Equilibria and Stabilities for Competing-Species Reaction-Diffusion Equations with Dirichlet Boundary Data

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1. INTRODUCTION AND PRELIMINARIES

This article is concerned with the study of the reaction-diffusion equations

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \sigma_1 \Delta u_1 + u_1[a + f_1(u_1, u_2)] \\
\frac{\partial u_2}{\partial t} &= \sigma_2 \Delta u_2 + u_2[b + f_2(u_1, u_2)]
\end{align*}
\]

(1.1)

where \( \Delta = \sum_{i=1}^{n} \left( \frac{\partial^2}{\partial x_i^2} \right) \), \( a, b, \sigma_1, \sigma_2 \) are positive constants, \( f_i: \mathbb{R}^2 \to \mathbb{R} \) have Hölder continuous partial derivatives up to second order in compact sets, \( i = 1, 2 \). Further, we assume that

\[
f_1(0, 0) = f_2(0, 0) = 0.
\]

(1.2)

For \( (u_1, u_2) \) in the first open quadrant, the first partial derivatives of \( f_1, f_2 \) satisfy

\[
\frac{\partial f_i}{\partial u_j} < 0 \quad \text{for each } i, j = 1 \text{ or } 2; \quad (1.3)
\]

and there exist a positive constant \( C \), such that

\[
a + f_1(C, 0) \leq 0, \quad b + f_2(0, C) \leq 0. \quad (1.4)
\]

The system (1.1) together with assumptions (1.2), (1.3) and (1.4) is a model for biological competing species interaction, where \( u_i(x, t), i = 1, 2 \) represent the concentration of the two species at position \( x = (x_1, ..., x_n) \) and time \( t \geq 0 \). The parameters \( \sigma_1, \sigma_2 \) are diffusion rates; \( a \) and \( b \) are growth rates when no interaction occurs. The functions \( f_1 \) and \( f_2 \) describe the additional effect of interactions on the growth rates, and (1.2), (1.3) and (1.4) are general assumptions which include the classical Volterra–Lotka model.

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We will consider the Dirichlet problem \( u_i = g_i, \ i = 1, 2, \) on the boundary \( \partial \mathcal{D} \) of a bounded domain \( \mathcal{D} \). For the homogeneous problem \( (g_i = 0, \ i = 1, 2) \) or Neumann problem, some results can be found in [3], [10], [17], [18]. Such problems have recently received much attention of mathematicians and biologists. (See e.g. [2], [6], [8], [12], [13] and [18].) Mathematically, we will be considering stability of equilibrium solutions, non-uniqueness of equilibrium states in various parameter ranges, and asymptotic behavior of solutions. (An equilibrium is a time, \( t \), independent solution of (1.1). Biologically, the results can be interpreted as statements concerning extinction, coexistence and predictions of future behavior. A similar study for the Dirichlet problem \( u_i = g_i, \ i = 1, 2, \) is made in [11] for the prey-predator equations (i.e. \( b < 0, \partial g_i/\partial u_i > 0 \)). The difference in signs of the growth and interaction rates here gives rise to many different results, and necessitates a separate treatment.

The essential techniques used in this article are the method of comparison, and upper, lower solutions (see e.g. [14], [16]). Theorem 2.1 gives sufficient conditions for a stable equilibrium when both \( u_i, \ i = 1, 2, \) are positive in \( \mathcal{D} \). Examples of such cases are given. Section 3 considers cases when one or both \( u_i \) are held at zero identically on \( \partial \mathcal{D} \). Extinction, coexistence and bifurcation results are given. It turns out that the relative sizes of the growth rates in relation to the diffusion rates are important in determining the stable asymptotic states. Most of the results have no analogs in the case where there is no diffusion at the boundary. Many more questions remain open for investigation, but will be too lengthy for our present study.

We will clarify the notations and conventions. Let \( \rho_0 > 0 \) and \( l, 0 < l < 1, \) be fixed numbers. For an open set \( G \) in \( \mathbb{R}^n, \) let \( H^{2+l}(G) \) denote the Banach space of all real-valued functions \( u \) continuous on \( G \) with all first and second derivatives also continuous in \( G, \) and with finite value for the norm

\[
|u|^{(2+l)} = \sum_{0 < |\alpha| < 2} \sup_{G} |D^\alpha u| + \sum_{|\alpha| = 2} \sup_{|x - y| < \rho_0} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^l}
\]

where \( \alpha = (\alpha_1, ..., \alpha_n) \) is a multi-index, \( |\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n, \) \( D^\alpha u = \partial^{(|\alpha|)}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}. \) The second supremum is taken over all \( x, y \) in \( G \) such that \( 0 < |x - y| \leq \rho_0. \) We will consider equation (1.1) for \( x = (x_1, ..., x_n) \in \mathcal{D}, \) where \( \mathcal{D} \) is a bounded open connected subset of \( \mathbb{R}^n, \) \( n \geq 1, \) with boundary \( \partial \mathcal{D}. \) We assume that \( \delta \mathcal{D} \in H^{2+l}, \) i.e., that there are positive constants \( \rho \) and \( M \) such that for any \( x^0 \in \delta \mathcal{D} \), there is a one-to-one map \( \phi \) of the closure of the set \( K = \{ x \in \mathbb{R}^n; x_1^2 + ... + x_{n-1}^2 < 1 \text{ and } |x_n| < 1 \} \) into \( \mathbb{R}^n, \) where \( \phi \in H^{2+l}(\bar{K}), |\phi|^{(2+l)} \leq M \), \( \phi(0,...,0) = x^0, \) \( \phi(x_n = 0) \cap K = \delta \mathcal{D} \cap \phi(K) \) \( \phi(x_n < 0) \cap K = \mathcal{D} \cap \phi(K), \) and \( \phi(K) \) contain a ball of radius \( \rho \) about \( x^0. \) For any \( T > 0, \) let \( \mathcal{D}_T = \mathcal{D} \times (0, T). \) \( H^{2+l, (2+l)/2}(\mathcal{D}_T) \) denotes the Banach space of all real-valued functions \( u \) having all derivatives of the form \( D^\alpha D_t^r u \) (\( \alpha \) is a multi-index, \( r \geq 0 \))
is an integer, $D_i = \partial / \partial t$ with $2r + |\alpha| \leq 2$ continuous on $\mathcal{D}_T$ and having finite norm

\[ |u|_{H^{2+1/2}}(\mathcal{D}_T) \]

\[ \leq \sum_{0 < 2r + |\alpha| \leq 2} \sup_{\mathcal{D}_T} |D^{2}D_{i}u| + \sum_{2r + |\alpha| = 2} \sup_{\mathcal{D}_T} \frac{|[D^{2}D_{i}u](x, t) - [D^{2}D_{i}u](y, t)|}{|x - y|^t} \]

\[ + \sum_{1 < 2r + |\alpha| \leq 2} \sup_{\mathcal{D}_T} \frac{|[D^{2}D_{i}u](x, t) - [D^{2}D_{i}u](x, t')|}{|t - t'|^{(1 + 2 - 2r - |\alpha|)/2}} \]

where the second supremum is extended over all $(x, t), (y, t)$ in $\mathcal{D}_T$ with $0 < |x - y| \leq \rho_0$, and the third supremum is extended over all $(x, t), (x, t')$ in $\mathcal{D}_T$ with $0 < |t - t'| \leq \rho_0$.

2. Stability of Positive Spatially Dependent Equilibria

This section gives a sufficient condition when an equilibrium with $u_i > 0$ in $\mathcal{D}, i = 1, 2$, is stable. A series of lemmas leads to the main result in Theorem 2.1. Examples are given following the theorem.

**Lemma 2.1.** Let $v_i(x, t), w_i(x, t), (x, t) \in \mathcal{D} \times [0, \infty)$, $i = 1, 2$ be functions in $H^{2+1/2}(\mathcal{D} \times [0, T])$, each $T > 0$, satisfying the inequalities:

\[ 0 < v_i < w_i, \quad i = 1, 2, \]

\[ \sigma_1 \Delta v_1 + v_1[a + f_1(v_1, w_2)] - \frac{\partial v_1}{\partial t} \geq 0, \]

\[ \sigma_1 \Delta w_1 + w_1[a + f_1(v_1, v_2)] - \frac{\partial w_1}{\partial t} \leq 0, \]

\[ \sigma_2 \Delta v_2 + v_2[b + f_2(v_1, v_2)] - \frac{\partial v_2}{\partial t} \geq 0, \]

\[ \sigma_2 \Delta w_2 + w_2[b + f_2(v_1, w_2)] - \frac{\partial w_2}{\partial t} \leq 0. \]

(2.1)

Let $(u_i(x, t), v_i(x, t))$, with $u_i \in H^{2+1/2}(\mathcal{D} \times [0, T])$, each $T > 0$, $i = 1, 2$, be a solution of the reaction diffusion equations (1.1) with initial-boundary conditions such that

\[ v_i(x, 0) \leq u_i(x, 0) \leq w_i(x, 0), \quad x \in \mathcal{D}, \quad i = 1, 2, \]

\[ v_i(x, t) \geq u_i(x, t) \leq w_i(x, t), \quad (x, t) \in \partial \mathcal{D} \times [0, \infty), \quad i = 1, 2. \]
Then \((u_1(x, t), u_2(x, t))\) will satisfy

\begin{equation}
\nu_i(x, t) \leq u_i(x, t) \leq v_i(x, t), \quad (x, t) \in \mathcal{D} \times [0, \infty)
\end{equation}

(2.3)

Proof. We first prove the lemma under the additional hypothesis that strict inequalities hold in (2.2). With this hypothesis and by continuity considerations, (2.3) holds with strict inequalities for \((x, t) \in \mathcal{D} \times [0, T)\), for some \(T > 0\), and holds with possibly non-strict inequalities for \(x \in \mathcal{D}, t = T\). We will show that actually strict inequalities hold for \(x \in \mathcal{D}, t = T\) as well. Thus the interval of \(t\) values where strict inequalities hold is non-empty, is both open and closed relative to \([0, \infty)\), and therefore is equal to \([0, \infty)\). We have

\[
\sigma_1 \Delta u_1 - w_1 + u_1[a + f_1(u_1, u_2)] - w_1[a + f_1(w_1, v_2)] - \frac{\partial}{\partial t}(u_1 - w_1) \geq 0,
\]

hence, setting \(z = u_1 - w_1\) and using the fact that \(\frac{\partial f_1}{\partial u_2} < 0\), for \((x, t) \in \mathcal{D} \times [0, T)\) we have

\[
\sigma_1 \Delta z + az + u_1f_1(u_1, u_2) - w_1f_1(w_1, u_2) - \frac{\partial}{\partial t}(u_1 - w_1) \geq 0,
\]

which, using the mean value theorem, may be written in the form

\[
\sigma_1 \Delta z + g(x, t) z - \frac{\partial z}{\partial t} \geq 0,
\]

where \(g\) is continuous in \(\mathcal{D} \times [0, T]\). From the above arguments, we have \(z < 0\) on \(\mathcal{D} \times [0, T]\), therefore the maximum principle for parabolic equations ([14], Remark (ii), p. 175) implies that \(z < 0\) on \(\mathcal{D} \times [0, T]\), i.e. \(u_1 < w_1\) on \(\mathcal{D} \times [0, T]\). In precisely the same manner, we may prove that \(v_1 < u_1, v_2 < u_2 < w_2\) on \(\mathcal{D} \times [0, T]\). As noted above, this shows that (2.3) holds (with strict inequalities) in the present case.

For the general case, we first write the differential inequatilities in (2.1) in the form

\begin{align}
\sigma_1 \Delta v_1 + v_1[a + f_1(v_1, w_2)] - \frac{\partial v_1}{\partial t} &= \phi_1(x, t) \\
\sigma_1 \Delta w_1 + w_1[a + f_1(w_1, v_2)] - \frac{\partial w_1}{\partial t} &= \psi_1(x, t) \\
\sigma_2 \Delta v_2 + v_2[b + f_2(v_1, v_2)] - \frac{\partial v_2}{\partial t} &= \phi_2(x, t) \\
\sigma_2 \Delta w_2 + w_2[b + f_2(v_1, w_2)] - \frac{\partial w_2}{\partial t} &= \psi_2(x, t)
\end{align}

(2.4)
where $\phi_i(x, t) \geq 0$, $\psi_i(x, t) \leq 0$, $i = 1, 2$. For any $T > 0$, we define $\hat{v}_i$, $\hat{w}_i$, $i = 1, 2$ to be the solution of the initial boundary value problem

$$
\sigma_1 \Delta \hat{v}_1 + \hat{v}_1[a + f_1(\hat{v}_1, \hat{w}_2)] - \frac{\partial \hat{v}_1}{\partial t} = \phi_1(x, t)
$$

$$
\sigma_1 \Delta \hat{w}_1 + \hat{w}_1[a + f_1(\hat{v}_1, \hat{w}_2)] - \frac{\partial \hat{w}_1}{\partial t} = \psi_1(x, t)
$$

$$
\sigma_2 \Delta \hat{v}_2 + \hat{v}_2[b + f_2(\hat{v}_1, \hat{w}_2)] - \frac{\partial \hat{v}_2}{\partial t} = \phi_2(x, t)
$$

$$
\sigma_2 \Delta \hat{w}_2 + \hat{w}_2[b + f_2(\hat{v}_1, \hat{w}_2)] - \frac{\partial \hat{w}_2}{\partial t} = \psi_2(x, t)
$$

for $(x, t) \in \mathcal{D} \times [0, T]$,

$$
\hat{v}_i(x, 0) = v_i(x, 0) - \mu_i, \quad x \in \mathcal{D}, \quad i = 1, 2
$$

$$
\hat{w}_i(x, 0) = w_i(x, 0) + \mu_i
$$

(2.5)

$$
\hat{v}_i(x, t) = v_i(x, t) - \mu_{x_i}(x, t)
$$

$$
\hat{w}_i(x, t) = w_i(x, t) + \mu_{x_i}(x, t)
$$

(2.6)

where $0 < \alpha_i \leq 2$, $0 < \beta_i \leq 2$, $i = 1, 2$ are chosen so that the compatibility conditions of order 1 are satisfied, and $\mu > 0$ is chosen sufficiently small so that the solution exists (see [9], p. 616–617). For more details of the smooth choice of $\alpha_i(x, t)$, $\beta_i(x, t)$ to insure the existence of solution to the boundary value problem (2.5), (2.6), see [11], Lemma 2.4, for a similar situation.

For arbitrary $\epsilon > 0$, we may choose $\mu > 0$ smaller, if necessary, so that the solution of (2.5), (2.6) satisfies $|v_i(x, t) - \hat{v}_i(x, t)| < \epsilon$, $|w_i(x, t) - \hat{w}_i(x, t)| < \epsilon$, $i = 1, 2$, $(x, t) \in \mathcal{D} \times [0, T]$. We then apply the same procedure as in the first part of this proof to show that

$$
\hat{v}_i(x, t) < v_i(x, t)
$$

$$
\hat{w}_i(x, t) > w_i(x, t)
$$

(2.7)

What we have in the form of $\hat{v}_i$, $\hat{w}_i$, $i = 1, 2$, is a solution of the inequality (2.1) and the strict inequalities (2.2) with $t \in [0, T]$. The first part of this proof is valid if we are dealing with the interval $[0, T]$ rather than $[0, \infty)$, hence

$$
\hat{v}_i(x, t) < u_i(x, t) < \hat{w}_i(x, t), \quad (x, t) \in \mathcal{D} \times [0, T], \quad i = 1, 2.
$$

(2.8)

Since both $T$ and $\epsilon$ were arbitrary, we deduce from (2.8) that (2.3) holds.
Remark. We shall refer to \( v_1, v_2 \) as lower solutions and \( w_1, w_2 \) as upper solutions of the respective equations in (1.1) and initial-boundary conditions associated with (2.2).

**Lemma 2.2.** Let \((\bar{u}_1(x), \bar{u}_2(x))\), with \(\bar{u}_i(x) \in H^{2+i}(\bar{D})\), \(i = 1, 2\), be a solution of the boundary value problem.

\[
\begin{align*}
\sigma_1 \Delta u_1 + u_1[a + f_1(u_1, u_2)] &= 0 && x \in \bar{D}, \\
\sigma_2 \Delta u_2 + u_2[b + f_2(u_1, u_2)] &= 0 \\
u_i(x) = g_i(x) &> 0, \quad \neq 0 \text{ on } \partial \bar{D}, \quad i = 1, 2.
\end{align*}
\]

(2.9)

Suppose that \(\bar{u}_i(x) \geq 0\) in \(\bar{D}\), \(i = 1, 2\), then \(\bar{u}_i(x) > 0\) in \(D\), \(i = 1, 2\).

**Proof.** Let \(w = -\bar{u}_1(x)e^{-at}\). Direct computation yields \(\sigma_i \Delta w - \partial w / \partial t + \omega f_1(\bar{u}_1, \bar{u}_2) = 0\), where \(f_2(\bar{u}_1, \bar{u}_2) \leq 0\). If \(\bar{u}_i(\bar{x}) = 0\), for \(\bar{x} \in \bar{D}\), then \(w(\bar{x}, t) = 0\) for a positive \(t\). The maximum principle ([14], Theorem 7, p. 174) therefore implies that \(g_1(x) = 0\), contradicting the assumption. Consequently, \(\bar{u}_i(x) > 0\) in \(\bar{D}\). Similarly, we can prove \(\bar{u}_i(x) > 0\) in \(D\).

**Lemma 2.3.** Consider the boundary value problem (2.9) described in Lemma 2.2, while we further assume that \(g_i(x) > 0\) on \(\partial \bar{D}\) and has an extension \(\hat{g}_i \in H^{2+i}(\bar{D})\), \(i = 1, 2\). Then there exists a solution \((\bar{u}_1(x), \bar{u}_2(x))\), with \(\bar{u}_i(x) \in H^{2+i}(\bar{D})\), \(0 < \bar{u}_i(x) \leq K_i\), \(x \in \bar{D}\), \(i = 1, 2\). Here, \(K_1, K_2\) are positive constants satisfying \(a + f_1(K_1, 0) \leq 0\), \(b + f_2(0, K_2) \leq 0\), \(g_i(x) \leq K_i\) on \(\partial \bar{D}\), \(i = 1, 2\).

**Proof.** For each \(0 < u_2 < K_2\), the function \(\phi_1(x) = 0\), \(\psi_1(x) = K_1\) are respectively lower and upper solutions of the boundary value problem \(\sigma_1 \Delta u + a[u + f_1(u, u_2)] = 0\) in \(D\), \(u = g_1\) on \(\partial D\). Similarly, \(\phi_2 = 0\), \(\psi_2 = K_2\) are respectively lower and upper solutions for the boundary value problem \(\sigma_2 \Delta u + b[u + f_2(u_1, u)] = 0\) in \(D\), \(u = g_2\) on \(\partial D\), for each \(0 < u_1 < K_1\). By [5] or [17] there exists a solution \((\bar{u}_1(x), \bar{u}_2(x))\), with \(\bar{u}_i \in H^{2+i}(\bar{D})\), \(0 \leq \bar{u}_i(x) \leq K_i\), \(i = 1, 2\) to the boundary value problem described in the lemma. By Lemma 2.2, \(0 < \bar{u}_i(x)\) in \(\bar{D}\).

**Theorem 2.1.** Let \((\bar{u}_i(x), \bar{u}_2(x))\) be an equilibrium solution to (2.9) as described in Lemma 2.3 \((g_i > 0\), with extension \(\hat{g}_i \in H^{2+i}(\bar{D})\), \(i = 1, 2\)). Suppose that

\[
\left| \begin{array}{c}
\bar{u}_i(x) \\
\bar{u}_2(x)
\end{array} \right| \left( \begin{array}{c}
\hat{f}_1 / \hat{u}_i (\bar{u}_1(x), \bar{u}_2(x)) \\
\hat{f}_2 / \hat{u}_2 (\bar{u}_1(x), \bar{u}_2(x))
\end{array} \right) < \min_{z \in \bar{D}} \left| \begin{array}{c}
\bar{u}_i(z) \\
\bar{u}_2(z)
\end{array} \right| \left( \begin{array}{c}
\hat{f}_1 / \hat{u}_i (\bar{u}_1(x), \bar{u}_2(x)) \\
\hat{f}_2 / \hat{u}_2 (\bar{u}_1(x), \bar{u}_2(x))
\end{array} \right)
\]

(2.10)

for each \(x \in \bar{D}\), \(i \neq j\), \(1 \leq i, j \leq 2\), then \((\bar{u}_i(x), \bar{u}_2(x))\) is asymptotically stable. (Here, asymptotic stability is interpreted to mean that for any solution \((u_i(x, t), u_2(x, t))\) with \(u_i \in H^{2+i+1/2}(\bar{D} \times [0, T])\), each \(T > 0\), \(i = 1, 2\), of the reaction diffusion equations (1.1) with boundary conditions \(u_i(x, t) = g_i(x)\) and initial condi-
tions $u_i(x, 0)$ close enough to $\bar{u}_i(x)$ for all $x \in \mathcal{D}$, $i = 1, 2$, one has $u_i(x, t) \to \bar{u}_i(x)$ uniformly as $t \to +\infty$, $i = 1, 2$.)

Proof. Assumption (2.10) implies that there are $\rho_1, \rho_2$ close enough to 1 with $\rho_1 < 1 < \rho_2$ such that

$$\frac{\bar{u}_i(x) \max_{\rho_1 < r < \rho_2} \left| \frac{\partial f_i}{\partial u_i} (s\bar{u}_1(x), \tau \bar{u}_2(x)) \right|}{\overline{u}_i(x) \min_{\rho_1 < r < \rho_2} \left| \frac{\partial f_i}{\partial u_i} (s\overline{u}_1(x), \overline{u}_2(x)) \right|} \leq \min_{x \in \mathcal{D}} \left\{ \frac{\bar{u}_i(x)}{\overline{u}_i(x)} \frac{\min_{\rho_1 < r < \rho_2} \left| \frac{\partial f_i}{\partial u_i} (s\bar{u}_1(x), \tau \bar{u}_2(x)) \right|}{\max_{\rho_1 < r < \rho_2} \left| \frac{\partial f_i}{\partial u_i} (s\overline{u}_1(x), \overline{u}_2(x)) \right|} \right\} - \bar{\varepsilon}_1$$

for each $x \in \mathcal{D}$, $i \neq j$, $1 \leq i, j \leq 2$, where $\bar{\varepsilon}_1$ is a small positive number. (Recall that Lemma 2.2 implies that $\bar{u}_i(x) > 0$ in $\mathcal{D}$, $i = 1, 2$). We will construct appropriate lower and upper solutions $v_i$, $w_i$, $i = 1, 2$, and apply Lemma 2.1.

Let $G(x) = \overline{u}_2(x) \min_{\rho_1 < r < \rho_2} \left| \frac{\partial f_2}{\partial u_2} (s\overline{u}_1(x), \tau \overline{u}_2(x)) \right| \cdot \left( \overline{u}_1(x) \max_{\rho_1 < r < \rho_2} \left| \frac{\partial f_2}{\partial u_1} (s\overline{u}_1(x), \overline{u}_2(x)) \right| \right)^{-1},$

and $\alpha$ be a number satisfying $1 < \alpha < \rho_2$, $(1 - \rho_1) (\alpha - 1)^{-1} > G(x)$ for all $x \in \mathcal{D}$. Define $w_2 = \left[ 1 + (\alpha - 1) e^{-n\tau} \right] \bar{u}_2(x)$ where $n > 0$ will be determined later so that $w_2$ becomes an upper solution. On the other hand define $v_2 = \left[ 1 - (1 - \beta) e^{-n\tau} \right] \bar{u}_2(x)$, where $\beta = 1 - \min_{x \in \mathcal{D}} (\alpha - 1) G(x) + \bar{\varepsilon}_2 (\alpha - 1)$ where $0 < \bar{\varepsilon}_2 < \min_{x \in \mathcal{D}} G(x)$, so $\rho_1 < \beta < 1$. We have

$$\alpha_2 \Delta w_2 + w_2 [b + f_2 (v_1, w_2)] - \frac{\partial w_2}{\partial t} = \left[ 1 + (\alpha - 1) e^{-n\tau} \right] \bar{u}_2 \left[ f_2 (v_1, w_2) - f_2 (v_1, \bar{u}_2) + f_2 (v_1, \bar{u}_2) - f_2 (\bar{u}_1, \bar{u}_2) \right] + n(\alpha - 1) e^{-n\tau} \bar{u}_2$$

$$\leq \left[ 1 + (\alpha - 1) e^{-n\tau} \right] \bar{u}_2 \left( \max_{\rho_1 < r < \rho_2} \left\{ \frac{\partial f_2}{\partial u_2} (v_1, \tau \bar{u}_2) \right\} \left( \alpha - 1 \right) \bar{u}_2 e^{-n\tau} \right.$$

$$- \left. \min_{\rho_1 < r < \rho_2} \left\{ \frac{\partial f_2}{\partial u_1} (s\bar{u}_1, \bar{u}_2) \right\} (1 - \beta) \bar{u}_2 e^{-n\tau} \right] + n(\alpha - 1) e^{-n\tau} \bar{u}_2$$

$$\leq e^{-n\tau} \bar{u}_2 \left[ 1 + (\alpha - 1) e^{-n\tau} \right] k + n(\alpha - 1) \right\}$$

where $k$ is a negative number, because

$$\left( 1 - \beta \right) \bar{u}_1(x) \left| \min_{\rho_1 < r < \rho_2} \left\{ \frac{\partial f_2}{\partial u_1} (s\bar{u}_1(x), \bar{u}_2(x)) \right\} \right| \leq \left[ (\alpha - 1) G(x) - \bar{\varepsilon}_2 (\alpha - 1) \right] \bar{u}_1(x) \max_{\rho_1 < r < \rho_2} \left| \frac{\partial f_2}{\partial u_1} (s\bar{u}_1(x), \bar{u}_2(x)) \right|$$
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\[
= (\alpha - 1) \bar{u}_2(x) \max_{\rho_1 \leq t, t \leq \rho_2} \frac{\partial f_2}{\partial u_2} (s\bar{u}_1(x), \tau\bar{u}_2(x)) - \epsilon_1(x)
\]
\[
\leq (\alpha - 1) \bar{u}_2(x) \max_{t \leq \rho_1} \frac{\partial f_2}{\partial u_2} (v_1(x, t), \tau\bar{u}_2(x)) - \epsilon_1(x),
\]
for each \( x \in \mathcal{D}, \ t > 0, \)

(here \( \epsilon_1(x) = \bar{\epsilon}_2(\alpha - 1) \bar{u}_2(x) \max_{t \leq \rho_1} \|\frac{\partial f_2}{\partial u_2}(s\bar{u}_1(x), \bar{u}_2(x))\| > 0, \) for \( x \in \mathcal{D} \)).

Choosing \( n \) to satisfy \( 0 < n(\alpha - 1) < -\lambda \), we have \( w_1 \) as an upper solution.

For \( v_1 \), we have the differential inequality:

\[
\sigma_1 \Delta v_1 + v_1[a + f_1(v_1, w_2)] - \frac{\partial v_1}{\partial t} = [1 - (1 - \beta) e^{-nt}] \bar{u}_1[f_1(v_1, w_2) - f_1(v_1, \bar{u}_2) + f_1(v_1, \bar{u}_2) - f_1(\bar{u}_1, \bar{u}_2)] - n(1 - \beta) e^{-nt}\bar{u}_1
\]
\[
\geq [1 - (1 - \beta) e^{-nt}] \bar{u}_1 \left[ \min_{t \leq \rho_1} \left\{ \frac{\partial f_1}{\partial u_1}(v_1, \tau\bar{u}_2) \right\} (\alpha - 1) \bar{u}_1 e^{-nt} - n(1 - \beta) e^{-nt}\bar{u}_1 \right.
\]
\[
\geq e^{-nt}\bar{u}_1 \left[ (1 - (1 - \beta) e^{-nt}) \bar{u}_1 - n(1 - \beta) \right],
\]
where \( \rho \) is a positive number, because

\[
\bar{u}_2(x)(\alpha - 1) \max_{t \leq \rho_1} \left\{ \frac{\partial f_2}{\partial u_2}(v_1(x, t), \tau\bar{u}_2(x)) \right\}
\]
\[
\geq (\alpha - 1) \bar{u}_2(x) \max_{t \leq \rho_1} \left\{ \frac{\partial f_1}{\partial u_1}(v_1(x, t), \tau\bar{u}_2(x)) \right\}
\]
\[
\leq (\alpha - 1) \bar{u}_2(x) \max_{t \leq \rho_1} \left\{ \frac{\partial f_2}{\partial u_2}(s\bar{u}_1(x), \tau\bar{u}_2(x)) \right\}
\]
\[
< (\alpha - 1) \left[ \min_{x \in \mathcal{D}} \left\{ \frac{\bar{u}_2(x)}{\bar{u}_1(x)} \right\} \max_{t \leq \rho_1} \left\{ \frac{\partial f_2}{\partial u_2}(s\bar{u}_1(x), \tau\bar{u}_2(x)) \right\} \right] - \bar{\epsilon}_1
\]
\[
\cdot \bar{u}_1(x) \min_{t \leq \rho_1} \left\{ \frac{\partial f_1}{\partial u_1}(s\bar{u}_1(x), \bar{u}_2(x)) \right\}
\]
\[
= [\min_{x \in \mathcal{D}}(\alpha - 1) G(x) - (\alpha - 1) \epsilon_1] \bar{u}_1(x) \max_{t \leq \rho_1} \frac{\partial f_1}{\partial u_1}(s\bar{u}_1(x), \bar{u}_2(x)) + \epsilon_1(x),
\]
for each \( x \in \mathcal{D}, \ t > 0, \)
(here $\bar{e}_a = (\bar{e}_2 - \bar{e}_1)(x - 1)$ | $\max_{x \in \mathcal{D}} (\partial f_1/\partial u_1)(x) > 0$, $\bar{u}_1(x) < 0$, for $x \in \mathcal{D}$, since we may reduce $\bar{e}_a$ so that $0 < \bar{e}_a < \bar{e}_1$. Reducing the choice of $n$ if necessary, so that $n(1 - \beta) < \beta p$, we have $v_1$ as a lower solution.

Since all the first partial derivatives of $f_1$ and $f_2$ have the same sign, we can interchange the role of $\bar{u}_1, f_1$ with $\bar{u}_2, f_2$ respectively and construct lower and upper solutions $v_2, w_1$ in exactly the same manner as before. Here $v_2, w_1$ are of the form $v_2 = [1 - (1 - \beta) e^{-m^t}] \bar{u}_2(x), w_1 = [1 + (\bar{\alpha} - 1) e^{-m^t}] \bar{u}_1(x)$ where $\bar{\beta}, \bar{\alpha}, m$ are chosen constants with $\rho_1 < \bar{\beta} < 1, 1 < \bar{\alpha} < \rho_2, m > 0$.

Finally, we observe that $v_1(x, t) \rightarrow \bar{u}_1(x), v_2(x, t) \rightarrow \bar{u}_2(x)$ from below as $t \rightarrow +\infty$, and $w_1(x, t) \rightarrow \bar{u}_2(x), w_2(x, t) \rightarrow \bar{u}_1(x)$ from above as $t \rightarrow +\infty$. Applying Lemma 2.1, we clearly have $(\bar{u}_1(x), \bar{u}_2(x))$ as an asymptotically stable solution as described in the theorem.

**Remark.** For the existence of a solution $(u_1(x, t), u_2(x, t))$ of the initial boundary value problem for equations (1.1), see e.g. [18].

As an example to an application of Theorem 2.1, we consider the boundary value problem

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \sigma_1 \Delta u_1 + u_1[10 - 4u_1 - u_2] \\
\frac{\partial u_2}{\partial t} &= \sigma_2 \Delta u_2 + u_2[10 - u_1 - 4u_2] \\
u_1 &= g_1(x), \quad \text{for } x \in \mathcal{D}, \quad t \geq 0, \quad i = 1, 2
\end{align*}
\]

where $1.5 \leq g_1(x) \leq 4$. Let $\phi_1(x) = 1.5, \psi_1(x) = 4$. We have $\sigma_1 \Delta \phi_1 + \phi_1[10 - 4\phi_1 - u_2] = 1.5[4 - u_2] \geq 0$ for each $1.5 \leq u_2 \leq 4$, and $\sigma_1 \Delta \psi_1 + \psi_1[10 - 4\psi_1 - u_2] = 4[-6 - u_2] \leq 0$ for each $1.5 \leq u_2 \leq 4$. Similarly, we let $\phi_2(x) = 1.5, \psi_2(x) = 4$ and prove as in Lemma 2.3 that there is an equilibrium $(\bar{u}_1(x), \bar{u}_2(x))$, with $\bar{u}_i(x) \in H^2(\mathcal{D}), 1.5 \leq \bar{u}_i(x) \leq 4$, for $x \in \mathcal{D}, i = 1, 2$. For (2.11), $\partial f_1/\partial u_1 = -4, \partial f_1/\partial u_2 = -1, \partial f_2/\partial u_1 = -1, \partial f_2/\partial u_2 = -4$. Therefore

\[
\left| \bar{u}_1(x) \left( \frac{\partial f_1}{\partial u_1} \right) (\bar{u}_1(x), \bar{u}_2(x)) \right| \cdot \left| \bar{u}_2(x) \left( \frac{\partial f_1}{\partial u_2} \right) (\bar{u}_1(x), \bar{u}_2(x)) \right|^{-1} \geq \frac{(1.5)(4)}{(4)(1)} = 1.5,
\]

and

\[
\left| \bar{u}_1(x) \left( \frac{\partial f_2}{\partial u_1} \right) (\bar{u}_1(x), \bar{u}_2(x)) \right| \cdot \left| \bar{u}_2(x) \left( \frac{\partial f_2}{\partial u_2} \right) (\bar{u}_1(x), \bar{u}_2(x)) \right|^{-1} \leq \frac{(4)(1)}{(1.5)(4)} = \frac{2}{3}.
\]

We see that (2.10) is satisfied for $i = 1, j = 2$. Similarly, one checks that (2.10)
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is also satisfied for \( i = 2, j = 1 \). Theorem 2.1 then implies that \((\hat{u}_1(x), \hat{u}_2(x))\) is asymptotically stable.

For a less restrictive example, consider equation (1.1). Suppose that there exist positive constants \( 0 < k_1 < K_1, 0 < k_2 < K_2 \) such that \( a + f_1(K_1, 0) < 0, b + f_2(0, K_2) < 0, a + f_1(k_1, K_2) > 0, b + f_2(K_1, k_2) > 0 \). Then for boundary values \( k_i \leq q_i(x) \leq K_i \), \( i = 1, 2 \), the boundary value problem (2.9) has an equilibrium solution \((\hat{u}_1(x), \hat{u}_2(x))\), \( k_i \leq \hat{u}_i \leq K_i \), \( i = 1, 2 \), as it is proved in the last paragraph. Let \( r_1 = \max \left[ (\partial f_1/\partial u_2) (u_1, u_2) (\min |\partial f_1/\partial u_1|)^{-1}, R_1 = \min |\partial f_2/\partial u_2| (\max |\partial f_2/\partial u_1|)^{-1}, r_2 = \max |\partial f_2/\partial u_1| (\min |\partial f_2/\partial u_2|)^{-1}, \right. \) and \( R_2 = \min |\partial f_1/\partial u_1| (\max |\partial f_1/\partial u_2|)^{-1} \); here max. and min. are taken over the rectangle \( \{(u_1, u_2): k_i \leq u_i \leq K_i, i = 1, 2 \} \). If one has \( k_1 K_2 r_i < k_1 k_2 R_i, i = 1, 2, \) then Theorem 2.1 implies that \((\hat{u}_1(x), \hat{u}_2(x))\) is asymptotically stable. (Roughly speaking, the condition that \( |\partial f_i/\partial u_j|, i \neq j, 1 \leq i, j \leq 2, \) is “small” compared with \( |\partial f_i/\partial u_i|, h = 1, 2 \) in the region of values of the solution will imply asymptotic stability.)

3. SOME HOMOGENEOUS BOUNDARY CONDITIONS

We next consider boundary value problems when one or both \( u_i \) are held at zero identically on \( \partial \Omega \). It turns out that the relative sizes of the growth rates \( a, b \) in relation to the diffusion rates \( \sigma_1, \sigma_2 \) are important in determining the stable asymptotic states. Let \( \lambda = \lambda_1 > 0 \) be the principal eigenvalue of the eigenvalue problem

\[ \Delta u + \lambda u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega. \]

**Theorem 3.1.** Let \( a < \sigma_1 \lambda_1 \). Suppose \( u^*_2(x) > 0 \) in \( \mathcal{G} \) is a solution of the boundary value problem

\[ \sigma_2 \Delta u + u [b + f_2(0, u)] = 0 \quad \text{in } \Omega \]
\[ u = g(x) > 0 \quad \text{on } \partial \Omega, \quad (3.1) \]

where \( g \) has an extension \( \hat{g} \in H^{2+\frac{1}{2}}(\hat{\mathcal{G}}) \), (such solution will be shown to exist). Let \( (u_1(x, t), u_2(x, t)) \) with \( u_i \in H^{2+1+\frac{1}{2}}(\mathcal{G} \times [0, T]) \), each \( T > 0, i = 1, 2, \) be a solution of the reaction-diffusion equations (1.1) with initial boundary conditions

\[ u_1(x, 0) = \theta_1(x) \geq 0, \quad x \in \mathcal{G} \]
\[ u_2(x, 0) = \theta_2(x) \geq 0 \]
\[ u_1(x, t) = 0 \quad (x, t) \in \partial \mathcal{G} \times [0, \infty) \]
\[ u_2(x, t) = g(x) > 0 \quad (x, t) \in \partial \Omega \times [0, \infty) \]
\[ u_1(x, 0) = \theta_1(x) \geq 0, \quad x \in \mathcal{G} \]
\[ u_2(x, 0) = \theta_2(x) \geq 0 \]
\[ u_1(x, t) = 0 \quad (x, t) \in \partial \mathcal{G} \times [0, \infty) \]
\[ u_2(x, t) = g(x) > 0 \quad (x, t) \in \partial \Omega \times [0, \infty) \]
where \( \theta_1, \theta_2, g \) satisfy the compatibility conditions of order 1 at \( t = 0 \) as described in [9], p. 319. Then \((u_1(x, t), u_2(x, t)) \to (0, u_2^*(x))\), as \( t \to \infty, x \in \mathscr{D} \).

**Remark.** An immediate consequence is that a positive solution of (3.1) is unique.

**Proof.** The zero function and large enough positive constant functions are respectively lower and upper solutions of the boundary value problem (3.1), therefore there exists a solution \( u_2^*(x) \) between them. To see that \( u_2^* > 0 \) in \( \mathscr{D} \), we observe that \( w = -u_2^*(x) e^{bt} (\leq 0) \) satisfies \( \sigma_2 \Delta w - \alpha w + f_2(0, u_2^*) = 0 \), where \( f_2(0, u_2^*) \leq 0 \). By the maximum principle we see that \( u_2^*(x) > 0 \) in \( \mathscr{D} \) as in Lemma 2.2. We now proceed to apply Lemma 2.1 by constructing appropriate \( v_i, w_i, i = 1, 2 \). Let \( v_i(x, t) \equiv 0 \) for \((x, t) \in \mathscr{D} \times [0, \infty)\), and define \( w_i(x, t) \) as the solution of the initial value problem

\[
\begin{align*}
\sigma_1 \Delta w_1 + w_1[a + f_1(w_1, 0)] - \frac{\partial w_1}{\partial t} &= 0, \\
(x, t) &\in \mathscr{D} \times (0, \infty), \\
w_1(x, 0) &= \theta_1(x), \\
w_1(x, t) &= 0, \\
(x, t) &\in \delta \mathscr{D} \times [0, \infty).
\end{align*}
\]

Existence of the solution is by [9], Theorem 4.1, p. 558. Further, by the maximum principle for parabolic equations ([14], Theorem 7, p. 174) we have \( w_1 > 0 \) in \( \mathscr{D} \times (0, \infty) \). Let \( \alpha_1 < \lambda_1^* \), and \( \mathscr{D}' \) be a domain containing \( \mathscr{D} \), and \( \psi(x) \) be a function satisfying \( \Delta \psi + \lambda_1^* \psi = 0 \) in \( \mathscr{D}' \), \( \psi|_{\partial \mathscr{D}'} = 0 \), \( \psi(x) > 0 \) in \( \mathscr{D}' \), \( \sup_{x \in \mathscr{D}'} |\psi(x)| = 1 \). Define \( z(x, t) \) by \( w_i(x, t) = z(x, t) \psi(x) e^{-\alpha_1 t} \), where \( \alpha_1 > 0 \) is chosen to satisfy \( a - \sigma_1 \lambda_1^* + \alpha_1 < 0 \). We have

\[
\sigma_1 \Delta z + 2\sigma_1 \frac{\nabla \psi \cdot \nabla z - \partial z}{\psi} - \frac{\partial z}{\partial t} = -z[a - \sigma_1 \lambda_1^* + \alpha_1 + f_1(w_1, 0)] > 0
\]
in \( \mathscr{D} \times (0, \infty) \), \( z = 0 \) on \( \partial \mathscr{D} \times [0, \infty) \), \( z(x, 0) \geq 0, \neq 0, x \in \mathscr{D} \). The maximum principle implies that \( 0 \leq z(x, t) \leq \sup\{\theta_1(x)/\psi(x): x \in \mathscr{D}\} \), and hence \( 0 \leq w_i(x, t) \leq Ke^{-\alpha_1 t} \) for \((x, t) \in \mathscr{D} \times [0, \infty)\), and some constant \( K \).

We now make a preliminary choice of \( v_2(x, t) \equiv 0, u_2(x, t) \equiv C \) where \( C > 0 \) is large enough. Lemma 2.1 implies that \( v_1(x, t) \leq u_1(x, t) \leq w_1(x, t) \) and \( 0 \leq u_4(x, t) \leq C \) for \((x, t) \in \mathscr{D} \times [0, \infty)\). Also, by the equation for \( u_2(x, t) \) and the maximum principle ([14], p. 175, Remark (ii)), we have \( u_2(x, t) > 0, (x, t) \in \mathscr{D} \times (0, \infty) \). To improve these estimates, we first observe that Lemma 2.1 can readily be generalized slightly, so as to apply to solutions of (1.1) considered on domains of the form \( \mathscr{D} \times [\tau, \infty) \), where \( \tau \geq 0 \). The 0's in (2.2) are to be replaced by \( \tau \)'s. We next change only the choice of \( v_2(x, t) \) to be identically equal to \( \delta > 0 \), which is sufficiently small so that \( \sigma_2 \Delta \psi + \psi[a + f_2(w_1, \psi, v_2)] - \partial \psi \partial t \geq 0 \) for \((x, t) \in \mathscr{D} \times [\tau, \infty) \), \( \tau \) sufficiently large (this is possible since \( w_1 \to 0 \) as \( t \to \infty \)).
uniformly in \( x \) as \( t \to \infty \), and \( f_2(0, 0) = 0, b > 0 \). Since \( \frac{\partial f_1}{\partial u_2} < 0 \), we have \( \sigma_1 \Delta w_1 + w_1[a + f_1(w_1, \delta)] - \frac{\partial w_1}{\partial t} \leq 0 \) for \( (x, t) \in \mathcal{D} \times [\tau, \infty) \). By reducing the choice of \( \delta > 0 \), if necessary, we also have (2.2) where all the 0's are replaced by \( \tau \)'s. The general version of Lemma 2.1 now implies that \( 0 < \delta \leq u_2(x, t) \leq C \) for \( (x, t) \in \mathcal{D} \times [\tau, \infty) \).

Finally, we make a last set of choice for \( v_2 \) and \( w_2 \). We take \( v_2(x, t) \) to be \( \beta(t) u^*_2(x) \), \( (x, t) \in [\tau_1, \infty) \), where \( \beta(t) = 1 - re^{-st} \) with \( \tau_1 > \tau \), and \( r, s > 0 \) to be chosen. We have

\[
\sigma_2 \Delta v_2 + v_2[b + f_2(w_1, v_2)] - \frac{\partial v_2}{\partial t} \geq \beta u^*_2[f_2(0, \beta u^*_2)] - rse^{-st}u^*_2
\]

for certain positive constants \( M_1, M_2 \). For convenience, we make the restriction \( \tau_1 = \rho \), a fixed number. For \( s < \alpha \) small enough, \( -M_1e^{-\alpha t} + M_2 > c \alpha \) for each \( x \in \mathcal{D} \), \( t \geq \tau_1 \), where \( c \) is a positive constant (here, no condition is imposed on \( r \)). Choose \( r, 0 < r < \rho \) such that \( (1 - r\rho) u^*_2(x) < \delta \) for all \( x \in \mathcal{D} \). Then reduce \( s \), if necessary, so that \( (1 - r\rho) > s \). Consequently, for \( (x, t) \in \mathcal{D} \times [\tau_1, \infty) \), \( \sigma_2 \Delta v_2 + v_2[b + f_2(w_1, v_2)] - \frac{\partial v_2}{\partial t} \geq u^*_2 \beta t (\beta(t) c - s) > 0 \); \( v_2(x, \tau_1) = (1 - r\rho) u^*_2(x) < \delta \leq u_2(x, \tau_1) \) and \( v_2(x, t) \leq u_2(x, t) \) on \( \delta \mathcal{D} \times [\tau_1, \infty) \). Thus \( v_2 \) is the required lower solution on \( \mathcal{D} \times [\tau_1, \infty) \).

Define \( w_2(x, t) \) to be \( \alpha(t) u^*_2(x) \), \( (x, t) \in \mathcal{D} \times [\tau_1, \infty) \), where \( \alpha(t) = 1 + pqe^{-st} \) with \( q, \rho \) to be chosen positive numbers. We have

\[
\sigma_2 \Delta w_2 + w_2[b + f_2(0, w_2)] - \frac{\partial w_2}{\partial t} = \sigma_2 \alpha \Delta u^*_2 + \alpha u^*_2[b + f_2(0, \alpha u^*_2)] + pqe^{-st}u^*_2
\]

for some \( k < 0 \) (which can be assumed to depend on the choice of \( r \) only), for \( (x, t) \in \mathcal{D} \times (\tau_1, \infty) \). Choose \( p > 0 \) such that \( w_2(x, \tau_1) = (1 + pqe^{-\tau_1}) u^*_2(x) \geq C \) for \( x \in \mathcal{D} \); then choose \( q > 0 \) small enough so that \( u^*_2k + q < 0 \), \( x \in \mathcal{D} \). Thus, we have \( w_2(x, t) \) is the required upper solution on \( \mathcal{D} \times (\tau_1, \infty) \).

Applying the general version of Lemma 2.1, we have \( 0 = v_1(x, t) \leq u_2(x, t) \leq w_1(x, t), \beta(t) u^*_2(x) \leq u_2(x, t) \leq \alpha(t) u^*_2(x) \), for \( (x, t) \in \mathcal{D} \times [\tau_1, \infty) \). Since \( w_1(x, t) \to 0, \alpha(t) \to 1, \beta(t) \to 1, \) as \( t \to \infty \), the theorem is proved.
Remark. Analogous result is true when \( b < \sigma_2 \lambda_1 \) and \( u_1^* > 0 \) in \( \Omega \) is a solution of \( \sigma_1 \Delta u + u[a + f_1(u, 0)] = 0 \) in \( \Omega \), \( u = g(x) > 0 \) on \( \partial \Omega \). That is, we have \( (u_1(x, t), u_2(x, t)) \rightarrow (u_1^*(x), 0) \), as \( t \rightarrow \infty \), \( x \in \Omega \).

**THEOREM 3.2.** Let \( a < \sigma_1 \lambda_1 \), \( b < \sigma_2 \lambda_1 \), and \( (u_1(x, t), u_2(x, t)) \) with \( u_i \in H^{2+1/2}([\Omega \times [0, T]]) \), each \( T > 0 \), \( i = 1, 2 \), be a solution of the reaction-diffusion equations (1.1) with initial boundary conditions

\[
\begin{align*}
\phi_i(x, 0) &= \phi_i(x) \geq 0, \quad i = 1, 2, \quad x \in \Omega \\
u_i(x, t) &= 0, \quad i = 1, 2, \quad (x, t) \in \partial \Omega \times [0, \infty)
\end{align*}
\]

where \( \phi_i \), \( i = 1, 2 \) satisfy the compatibility conditions of order 1 at \( t = 0 \) as described in [9], p. 319. Then \( (u_1(x, t), u_2(x, t)) \rightarrow (0, 0) \) uniformly for \( x \in \Omega \), as \( t \rightarrow \infty \).

**Proof.** Let \( v_1(x, t) = v_2(x, t) = 0 \) for \( (x, t) \in \partial \Omega \times [0, \infty) \). Define \( w_1(x, t) \) exactly as it is in Theorem 3.1. Let \( w_2(x, t) \) be the solution of the initial value problem

\[
\begin{align*}
\sigma_2 \Delta w_2 + w_2[b + f_2(0, w_2)] - \frac{\partial w_2}{\partial t} &= 0, \quad (x, t) \in \Omega \times (0, \infty), \\
w_2(x, 0) &= \phi_2(x), \quad x \in \Omega \\
w_2(x, t) &= 0, \quad (x, t) \in \partial \Omega \times [0, \infty).
\end{align*}
\]

As for \( w_1 \), we prove in the same way as in the first part of the proof in Theorem 3.1 that \( w_2(x, t) \rightarrow 0 \) uniformly too, for \( x \in \Omega \), as \( t \rightarrow \infty \). By Lemma 2.1, the theorem is proved.

In view of the last theorem, it is interesting to note that in certain cases, nontrivial equilibrium states exist even under the homogeneous boundary conditions \( u_i(x, t) = 0 \), \( i = 1, 2 \), \( (x, t) \in \partial \Omega \times [0, \infty) \). Suppose that

\[
a > \sigma_1 \lambda_1, \quad b > \sigma_2 \lambda_1
\]

and there are positive constants \( \bar{k}_1, \bar{k}_2 \) such that

\[
\begin{align*}
a - \sigma_1 \lambda_1 + f_1(0, \bar{k}_2) &> 0 \\
b + f_2(0, \bar{k}_2) &< 0 \\
b - \sigma_2 \lambda_1 + f_2(\bar{k}_1, 0) &> 0 \\
a + f_1(\bar{k}_1, 0) &< 0.
\end{align*}
\]

(Note that this can happen when \( |\partial f_i/\partial u_i|, \ i \neq j \) is small compared with \( |\partial f_i/\partial u_i|, \ 1 \leq i, j \leq 2 \), the following theorem holds.
Theorem 3.3. Under assumptions (3.2) and (3.3), the boundary value problem

\begin{align*}
\sigma_1 \Delta u_1 + u_1[a + f_1(u_1, u_2)] &= 0, \quad x \in \mathcal{D}, \\
\sigma_2 \Delta u_2 + u_2[b + f_2(u_1, u_2)] &= 0, \\
\quad u_i(x) &= 0, \quad x \in \partial \mathcal{D}, \quad i = 1, 2
\end{align*}

(3.4)

has a solution \((u_1^0(x), u_2^0(x))\), with \(u_i^0(x) \in H^{2+1}(\mathcal{D}), u_i^0(x) > 0, \) for \(x \in \mathcal{D}, i = 1, 2\).

Proof. Let \(\omega_1(x)\) be the principal eigenfunction for the eigenvalue problem \(\Delta u + \lambda u = 0\) in \(\mathcal{D}\), \(u = 0\) on \(\partial \mathcal{D}\), with \(\lambda = \lambda_1\) as the principal eigenvalue (thus \(\omega_1(x) > 0\) in \(\mathcal{D}\)). For \(k_1 > 0\) small enough, we have \(\sigma_1 \Delta (k_1 \omega_1) + k_1 \omega_1[a + f_1(k_1 \omega_1, u_2)] = k_1 \omega_1[a - \sigma_1 \lambda_1 + f_1(k_1 \omega_1, u_2)] > 0\), and \(\sigma_1 \Delta k_1 + k_1[a + f_1(k_1, u_2)] < 0\) for \(0 \leq u_2 \leq k_2\). Also for \(k_2 > 0\) small enough, we have \(\sigma_2 \Delta (k_2 \omega_1) + k_2 \omega_1[b + f_2(u_1, k_2 \omega_1)] = k_2 \omega_1[b - \sigma_2 \lambda_1 + f_2(u_1, k_2 \omega_1)] > 0\), and \(\sigma_2 \Delta k_2 + k_2[b + f_2(u_1, k_2 \omega_1)] < 0\) for \(0 \leq u_1 \leq k_1\). Thus by [5] or [17], there exists a solution \((u_1^0(x), u_2^0(x))\) to (3.4), with \(k_i \omega_1(x) \leq u_i^0(x) \leq k_i, u_i^0(x) \in H^{2+1}(\mathcal{D}), i = 1, 2\). Since \(\omega_1(x) > 0\) in \(\mathcal{D}\), the theorem is proved.

Remark (i). In the case \(a < \sigma_1 \lambda_1\) and \(b < \sigma_2 \lambda_1\), let \((\hat{u}_1(x), \hat{u}_2(x))\) be a solution of the boundary value problem (3.4), with \(\hat{u}_1(x) \geq 0\) in \(\mathcal{D}\). The function \(u = k \omega_1(x)\) satisfies \(\sigma_1 \Delta u + u[a + f_1(u, \hat{u}_2)] = k \omega_1[a - \sigma_1 \lambda_1 + f_1(k \omega_1, \hat{u}_2)] \leq 0\) in \(\mathcal{D}\) and \(u = 0\) on \(\partial \mathcal{D}\), for each \(k \geq 0\). Further, it is well known that \(\partial \omega_1/\partial n < 0\) where \(n\) is the outward unit normal at the boundary. Therefore, by means of Serrin’s sweeping principle (see e.g. [16], p. 40), we must have \(\hat{u}_1(x) = 0\) in \(\mathcal{D}\) (i.e. corresponding to \(k = 0\)). Similarly, we have \(\hat{u}_2(x) = 0\) in \(\mathcal{D}\). In view of Theorem 3.3, one might say that bifurcation occurs when \(a\) and \(b\) increase pass \(\sigma_1 \lambda_1\) and \(\sigma_2 \lambda_1\) respectively.

Remark (ii). In the case \(a < \sigma_1 \lambda_1\) and \(b > \sigma_2 \lambda_1\), the boundary value problem (3.4) has a nontrivial solution of the form \((u_1(x), u_2(x)) = (0, \bar{u}_2(x))\), where \(k \omega_1(x) \leq \bar{u}_2(x) \leq K, k, K\) are respectively small and large positive constants. This can be readily proved as in Theorem 3.3.

References