mapping directed graph $G(F)$ to a combinatorially dual graph $H$, and then reversing the orientation of edge $e$ in $H$, if $e \in E(H)$. Of course, it can be carried out on multiple edges, and corresponds to reversing signs in the associated oriented matroids.

Therefore we can specify exactly the role played by the isomorphism-preserving condition in Theorem 1.5. In these settings every map $D'$ satisfying (1.1)–(1.3) arises from the unique map $D$ satisfying (1.1)–(1.3) and (1.6) by reversing signs on a fixed subset of elements (see the Appendix for an outline of a proof).

The inclusion of (1.6) with (1.1)–(1.3) also enables us to extend Theorem 1.1 to vector spaces coordinatized over fields other than $GF(2)$. First note that for any field $K$, all of $\mathcal{F}_K$, /, \, and $D_K$ remain well-defined. The condition (1.6) on preserving isomorphisms can be formulated as follows: Let $F_1, F_2 \in \mathcal{F}_K$. Every $x^1 \in \mathcal{V}(F_1)$ is a map from $E(F_1)$ to $K$ and every $x^2 \in \mathcal{V}(F_2)$ is a map from $E(F_2)$ to $K$. Given a bijection $\psi$ from $E(F_2)$ to $E(F_1)$, we can examine the maps from $E(F_2)$ to $K$ that arise by composing $x \circ \psi$ for all $x \in \mathcal{V}(F_1)$. If $\{x \circ \psi : x \in \mathcal{V}(F_1)\} = \mathcal{V}(F_2)$, we say that $\psi$ is an *isomorphism* from $F_2$ to $F_1$. Condition (1.6) requires that for all $F_1, F_2 \in \mathcal{F}_K$ and for all isomorphisms $\psi$ from $F_2$ to $F_1$

$$\mathcal{V}(D(F_2)) = \{y \circ \psi \mid y \in \mathcal{V}(D(F_1))\}.$$

**Theorem 1.6.** *For any field $K$ having no nontrivial involutary automorphisms, (e.g., $K = \mathbb{R}$, $\mathbb{Q}$, or $GF(p^n)$, for $p$ prime and $n$ odd) conditions* (1.1)–(1.3) *and* (1.6) *characterize the orthogonality relation $D_K$ on $\mathcal{F}_K$.*

The results outlined in this section indicate that properties (1.1)–(1.3) and (1.6) characterize the duality relations in each of several different settings, *when we take the operations / and \ in* (1.3) *to be the standard contraction and deletion operations in the relevant setting*. We shall see that there is a common description of the contraction and deletion operations across these examples, which leads to the definition of an abstract duality relation.

Section 2 defines the notion of *abstract duality* after introducing a general notation embracing all of the examples. The main result of this paper is that each of the structures noted above has a unique abstract duality. Section 3 concerns some fundamental properties of abstract dualities that are used in Section 4 to prove the uniqueness results for matroids and clutters. The uniqueness proofs for oriented matroids and weakly oriented matroids are presented in the Appendix. The results for coordinatized vector spaces are in the companion paper [4]. This study was initially prompted by the question of whether *anti matroids* (also known as convex geometries or anti-exchange closures) [10, 12,13,18,19] admit a relation reminiscent of the duality relation in matroids. Anti matroids have a natural description in the notation of Section 2, which determines contraction and deletion operations for anti matroids. However, with these operations anti matroids have no abstract duality (see Dietrich [9,11] and [3]).

Another example of this sort arises from polymatroids. Whittle [28] has shown that there is no involution on the family of polymatroids that preserves the ground set, preserves isomorphisms, and interchanges natural contraction and deletion operations on polymatroids. However, he also shows that for every positive integer $k$, there is a unique map of this sort on the subfamily of $k$-polymatroids, and, in the special case of $k = 1$, it reduces to matroid duality.

The research announcement [2] gave a brief discussion of this work, which was first presented in the Ph.D. dissertation of the second author [9]. Additional details on the combinatorial structures studied here can be found in [1,5–7,9,14–17,23,25,26]. Note that our attention is limited to structures $F$ with $E(F)$, the underlying set of elements, or coordinates, finite.

## 2. Abstract dualities

In the introduction we discussed briefly families $\mathcal{F}_K$ of vector spaces coordinatized over a field $K$, and the special case of $\mathcal{F}_{GF(2)}$. The contraction (1.4) and deletion (1.5) operations for $\mathcal{F}_K$ have very simple and closely related forms in terms of $(E(F), \mathcal{V}(F))$. In this section we shall see how (1.4) and (1.5) can serve as the definitions of contraction and deletion across all of the families of combinatorial and algebraic examples of interest here, and in the general context of abstract duality. Contraction and deletion will still correspond to projection and intersection with coordinate hyperplanes from a set of vectors (maps) $\mathcal{V}(F)$. However, for the combinatorial families, the set $\mathcal{V}(F)$ will not necessarily be a vector space. We illustrate the approach with matroids.

Any matroid $F \in \mathcal{F}_M$ can be described (encoded) in many different, but equivalent ways: in terms of circuits, bases, independent sets, rank function, closure operator, etc. The matroid contraction and deletion operations can be characterized in terms of any one of these encodings. Note that the properties in Theorem 1.3 do not depend on the notation of any one encoding. The form of the duality relation as a map from $\mathcal{F}_M$ to $\mathcal{F}_M$ does not depend on a specific choice of how to encode matroids, nor does the form of the contraction (or deletion) operation as a map that takes each $F \in \mathcal{F}_M$ and $e \in E(F)$ to some *minor* $G \in \mathcal{F}_M$ with $E(G) = E(F)\backslash\{e\}$.

For $F \in \mathcal{F}_M$ and $e \in E(F)$, let $F/e$ and $F\backslash e$ denote the standard matroid contraction and deletion operations, respectively. Our immediate aim is to give an encoding of matroids $F \in \mathcal{F}_M$ as $(E(F), \mathcal{V}(F))$ such that $\mathcal{V}(F)$ is a set of vectors and:

$$\mathcal{V}(F/e) \quad \text{and} \quad \mathcal{V}(F\backslash e) \quad \text{are given by (1.4) and (1.5), respectively.} \tag{2.1}$$

(Note that an encoding with these features is not necessarily unique.) For every matroid $F \in \mathcal{F}_M$ let $\mathcal{V}(F)$ be the set of (0, 1)-incidence vectors of unions of circuits of $F$. Given $E(F)$ and $\mathcal{V}(F)$ it is easy to recover the circuits of $F$, so matroids can be encoded by $(E(F), \mathcal{V}(F))$. Moreover, it is easy to verify that this choice of $\mathcal{V}(F)$ for matroids satisfies (2.1).

For each family $\mathcal{F}$ of interest, there will be an associated *target set* $T$ with a distinguished (zero) element $o$ and an encoding $(E(F), \mathcal{V}(F))$ of the $F \in \mathcal{F}$, where $\mathcal{V}(F)$ is a set of maps (vectors) from $E(F)$ to $T$. For $\mathcal{F}_M$, $T = \{0, 1\}$ and $o = 0$.

Typically the set of vectors $\mathcal{V}(F)$ is not a vector space. In some of the combinatorial examples, including matroids and clutters, it is the set of incidence vectors of a set of subsets of $E(F)$.

We are now ready to develop formally the notion of an abstract duality.

$$\begin{array}{l}\text{Let } T \text{ be a nonempty set with a distinguished element } o, \text{ and let the nonempty family } \mathcal{F} \text{ have}\\ \quad \text{for every } F \in \mathcal{F} \text{ a set } \mathcal{V}(F) \text{ of maps from a finite set } E(F) \text{ to } T.\end{array} \tag{2.2}$$

Suppose further that $\mathcal{F}$ is closed under the operations of *contraction* (/) and *deletion* (\), which are defined as follows. For any $F \in \mathcal{F}$ and $e^* \in E(F)$,

$$E(F/e^*) = E(F\backslash e^*) = E - \{e^*\} \tag{2.3}$$

Table 1
The unifying framework

| $\mathcal{F}$ | $\mathcal{V}(F)$ | $T$ | $o$ |
|---|---|---|---|
| $\mathcal{F}_M$: Matroids | Incidence vectors of unions of circuits of matroid $F \in \mathcal{F}_M$ | $\{0, 1\}$ | 0 |
| $\mathcal{F}_S$: Clutters | Incidence vectors of supersets of minimal dependent sets of $F \in \mathcal{F}_S$ | $\{0, 1\}$ | 0 |
| $\mathcal{F}_O$: Oriented matroids | Signed incidence vectors of conformal unions of signed circuits of $F \in \mathcal{F}_O$ | $\{-, 0, +\}$ | 0 |
| $\mathcal{F}_W$: Weakly oriented matroids | Signed incidence vectors of consistent unions of signed circuits of $F \in \mathcal{F}_W$ | $\{-, 0, +\}$ | 0 |
| $\mathcal{F}_K$: Vector spaces coordinatized over field $K$ | Vectors in $F \in \mathcal{F}_K$ | $K$ | 0 |

$$\mathcal{V}(F/e^*) = \{x : E(F/e^*) \to T \mid \exists x' \in F \text{ with } x'(e) = x(e),\ \forall e \in E(F/e^*)\},$$
$$\mathcal{V}(F \backslash e^*) = \{x : E(F \backslash e^*) \to T \mid \exists x' \in F \text{ with } x'(e^*) = o, \text{ and } x'(e) = x(e),\ \forall e \in E(F \backslash e^*)\}.$$

A function $D : \mathcal{F} \to \mathcal{F}$ is a *weak abstract duality* on $\mathcal{F}$ if it preserves the ground set:

$$E(D(F)) = E(F) \quad (\forall F \in \mathcal{F}), \tag{2.4}$$

is an involution:

$$D(D(F)) = F \quad (\forall F \in \mathcal{F}), \tag{2.5}$$

and interchanges contraction and deletion:

$$D(F/e) = D(F) \backslash e \quad (\forall F \in \mathcal{F},\ e \in E(F)), \tag{2.6}$$
$$D(F \backslash e) = D(F)/e \quad (\forall F \in \mathcal{F},\ e \in E(F)).$$

An isomorphism from $F_2 \in \mathcal{F}$ to $F_1 \in \mathcal{F}$ is a bijection $\phi : E(F_2) \to E(F_1)$ such that $\mathcal{V}(F_2) = \{x \circ \phi \mid x \in \mathcal{V}(F_1)\}$. A weak abstract duality on $\mathcal{F}$ is an *abstract duality* if it preserves isomorphisms:

$$\mathcal{V}(D(F_2)) = \{y \circ \phi \mid y \in \mathcal{V}(D(F_1))\} \quad \forall F_1, F_2 \in \mathcal{F}, \text{ and } \phi \text{ an isomorphism from } F_2 \text{ to } F_1. \tag{2.7}$$

Table 1 indicates how the earlier examples fit into this notation.

Note that (2.2) allows for the possibility that $E(F)$ is empty. In this case, the set $\mathcal{V}(F)$ may also be empty, but we allow for another possibility: $\mathcal{V}(F)$ may consist of the *empty map*. For example, in the case of clutters, this allows for the two possible clutters on $E = \emptyset$: $F_1$ and $F_2$ having $E(F_1) = E(F_2) = \emptyset$ and $\mathcal{P}(F_1) = \emptyset$, and $\emptyset \in \mathcal{P}(F_2)$.

It should be evident that each of $\mathcal{F}_M, \mathcal{F}_S, \mathcal{F}_O, \mathcal{F}_W$, and $\mathcal{F}_K$ (for arbitrary field $K$) has at least one abstract duality, namely the standard duality in that setting: $D_M$, $D_S$, $D_O$, $D_W$, and $D_K$, respectively. We can now rephrase Theorems 1.3–1.6.

**Theorem 2.1.** (a) *$D_S$ is the unique weak abstract duality on $\mathcal{F}_S$;*
(b) *$D_M$ is the unique weak abstract duality on $\mathcal{F}_M$;*
(c) *$D_O$ is the unique abstract duality on $\mathcal{F}_O$;*
(d) *$D_W$ is the unique abstract duality on $\mathcal{F}_W$;*
(e) *for every field $K$ having no nontrivial involutary automorphism $D_K$ is the unique abstract duality on $\mathcal{F}_K$.*

We noted earlier that there may be distinct encodings for a family having an abstract duality. The example of $\mathcal{F}_b$, binary matroids, has distinct encodings, each of which is very natural, but from different perspectives. We observed that there is a natural bijection between $\mathcal{F}_b$ and $\mathcal{F}_{GF(2)}$, so we could use the encoding from Table 1 for $\mathcal{F}_K$ with $K = GF(2)$. Alternatively, since $\mathcal{F}_b$ is a subfamily of $\mathcal{F}_M$, a subfamily closed under duality, contraction and deletion, we could use the encoding for $\mathcal{F}_M$. These distinct encodings share the same characterization of contraction and deletion on binary matroids.

Another example of this phenomenon comes from oriented matroids and weakly oriented matroids. Since $\mathcal{F}_O$ is a subfamily of $\mathcal{F}_W$, we could have used the same encoding (the $\mathcal{F}_W$ encoding in Table 1) for both. We have listed a different encoding for $\mathcal{F}_O$, because it relates closely to ideas that are fundamentally important in certain applications of oriented matroid theory.

Note also that we need to avoid ambiguity in what we mean by $\mathcal{V}(F)$ when $F$ is a member of more than one family, e.g., if $F$ is an oriented matroid then it is also a weakly oriented matroid, but we are using different encodings for $\mathcal{F}_O$

and $\mathcal{F}_W$. We will only resort to more cumbersome notation of the form $\mathcal{V}^O(F)$, $\mathcal{V}^W(F)$, to distinguish the different encodings when it is not clear from the context which family, and, therefore, which encoding, is under consideration.

There is a common approach to proving the five parts of Theorem 2.1, based on a reconstructibility property.

## 3. Reconstructibility

For each of the five examples in Table 1 of Section 2, it is easy to show that for all $F \in \mathcal{F}$ with $E(F)$ sufficiently large, $F$ is determined uniquely by its set of *simple minors*:

$$\{F/e \mid e \in E(F)\} \cup \{F\backslash e \mid e \in E(F)\}. \tag{3.1}$$

Such an $F$ is called *reconstructible*. It is not difficult to see why one might expect reconstructibility when $|E(F)|$ is sufficiently large. For $F \in \mathcal{F}$ having $|E(F)| \geq 2$, let $\mathcal{M}$ be the set (3.1) of simple minors of $F$. First note that $E(F)$ is just the union of $E(F')$ over $F' \in \mathcal{M}$. Also note that for any $e \in E(F)$ there are at most two $F' \in \mathcal{M}$ having $E(F') = E(F) - \{e\}$, namely $F/e$ and $F\backslash e$. In the *degenerate* case where $\mathcal{V}(F)/e = \mathcal{V}(F)\backslash e$, then $F/e = F\backslash e$ is the unique $F' \in \mathcal{M}$ having $E(F') = E(F) - \{e\}$. Otherwise, $F/e \neq F\backslash e$, and it is easy to distinguish between these two because $\mathcal{V}(F\backslash e) \subsetneq \mathcal{V}(F/e)$. Hence given $\{(E(F'), \mathcal{V}(F')) : \forall F' \in \mathcal{M}\}$ we can determine which of these simple minors of $F$ arose by contraction and which by deletion, and, in each case, on which element. Next note that if we extend each $x \in \mathcal{V}(F)\backslash e$ to a map $x' : E(F) \to T$ having $x'(e) = o$, then $x' \in F$. Typically $F$ can be generated unambiguously from these $x'$, except in the degenerate cases, and then the structure of $F$ can be determined from the contraction minors. For example, if $F \in \mathcal{F}_{GF(2)}$, then the only ambiguity arises when for every choice of $e \in E(F)$, $\mathcal{V}(F\backslash e)$ contains only the zero vector $(0, \ldots, 0)$. In this circumstance, $\mathcal{V}(F)$ could be either $\{(0, \ldots, 0)\}$ or $\{(0, \ldots, 0), (1, \ldots, 1)\}$, but the contraction minors $F/e$ immediately reveal which of the two possibilities holds. Note that if $|E(F)| = 1$, then $F$ cannot be reconstructed, since we cannot even recover $E(F)$.

Let $r(\mathcal{F})$ be the least integer $r$ such that every $F \in \mathcal{F}$ having $|E(F)| \geq r$ is reconstructible; write $r(\mathcal{F}) = \infty$ if there exists no such integer $r$. As we shall see $r(\mathcal{F}_M) = r(\mathcal{F}_S) = 2$; $r(\mathcal{F}_O) = r(\mathcal{F}_W) = 3$; $r(\mathcal{F}_{GF(2)}) = 2$. In [4] we show that in general $r(\mathcal{F}_K) \leq 3$.

We will say that $F \in \mathcal{F}$ is *small* if $|E(F)| < r(\mathcal{F})$.

**Theorem 3.1.** *If $D_1$ and $D_2$ are distinct weak abstract dualities on $\mathcal{F}$, then $D_1(F) \neq D_2(F)$ for a small $F \in \mathcal{F}$.*

The proof of Theorem 3.1 uses the following lemma. For $q \in Z_+$, denote by $\mathcal{F}^q$ the subfamily $\{F \in \mathcal{F} : q \geq |E(F)|\}$ of $\mathcal{F}$.

**Lemma 3.2.** *Suppose $D_1$ and $D_2$ are weak abstract dualities on $\mathcal{F}$ and for some nonnegative integer $q$:* (i) $D_1(F) = D_2(F)$, $\forall F \in \mathcal{F}^q$; (ii) *every $\hat{F} \in \mathcal{F} - \mathcal{F}^q$ is reconstructible. Then $D_1 = D_2$.*

**Proof.** Suppose that $\hat{F} \in \mathcal{F}$ has $|E(\hat{F})| = q + 1$. Then $D_1(\hat{F})$ is determined uniquely by

$$\mathcal{M} = \{D_1(\hat{F})\backslash e \mid e \in E(\hat{F})\} \cup \{D_1(\hat{F})/e \mid e \in E(\hat{F})\}. \tag{3.2}$$

By (2.6) and (3.2)

$$\mathcal{M} = \{D_1(\hat{F}/e) \mid e \in E(\hat{F})\} \cup \{D_1(\hat{F}\backslash e) \mid e \in E(\hat{F})\}. \tag{3.3}$$

Now each simple minor of $\hat{F}$ in (3.3) has $q$ elements so by (i)

$$\mathcal{M} = \{D_2(\hat{F}/e) \mid e \in E(\hat{F})\} \cup \{D_2(\hat{F}\backslash e) \mid e \in E(\hat{F})\}. \tag{3.4}$$

Again using (2.6), with (3.4) we get

$$\mathcal{M} = \{D_2(\hat{F})\backslash e \mid e \in E(\hat{F})\} \cup \{D_2(\hat{F})/e \mid e \in E(\hat{F})\}. \tag{3.5}$$

Since $E(D_1(\hat{F}))$ and $E(D_2(\hat{F}))$ both have $q + 1$ elements, by (ii) $D_1(\hat{F})$ and $D_2(\hat{F})$ are reconstructible from their sets of simple minors, which, by (3.2) and (3.5), are identical. Therefore, $D_1(\hat{F}) = D_2(\hat{F})$ for all $\hat{F} \in \mathcal{F}^{q+1}$, i.e., (i) holds with $q$ replaced by $q + 1$. Since $\mathcal{F} - \mathcal{F}^{q+1} \subseteq \mathcal{F} - \mathcal{F}^q$, (ii) also holds with $q$ replaced by $q + 1$. By induction, $D_1 = D_2$. □

**Proof of Theorem 3.1.** If $r(\mathcal{F}) = \infty$, then there is nothing to prove, since all $F \in \mathcal{F}$ are small. So assume that $r(\mathcal{F})$ is finite.

We will attend first to the case where $r(\mathcal{F}) > 0$. Let $D_1$ and $D_2$ be weak abstract dualities on $\mathcal{F}$, and suppose that the conclusion of Theorem 3.1 fails. Then $D_1(F) = D_2(F)$ for all $F \in \mathcal{F}^q$, with $q = r(\mathcal{F}) - 1$. Conditions (i) and (ii) of Lemma 3.2 both hold for this choice of $q$. Hence, $D_1 = D_2$, contradicting the hypothesis that $D_1$ and $D_2$ are distinct.

Now consider the case where $r(\mathcal{F}) = 0$. This implies that any $F \in \mathcal{F}$ with $E(F) = \emptyset$ is reconstructible from its empty set of simple minors. Thus there is a unique $F \in \mathcal{F}$ with $E(F) = \emptyset$. Hence all weak abstract dualities $D_1$ and $D_2$ on $\mathcal{F}$ agree on $\mathcal{F}^q$ with $q = r(\mathcal{F}) = 0$, and conditions (i) and (ii) of Lemma 3.2 both hold for this choice of $q$. Hence $D_1 = D_2$. □

**Corollary 3.3.** *If all (weak) abstract dualities $D_1$ and $D_2$ on $\mathcal{F}$ agree on all small $F \in \mathcal{F}$, then $\mathcal{F}$ has at most one (weak) abstract duality.*

Establishing Theorem 2.1 now reduces to establishing the hypothesis of Corollary 3.3 in each of the five families. In the combinatorial cases, $\mathcal{F}_M, \mathcal{F}_S, \mathcal{F}_O, \mathcal{F}_W$, this is facilitated by the modest size of $r(\mathcal{F})$ and by the following lemma. For any fixed finite set $E$ let $\mathcal{F}(E)$ be the subfamily $\{F \in \mathcal{F} \mid E(F) = E\}$.

**Lemma 3.4.** *Let $D$ be a weak abstract duality on $\mathcal{F}$, let $E$ be a finite set, and let $e^* \in E$ and $F^* \in \mathcal{F}(E - \{e^*\})$ be fixed. Then the restriction of $D$ to $\mathcal{F}_1 = \{F \in \mathcal{F}(E) \mid F \backslash e^* = F^*\}$ is a bijection from $\mathcal{F}_1$ to $\mathcal{F}_2 = \{F \in \mathcal{F}(E) \mid F/e^* = D(F^*)\}$.*

**Proof.** Suppose $F \in \mathcal{F}_1$. Then $F \backslash e^* = F^*$, so $D(F \backslash e^*) = D(F^*)$, and by (2.6), $D(F \backslash e^*) = D(F)/e^*$. Therefore, $D(F) \in \mathcal{F}_2$, so $D$ restricted to $\mathcal{F}_1$ has its range in $\mathcal{F}_2$. Also, it must be one-to-one, since $D$ is one-to-one on all of $\mathcal{F}$, by (2.5). Now suppose that $F \in \mathcal{F}_2$, which implies that $F/e^* = D(F^*)$. Then $D(F/e^*) = D(D(F^*))$, which is $F^*$ by (2.5). Furthermore, by (2.6), $D(F/e^*) = D(F)\backslash e^*$, so $D(F)\backslash e^* = F^*$, which implies that $D(F) \in \mathcal{F}_1$. By (2.5) $F$ is the image under $D$ of $D(F)$, so $D$ restricted to $\mathcal{F}_1$ is onto $\mathcal{F}_2$. □

In the next section we will complete the proof of parts (a) and (b) of Theorem 2.1, by establishing the hypothesis of Corollary 3.3 for clutters and matroids. The proofs for oriented matroids and weakly oriented matroids are given in the Appendix. The more difficult proof for part (e), coordinatized vector spaces, is in the companion paper [4].

Before proceeding, some additional notation will be useful in piecing together simple deletion minors. Let $T$ and $\mathcal{F}$ be as in (2.2), and let $F \in \mathcal{F}$. For an element $e \notin E(F)$ let $\mathcal{V}_e(F)$ denote the *extension* to $E(F) \cup \{e\}$ of $\mathcal{V}(F)$:

$$\mathcal{V}_e(F) = \{x : E(F) \cup \{e\} \to T \mid x(e) = o, \text{ and } \exists y \in \mathcal{V}(F) \text{ s.t. } x(e') = y(e'), \forall e' \in E(F)\}.$$

For any $t \in T$ let $t^E$ denote the map from $E$ to $T$ having $t^E(e) = t$, for all elements $e \in E$. In particular, $o^E$ denotes the zero map (vector) on $E$. The unique map from the empty set to any $T$ will be denoted by $o^\phi$. (When we examine some $F \in \mathcal{F}$ with $E(F) = \emptyset$ it is important to be able to distinguish between the case where the set of vectors $\mathcal{V}(F)$ is empty and the separate case where it contains only the empty vector $o^\phi$.)

## 4. Matroids and clutters: Uniqueness proofs

For matroids and clutters, the target set $T = \{0, 1\}$. Given any finite set $E$ and maps $x^i : E \to \{0, 1\}, i = 1, \ldots, k$, we will denote by $\bigsqcup_{i=1}^k x^i$ the incidence vector of the union of the subsets of $E$ underlying $x^1, \ldots, x^k$, i.e.,

$$\bigsqcup_{i=1}^{k} x^i(e) = \begin{cases} 0 & \text{if } x^i(e) = 0 \; \forall i = 1, \ldots, k \\ 1 & \text{otherwise.} \end{cases}$$

### 4.1. Matroids

The following lemma proves part (b) of Theorem 2.1 by establishing the hypothesis of Corollary 3.3 for $\mathcal{F}_M$.

**Lemma 4.1.** (a) *Every $F \in \mathcal{F}_M$ with $|E(F)| \geq 2$ is reconstructible.*
(b) *All weak abstract dualities on $\mathcal{F}_M$ agree on the subfamily $\mathcal{F}_M^1$.*

**Proof.** (a) Choose $F \in \mathcal{F}_M$ having $|E(F)| \geq 2$. Let $\mathcal{M}$ be the set of simple minors of $F$. Note that $E(F) = \cup\{E(G) : G \in \mathcal{M}\}$. Let $F'$ have $E(F') = E(F)$ and $\mathcal{V}(F') = \{\bigsqcup x^e : \forall e \in E(F) \;\; x^e \in \mathcal{V}_e(F \backslash e)\}$. If $\mathcal{V}(F') \neq \{0^{E(F)}\}$, then $\mathcal{V}(F) = \mathcal{V}(F')$. Suppose, on the other hand, $\mathcal{V}(F') = \{0^{E(F)}\}$. If $\mathcal{V}(F/e) = \{0^{E(F)-\{e\}}\}$ for any $e \in E(F)$, then $\mathcal{V}(F) = \{0^{E(F)}\}$; otherwise $\mathcal{V}(F) = \{0^{E(F)}, 1^{E(F)}\}$.
(b) Let $D : \mathcal{F}_M \to \mathcal{F}_M$ be a weak abstract duality.

In order to show that all weak abstract dualities agree on $\mathcal{F}^1_M$ we will need to examine the structure of all matroids on two or fewer elements. There is only one matroid $F_0 \in \mathcal{F}_M(\emptyset)$. By (2.4), $F_0$ must be its own image under $D$. For any fixed singleton $\{e\}$ there are two matroids $F_1(\{e\})$ and $F_2(\{e\})$ in $\mathcal{F}_M(\{e\})$. They have $\mathcal{V}(F_1(\{e\})) = \{(0), (1)\}$ and $\mathcal{V}(F_2(\{e\})) = \{(0)\}$. Here each map $x$ is denoted by the image of the element $e$ under $x$. Now, if we start with a matroid on two distinct elements, $e_1$ and $e_2$, and either delete or contract $e_2$, we must get as a minor one of $F_1(\{e_1\})$ or $F_2(\{e_1\})$. This will allow us to apply Lemma 3.4 to determine how $D$ behaves on $\mathcal{F}_M(\{e_1\})$.

There are five matroids, $F_1(\{e_1, e_2\}), \ldots, F_5(\{e_1, e_2\}) \in \mathcal{F}(\{e_1, e_2\})$ for any fixed pair of distinct elements $e_1$ and $e_2$, with

$$\mathcal{V}(F_1(\{e_1, e_2\})) = \{(0, 0)\} \qquad \mathcal{V}(F_4(\{e_1, e_2\})) = \{(0, 0), (1, 0), (0, 1), (1, 1)\}.$$
$$\mathcal{V}(F_2(\{e_1, e_2\})) = \{(0, 0), (0, 1)\} \qquad \mathcal{V}(F_5(\{e_1, e_2\})) = \{(0, 0), (1, 1)\}.$$
$$\mathcal{V}(F_3(\{e_1, e_2\})) = \{(0, 0), (1, 0)\}.$$

Here each map $x : \{e_1, e_2\} \to \{0, 1\}$ is denoted by the ordered pair $(x(e_1), x(e_2))$.

Let $\mathcal{Y} = \{F_i(\{e_1, e_2\}) \mid F_i(\{e_1, e_2\})/e_2 = F_1(\{e_1\})\}$, $\mathcal{Y}' = \{F_i(\{e_1, e_2\}) \mid F_i(\{e_1, e_2\})/e_2 = F_2(\{e_1\})\}$, $\mathcal{Z} = \{F_i(\{e_1, e_2\}) \mid F_i(\{e_1, e_2\})\backslash e_2 = F_1(\{e_1\})\}$, and $\mathcal{Z}' = \{F_i(\{e_1, e_2\}) \mid F_i(\{e_1, e_2\})\backslash e_2 = F_2(\{e_1\})\}$. Note that $\mathcal{Y} = \{F_3(\{e_1, e_2\}), F_4(\{e_1, e_2\}), F_5(\{e_1, e_2\})\}$, $\mathcal{Y}' = \{F_1(\{e_1, e_2\}), F_2(\{e_1, e_2\})\}$, $\mathcal{Z} = \{F_3(\{e_1, e_2\}), F_4(\{e_1, e_2\})\}$, $\mathcal{Z}' = \{F_1(\{e_1, e_2\}), F_2(\{e_1, e_2\}), F_5(\{e_1, e_2\})\}$. In particular, $|\mathcal{Y}| = 3$, $|\mathcal{Y}'| = 2$, $|\mathcal{Z}| = 2$, $|\mathcal{Z}'| = 3$. Lemma 3.4 with $e^* = e_2$ and $F^* = F_2(\{e_1\})$ implies that $D : \mathcal{Z}' \to \mathcal{Y}$, and, therefore, $D(F_2(\{e_1\})) = F_1(\{e_1\})$. By (2.5), $F_2(\{e_1\}) = D(F_1(\{e_1\}))$. This is independent of the choice of $e_1$, so $D$ is uniquely determined on $\mathcal{F}^1_M$. $\square$

Before we proceed to proving the part of Theorem 2.1 concerning clutters, note that the form of the proof for matroids implies a unique weak abstract duality for interesting subfamilies of $\mathcal{F}_M$.

Suppose that

$$\tilde{\mathcal{F}}_M \subseteq \mathcal{F}_M \text{ is closed under contraction, deletion, and } D_M. \tag{4.1}$$

Note that $D_M$ restricted to $\tilde{\mathcal{F}}_M$ is a weak abstract duality on $\tilde{\mathcal{F}}_M$. Also, reconstructibility of all $F \in \tilde{\mathcal{F}}_M - \tilde{\mathcal{F}}^1_M$ is immediate. If, in addition to (4.1),

$$\mathcal{F}^2_M \subseteq \tilde{\mathcal{F}}_M, \tag{4.2}$$

then it follows that all weak abstract dualities on $\tilde{\mathcal{F}}_M$ agree on $\tilde{\mathcal{F}}^1_M$, since the proof of Lemma 4.1b appealed only to $F \in \mathcal{F}^2_M$. Therefore, $D_M$ restricted to $\tilde{\mathcal{F}}_M$ is the unique weak abstract duality on $\tilde{\mathcal{F}}_M$. Among the subfamilies $\tilde{\mathcal{F}}_M$ of $\mathcal{F}_M$ that satisfy (4.1) and (4.2) are those that arise from planar graphic matroids, unimodular matroids, matroids representable over a particular field, matroids representable over all fields in some specified set, and all unions of subfamilies noted above.

### 4.2. Clutters

**Lemma 4.2.** (a) *Every $F \in \mathcal{F}_S$ with $|E(F)| \geq 2$ is reconstructible.*
(b) *All weak abstract dualities on $\mathcal{F}_S$ agree on the subfamily $\mathcal{F}^1_S$.*

**Proof.** (a) Choose $F \in \mathcal{F}_S$ with $|E(F)| \geq 2$.

Let $\mathcal{M}$ be the set of simple minors of $F$. Note that $E(F) = \cup\{E(G) : G \in \mathcal{M}\}$. Let $F'$ have $E(F') = E(F)$ and $\mathcal{V}(F') = \{x : E(F) \to \{0, 1\} \mid \exists e \in E(F)$ and $y \in \mathcal{V}_e(F\backslash e)$ with $y(e') = 0 \; \forall e' \in E(F\backslash e)$ s.t. $x(e') = 0\}$. If $\mathcal{V}(F') \neq \emptyset$, then $F = F'$.

Suppose that $\mathcal{V}(F') = \emptyset$. Then $\mathcal{V}(F) = \emptyset$ if $\exists e \in E(F)$ s.t.$\mathcal{V}(F/e) = \emptyset$; otherwise $\mathcal{V}(F) = \{1^{E(F)}\}$.
(b) There are two elements of $\mathcal{F}_S(\emptyset)$ : $F_1(\emptyset)$, and $F_2(\emptyset)$, with $\mathcal{V}(F_1(\emptyset)) = \emptyset$ and $\mathcal{V}(F_2(\emptyset)) = \{0^{\emptyset}\}$, where $\{0^{\emptyset}\}$ denotes the empty map (so, the set of minimal dependent sets of $F_1$ is empty, and the set of minimal

dependent sets of $F_2$ contains only the empty set). For a fixed singleton $E = \{e\}$, there are three elements of $\mathcal{F}_S(E) : F_1(\{e\}), F_2(\{e\}), F_3(\{e\})$, with $\mathcal{V}(F_1(\{e\})) = \emptyset, \mathcal{V}(F_2(\{e\})) = \{(0), (1)\}, \mathcal{V}(F_3(\{e\})) = \{(1)\}$. Taking the minors we find that

$$\mathcal{V}(F_1(\{e\})/e) = \emptyset \qquad \mathcal{V}(F_2(\{e\})/e) = \{0^{\emptyset}\} \qquad \mathcal{V}(F_3(\{e\})/e) = \{0^{\emptyset}\}$$
$$\mathcal{V}(F_1(\{e\})\backslash e) = \emptyset \qquad \mathcal{V}(F2(\{e\})\backslash e) = \{0^{\emptyset}\} \qquad \mathcal{V}(F_3(\{e\})\backslash e) = \emptyset.$$

Hence, $F_1(\emptyset) = F_1(\{e\})/e = F_1(\{e\})\backslash e = F_3(\{e\})\backslash e$ and $F_2(\emptyset) = F_2(\{e\})/e = F_3(\{e\})/e = F_2(\{e\})\backslash e$.

Let $D : \mathcal{F}_S \to \mathcal{F}_S$ be a weak abstract duality. Let $F_2(\emptyset)$ and $e$ play the roles of $F^*$ and $e^*$, respectively, in Lemma 3.4. Then the set $\mathcal{F}_1 = \{F \in \mathcal{F}(E) \mid F\backslash e^* = F^*\}$ is the singleton $\{F_2(\{e\})\}$. Hence the set $\mathcal{F}_2 = \{F \in \mathcal{F}(E) \mid F/e^* = D(F^*)\}$ must also be a singleton, which implies that $\mathcal{F}_2 = \{F_1(\{e\})\}$ and $D(F^*) = D(F_2(\emptyset)) = F_1(\emptyset)$. Since $D$ takes $\mathcal{F}_1$ to $\mathcal{F}_2$ we see that $D(F_2(\{e\})) = F_1(\{e\})$. By (2.5) $D(F_1(\emptyset)) = F_2(\emptyset)$, $D(F_1(\{e\})) = F_2(\{e\})$, and $D(F_3(\{e\})) = F_3(\{e\})$. This is independent of the choice of $e$, so $D$ is uniquely determined on $\mathcal{F}_S^1$. □

This establishes the hypothesis of Corollary 3.3 for $\mathcal{F}_S$, and, therefore, proves part (a) of Theorem 2.1.

The specific encodings $\{E(F), \mathcal{V}(F)\}$ in Table 1 bring greater unity to the notion of abstract duality, allowing for generic definitions of contraction and deletion. It is important to keep in mind that each of the families $\mathcal{F}$, the abstract duality map on $\mathcal{F}$, and the contraction and deletion operations on $\mathcal{F}$ all exist independent of any specific encoding of $\mathcal{F}$. For example on $\mathcal{F}_M$, matroids, the duality map has a different expression in terms of bases ($\mathcal{B}(D_M(F)) = \{E(F) - B : B \in \mathcal{B}(F)\}$) than in terms of circuits ($\mathcal{C}(D_M(F))$ is the set of minimal nonempty members of $\{Y \subseteq E(F) : |X \cap Y| \neq 1\ \forall X \in \mathcal{C}(F)\}$), or in terms of unions of circuits ($\mathcal{U}(D_M(F)) = \{Y \subseteq E(F) : |X \cap Y| \neq 1\ \forall X \in \mathcal{U}(F)\}$). (For a matroid $F$, $\mathcal{V}(F)$ is the set of incidence vectors of $\mathcal{U}(F)$). However, the map $D_M$ from $\mathcal{F}_M$ to $\mathcal{F}_M$ is independent of its expression in terms of an encoding. There are many expressions of matroid duality, including *mixed* expressions, e.g., relating circuits of $F$ and bases of $D_M(F)$: $\mathcal{B}(D_M(F))$ is the set of minimal members of $\{Y \subseteq E(F) : X \cap Y \neq \emptyset\ \forall X \in \mathcal{C}(F)\}$.

Similarly for clutters, $\mathcal{F}_S$, there are distinct encodings, which yield distinct expressions of the unique abstract duality $D_S$. For $F \in \mathcal{F}_S$, recall that $\mathcal{P}(F)$ is the set of minimal dependent subsets of $E(F)$; let $\mathcal{B}(F)$ denote the set of maximal independent subsets of $F$, i.e., maximal subsets containing no $X \in \mathcal{P}(F)$. We can express $D_S$ in terms of one of these encodings, or in terms of $\mathcal{V}(F)$, among others. Since $\mathcal{P}(D_S(F)) = \{E(F) - X : X \in \mathcal{B}(F)\}$, we can also get simple mixed expressions of $D_S$, just as we did for $D_M$. An additional wrinkle arises here – for a fixed $F \in \mathcal{F}_S$ the set $\mathcal{P}(F)$ is also $\mathcal{B}(F')$ for some $F' \in \mathcal{F}_S$, so we get identities such as $\mathcal{P}(D_S(F')) = \{E(F) - X : X \in \mathcal{P}(F)\}$.

Also note that the result on subfamilies of matroids satisfying (4.1) and (4.2) extends to clutters. On subfamilies $\tilde{\mathcal{F}}_S$ of $\mathcal{F}_S$ closed under $D_S$, contraction, and deletion, and containing all clutters on two or fewer elements, the restriction of $D_S$ to $\tilde{\mathcal{F}}_S$ is the unique weak abstract duality.

In [8] Cordovil, Fukuda, and Moreira examine the blocking duality $D_S$ on $F_S$ and they also examine the subfamily of $\mathcal{F}_S$ consisting of what they call *nonempty clutters*, i.e., $\{F \in \mathcal{F}_S : \mathcal{B}(F) \neq \emptyset\}$. We will denote this subfamily by $\mathcal{F}_{S2}$. Note that $\mathcal{F}_{S2}$ is not closed under the usual contraction and deletion operations on $\mathcal{F}_S$. For example, consider the clutter $F$ with $E(F) = \{a, b\}$, and $\mathcal{B}(F) = \{a\}$ ($\mathcal{P}(F) = \{b\}$). Then $\mathcal{B}(F/b) = \emptyset$ ($\mathcal{P}(F/b) = \{\emptyset\}$). Hence, restriction of $D_S$ is not an abstract duality on $\mathcal{F}_{S2}$. In [8] Cordovil, Fukuda, and Moreira devise slightly different contraction and deletion operations on $\mathcal{F}_{S2}$, which are interchanged by the the involution $D_{S2}$ that has $\mathcal{B}(D_{S2}(F)) = \{E(F) - X : X \in \mathcal{B}(F)\}$.

## 5. Conclusion

It has been appreciated previously that there are resemblances among the duality relations on the combinatorial structures examined here. For example, in each setting there is a variation on Minty's Coloring Property (see [22]) that characterizes the duality relation in that setting (see [14,16,22,7,5,17]). Here we have shown that in fact there is a common characterization of these duality relations, and in the sequel [4] we show how this extends to the orthogonality relation on vector spaces coordinatized over fields having no subfield of index 2.

Some of the results here can be strengthened. For example under conditions (2.4) and (2.5), either half of condition (2.6) implies the other. We included both parts of (2.6) to emphasize symmetry. The combinatorial parts, (a)–(d), of Theorem 2.1 remain valid when (2.5) is relaxed to require only that $D$ be one-to-one.

We have seen how the parts of Theorem 2.1 pertaining to matroids and clutters extend to interesting subfamilies of matroids and clutters. Similarly, the form of the proofs of the other parts of Theorem 2.1 yields uniqueness results for subfamilies of $\mathcal{F}_O, \mathcal{F}_W, \mathcal{F}_K$.

## Appendix. Oriented matroids and weakly oriented matroids

For oriented matroids and weakly oriented matroids the target set $T = \{-, 0, +\}$. Suppose $E$ is a finite set and $x^i : E \to \{-, 0, +\}$, $i = 1, \ldots, k$, such that there exist no $e \in E$ and $1 \le i, j \le k$ with $x^i(e) = -$ and $x^j(e) = +$. Then we say that $\{x^1, \ldots, x^k\}$ is *conformal* and we denote by $\biguplus_{i=1}^k x^i$ the *conformal sum of* $x^1, \ldots, x^k$, the map $\biguplus_{i=1}^k x^i : E \to \{-, 0, +\}$ such that for each $e \in E$

$$\biguplus_{i=1}^{k} x^i(e) = \begin{cases} 0 & \text{if } x^i(e) = 0 \ \forall i = 1, \ldots, k \\ - & \text{if } x^i(e) \in \{-, 0\} \ \forall i = 1, \ldots, k \text{ and } \exists j \text{ s.t. } x^j(e) = - \\ + & \text{if } x^i(e) \in \{0, +\} \ \forall i = 1, \ldots, k \text{ and } \exists j \text{ s.t. } x^j(e) = +. \end{cases}$$

### *A.1. Oriented matroids*

The following lemma proves part (c) of Theorem 2.1 by establishing the hypothesis of Corollary 3.3 for $\mathcal{F}_O$.

**Lemma A.1.** (a) *Every $F \in \mathcal{F}_O$ with $|E(F)| \ge 3$ is reconstructible.*
(b) *All abstract dualities on $\mathcal{F}_O$ agree on the subfamily $\mathcal{F}_0^2$.*

**Proof.** (a) Choose $F \in \mathcal{F}_O$ having $|E(F)| \ge 3$. Let $\mathcal{M}$ be the set of simple minors of $F$. Note that $E(F) = \bigcup\{E(G) : G \in \mathcal{M}\}$. Let $F'$ have $E(F') = E(F)$ and let $\mathcal{V}(F')$ be the set of all conformal sums of extensions (defined at the end of Section 3) of the form $\biguplus\{x^e \mid \forall e \in E(F), x^e \in \mathcal{V}_e(F \backslash e)\}$.

Case 1: if $\mathcal{V}(F') \ne \{0^{E(F)}\}$, then $F = F'$. Case 2: if $\mathcal{V}(F') = \{0^{E(F)}\}$, then $\mathcal{V}(F)$ is the union of $\{0^{E(F)}\}$ and the set of conformal sums of the form $\biguplus\{x^e \mid \forall e \in E(F),\ x^e \in \mathcal{V}_e(F/e),\ x^e \ne 0^{E(F)}\}$. (Note that in the second case, $\mathcal{V}(F)$ is either the singleton $\{0^{E(F)}\}$, or a triple $\{0^{E(F)},\ x, -x\}$ where the vector $x$ has $x(e) \ne 0 \ \forall e \in E(F)$ and the vector $-x$ has $-x(e) = -$ if $x(e) = +$ and $-x(e) = +$ if $x(e) = -$ for all $e \in E(F)$.

Note that this construction holds only for $|E(F)| \ge 3$. For $|E| = 2$, there are distinct oriented matroids in $\mathcal{F}_O(E)$ having the same set of simple minors.

(b) Let $D : \mathcal{F}_O \to \mathcal{F}_O$ be a weak abstract duality.

In order to show that all abstract dualities agree on $\mathcal{F}_O^2$ we will need to examine the structure of all oriented matroids on two or fewer elements. There is only one oriented matroid $F_0 \in \mathcal{F}_O(\emptyset)$. By (2.4), $F_0$ must be its own image under $D$. For any fixed singleton $\{e\}$ there are two oriented matroids $F_1(\{e\})$ and $F_2(\{e\})$ in $\mathcal{F}_O(\{e\})$. They have $\mathcal{V}(F_1(\{e\})) = \{(0)\}$ and $\mathcal{V}(F_2(\{e\})) = \{(0), (+), (-)\}$.

Here each map $x$ is denoted by the image of the element $e$ under $x$.

For any fixed pair of distinct elements $e_1$ and $e_2$ there are six oriented matroids, $F_1(\{e_1, e_2\}), \ldots, F_6(\{e_1, e_2\}) \in \mathcal{F}_O(\{e_1, e_2\})$, with

$$\mathcal{V}(F_1(\{e_1, e_2\})) = \{(0, 0)\}$$
$$\mathcal{V}(F_2(\{e_1, e_2\})) = \{(0, 0), (+, 0), (-, 0)\}$$
$$\mathcal{V}(F_3(\{e_1, e_2\})) = \{(0, 0), (0, +), (0, -)\}$$
$$\mathcal{V}(F_4(\{e_1, e_2\})) = \{(0, 0), (+, 0), (-, 0), (0, +), (0, -), (+, +), (-, -), (+, -), (-, +)\}$$
$$\mathcal{V}(F_5(\{e_1, e_2\})) = \{(0, 0), (+, +), (-, -)\}$$
$$\mathcal{V}(F_6(\{e_1, e_2\})) = \{(0, 0), (+, -), (-, +)\}.$$

Here each map $x : \{e_1, e_2\} \to \{-, 0, +\}$ is denoted by the ordered pair $(x(e_1), x(e_2))$. Now we apply Lemma 3.4 to deduce the behavior of $D$ on $F_1(\{e\}), F_2(\{e\}), F_1(\{e_1, e_2\}), F_2(\{e_1, e_2\}), F_3(\{e_1, e_2\}), F_4(\{e_1, e_2\})$ for all $e, e_1, e_2$. (We will not be able to deduce the behavior of $D$ on $F_5(\{e_1, e_2\}), F_6(\{e_1, e_2\})$ until we examine some oriented matroids on three elements.)

First we apply Lemma 3.4 with $e^* = e_2$ and $F^* = F_2(\{e_1\})$. Note that $\{F \in \mathcal{F}_O(\{e_1, e_2\}) \mid F \backslash e_2 = F_2(\{e_1\})\} = \{F_2(\{e_1, e_2\}), F_4(\{e_1, e_2\})\}$. Thus there are exactly two oriented matroids in $\mathcal{F}_O(\{e_1, e_2\})$ having $F^* = F_2(\{e_1\})$ as the minor resulting from deletion of $e^* = e_2$. Now note $\{F \in \mathcal{F}_O(\{e_1, e_2\}) \mid F/e_2 = F_2(\{e_1\})\} = \{F_2(\{e_1, e_2\}), F_4(\{e_1, e_2\}), F_5(\{e_1, e_2\}), F_6(\{e_1, e_2\})\}$. There are four oriented matroids in $\mathcal{F}_O(\{e_1, e_2\})$ having $F_2(\{e_1\})$ as the minor resulting from contraction of $e_2$. By Lemma 3.4 $F_2(\{e_1\}) \ne D(F_2(\{e_1\}))$. It must be, therefore, that $F_1(\{e_1\}) = D(F_2(\{e_1\}))$. Since the choice of $e_1$ was arbitrary, it follows that $D(F_2(\{e\})) = F_1(\{e\})$ for all singletons $\{e\}$, and by (2.5), $F_2(\{e\}) = D(F_1(\{e\}))$. It also follows from Lemma 3.4 that

$$\{D(F_2(\{e_1, e_2\})),\ D(F_4(\{e_1, e_2\}))\} = \{F_1(\{e_1, e_2\}),\ F_3(\{e_1, e_2\})\}. \tag{A.1}$$

(Note that $\{F_1(\{e_1, e_2\}),\ F_3(\{e_1, e_2\})\} = \{F \in \mathcal{F}_O(\{e_1, e_2\}) \mid F/e_2 = D(F_2(\{e_1\}))\}$.)

Now we apply Lemma 3.4 again, this time with $e^* = e_1$ and $F^* = F_2(\{e_2\})$. Observe that $\{F \in \mathcal{F}_O(\{e_1, e_2\}) \mid F\backslash e_1 = F_2(e_2)\} = \{F_3(\{e_1, e_2\}), F_4(\{e_1, e_2\})\}$ and $\{F \in \mathcal{F}_O(\{e_1, e_2\}) \mid F/e_1 = D(F_2(\{e_2\}))\} = \{F_1(\{e_1, e_2\}), F_2(\{e_1, e_2\})\}$, whereas $|\{F \in \mathcal{F}_O(\{e_1, e_2\}) \mid F\backslash e_1 = F_1(e_2)\}| = 4$. Therefore,

$$\{D(F_3(\{e_1, e_2\})),\ D(F_4(\{e_1, e_2\}))\} = \{F_1(\{e_1, e_2\}),\ F_2(\{e_1, e_2\})\}. \tag{A.2}$$

Now by (A.1) and (A.2) we have

$$D(F_4(\{e_1, e_2\})) = F_1(\{e_1, e_2\}), \qquad D(F_2(\{e_1, e_2\})) = F_3(\{e_1, e_2\}). \tag{A.3}$$

By (2.5) this implies

$$D(F_3(\{e_1, e_2\})) = F_2(\{e_1, e_2\}), \qquad D(F_1(\{e_1, e_2\})) = F_4(\{e_1, e_2\}). \tag{A.4}$$

We have not yet invoked the isomorphism-preserving condition (2.7). We will need it to determine $D(F_5(\{e_1, e_2\}))$ and $D(F_6(\{e_1, e_2\}))$.

The oriented matroids $F_5(\{e_1, e_2\})$ and $F_6(\{e_1, e_2\})$ have the same set of simple minors. Either

(i) $D(F_5(\{e_1, e_2\})) = F_6(\{e_1, e_2\})$ and $D(F_6(\{e_1, e_2\})) = F_5(\{e_1, e_2\})$

or

(ii) $D(F_5(\{e_1, e_2\})) = F_5(\{e_1, e_2\})$ and $D(F_6(\{e_1, e_2\})) = F_6(\{e_1, e_2\})$.

Let $E' = \{e'_1,\ e'_2,\ e'_3\}$ and consider the oriented matroid $F \in \mathcal{F}_O(E')$ having

$$\mathcal{V}(F) = \{(0, 0, 0), (+, +, +), (-, -, -)\}.$$

For each $i = 1, 2, 3$, $F/e'_i$, is isomorphic to $F_5(\{e_1, e_2\})$ and $F\backslash e'_i$ is isomorphic to $F_1(\{e_1, e_2\})$. By properties (2.6) and (2.7) there exists $F' = D(F) \in \mathcal{F}_O$ such that

$$F'\backslash e'_i = D(F/e'_i) \text{ is isomorphic to } D(F_5(\{e_1, e_2\})), \quad \text{for } i = 1, 2, 3. \tag{A.5}$$

By part (a) of this lemma $F'$ is reconstructible from its simple minors. Moreover, $\mathcal{V}(F')$ is the set of all conformal sums of the form $\biguplus\{x^e \mid \forall e \in E',\ \ x^e \in \mathcal{V}_e(F'\backslash e)\}$, unless this set $= \{0^{E'}\}$. If assumption (ii) holds, then $D(F_5(\{e_1, e_2\})) = F_5(\{e_1, e_2\})$, and by (A.5)

$$\mathcal{V}(F') = \{(+, +, +), (-, -, -), (+, +, 0), (-, -, 0), (+, 0, +), (-, 0, -), (0, +, +), (0, -, -), (0, 0, 0)\}.$$

However, this $F'$ is not an oriented matroid, contradicting the assumption (ii). Therefore, (i) holds and $D$ is determined on $\mathcal{F}_O(\{e_1, e_2\})$, for arbitrary pairs $\{e_1, e_2\}$ of distinct elements. Hence $D$ is determined for all $F \in \mathcal{F}^2_O$, i.e., all small oriented matroids. □

The proof of Lemma A.1 used the isomorphism-preserving property (2.7) to determine the behavior of $D$ on $\mathcal{F}^2_O$. The family $\mathcal{F}_O$ has many weak abstract dualities. Each of these weak abstract dualities arises from the usual oriented matroid duality by "reversing signs" on a set of elements. We will outline a proof.

We begin by formalizing the notion of reversing signs. Let $x$ be a map from a finite set $E$ to $\{-,\ 0,\ +\}$. For $j \in \{-, +\}$ denote by $x^j$ the subset $\{e \in E : x(e) = j\}$. Now if $S$ is a set, then ${}_{\overline{S}}x$ is the map from $E$ to $\{-,\ 0, +\}$ having $({}_{\overline{S}}x)^+ = (x^+ - S) \cup (x^- \cap S)$ and $({}_{\overline{S}}x)^- = (x^- - S) \cup (x^+ \cap S)$. The map ${}_{\overline{S}}x$ is said to be obtained from $x$ by *reversing signs* on $S$, and for any collection of maps $\mathcal{C}$ from $E$ to $\{-, 0, +\}$ we write ${}_{\overline{S}}\mathcal{C}$ for the collection $\{{}_{\overline{S}}x : x \in \mathcal{C}\}$. Note that we do not require $S \subseteq E$.

To characterize the weak abstract dualities on $\mathcal{F}_O$ we first note that reversing signs on a set $S$ commutes with oriented matroid duality, and with contraction and deletion in $\mathcal{F}_O$. It is also easy to see that the composition of sign-reversal on a set $S$ with the oriented matroid duality map gives a weak abstract duality for oriented matroids. The converse is also true – every weak abstract duality for oriented matroids arises from oriented matroid duality by sign-reversal – but more difficult to prove.

First note that since any $F \in \mathcal{F}_O$ having $|E(F)| \geq 3$ is reconstructible, it is sufficient to determine the behavior of a weak abstract duality $D$ on $\mathcal{F}^2_O$. For each two element set $E$, there are six $F \in \mathcal{F}_O$ having $E(F) = E$.

Properties (2.4)–(2.6), determine $D$ on four of these oriented matroids, as in the proof of Lemma A.1. It is only on the remaining members of $\mathcal{F}_O^2$ that the function $D$ can differ from oriented matroid duality, $D_O$. For each two element set $\{e_1, e_2\}$, let $F(\{e_1, e_2\})$ and $F'(\{e_1, e_2\})$ be given by $E(F(\{e_1, e_2\})) = E(F'(\{e_1, e_2\})) = \{e_1, e_2\}$ and $\mathcal{V}(F(\{e_1, e_2\})) = \{(0, 0), (+, +), (-, -)\}$ and $\mathcal{V}(F') = \{(0, 0), (+, -), (-, +)\}$. Either $D(F(\{e_1, e_2\})) = F'(\{e_1, e_2\})$ or $D(F(\{e_1, e_2\})) = F(\{e_1, e_2\})$; for $D = D_O$ it is the former. The set $S$ on which signs are reversed is determined from the behavior of $D$ on these members of $\mathcal{F}_O^2$.

For each singleton $\{e\}$, let $S(e) = \{e' \mid D(F(\{e, e'\})) = F(\{e, e'\})\}$. It is clear that $f \in S(e)$ if and only if $e \in S(f)$. It is tedious, but not difficult, to prove that for any singleton $e^*$ and any $\hat{F} \in \mathcal{F}_O^2$, $D(\hat{F}) = {}_{\overline{S(e^*)}}D_O(\hat{F})$. If $e^* \in E(\hat{F})$, this is straightforward. In the more general case where $e^* \notin E(\hat{F})$, we resolve the consistency issue by consideration of the behavior of $D$ on an oriented matroid $\tilde{F} \in \mathcal{F}_O$ and its minors, where $E(\tilde{F}) = E(\hat{F}) \cup \{e^*\}$ and $\mathcal{V}(\tilde{F}) = \{(0, 0, 0), (+, +, +), (-, -, -)\}$. Then by induction on the size of $E(F^*)$, $F^* \in \mathcal{F}_O$ we can prove that $D(F^*) = {}_{\overline{S(e^*)}}D_O(\hat{F}^*)$.

For oriented matroids, the unique abstract duality corresponds to oriented matroid duality. The analogous result for the family $\mathcal{F}_W$ of weakly oriented matroids follows from

**Lemma A.2.** (a) *Every $F \in \mathcal{F}_W$ with $|E(F)| \geq 3$ is reconstructible.*
(b) *All abstract dualities on $\mathcal{F}_W$ agree on the subfamily $\mathcal{F}_W^2$.*

The proof follows easily from the proof of Lemma A.1. Every weakly oriented matroid on three or more elements is reconstructible from its simple minors. (In the case $\mathcal{V}(F') \neq \{0^{E(F)}\}$, one follows the reconstruction for oriented matroids, but using consistent unions rather than conformal unions. In the other case, the reconstruction is exactly the same as for oriented matroids, using conformal unions). Every weakly oriented matroid on three or fewer elements is also an oriented matroid, i.e., $\mathcal{F}_O^3 = \mathcal{F}_W^3$, even though there exists $F \in \mathcal{F}_W^3$ such that $\mathcal{V}^O(F) \subsetneq \mathcal{V}^W(F)$ (of course $\mathcal{V}^W(F)$ can be deduced from $\mathcal{V}^O(F)$ in this case). Moreover, on $\mathcal{F}_O^3$ the contraction and deletion operations for oriented matroids behave the same as the related operations for weakly oriented matroids. Hence the second part of the proof of Lemma A.1 goes through.

Similarly, it is easy to deduce from the oriented matroid derivation that all weak abstract dualities on $\mathcal{F}_W$ arise from weakly oriented matroid duality by sign-reversal.

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