Compensated Compactness, Paracommutators, and Hardy Spaces*

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Let $B_1 : \mathbb{R}^n \times \mathbb{R}^{N_1} \to \mathbb{R}^m_1$, $B_2 : \mathbb{R}^n \times \mathbb{R}^{N_2} \to \mathbb{R}^m_2$ and $Q : \mathbb{R}^m \to \mathbb{R}^m$ be bilinear forms which are related as follows: if $u$ and $v$ satisfy $B_1(\zeta, u) = 0$ and $B_2(\zeta, v) = 0$ for some $\zeta \neq 0$, then $u'Qv = 0$. Suppose $p^{-1} + q^{-1} = 1$. Coifman, Lions, Meyer and Semmes proved that, if $u \in L^p(\mathbb{R}^n)$ and $v \in L^q(\mathbb{R}^n)$, and the first order systems $B_j(D, u) = 0$, $B_j(D, v) = 0$ hold, then $u'Qv$ belongs to the Hardy space $H^1(\mathbb{R}^n)$, provided that both (i) $p = q = 2$, and (ii) the ranks of the linear maps $B_j(\zeta, \cdot) : \mathbb{R}^N \to \mathbb{R}^m$ are constant. We apply the theory of paracommutators to show that this result remains valid when only one of the hypotheses (i), (ii) is postulated. The removal of the constant-rank condition when $p = q = 2$ involves the use of a deep result of Lojasiewicz from singularity theory.

1. INTRODUCTION

Recent discoveries tie the weak continuity of various nonlinear quantities in compensated compactness with the theory of harmonic analysis, showing that many of these quantities are in fact in well-known Hardy spaces. We refer readers to the paper [CLMS2] for more details. Further related results can be found in [D], [CG], [JJ], [M] and [Zk].

The problem we are concerned with is set up as follows. Let

- $B_1 : \mathbb{R}^n \times \mathbb{R}^{N_1} \to \mathbb{R}^m_1$
- $B_2 : \mathbb{R}^n \times \mathbb{R}^{N_2} \to \mathbb{R}^m_2$

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be two vector-valued bilinear forms. Therefore for every non-zero $\xi \in \mathbb{R}^n$, $B_j(\xi, \cdot)$ are linear maps from $\mathbb{R}^N_j$ to $\mathbb{R}^{m_j}$, $j = 1, 2$. Let $Q: \mathbb{R}^{m_2} \to \mathbb{R}^{m_1}$ be a linear map which satisfies $\mu^* Q v = 0$ whenever $B_1(\xi, \mu) = 0$, $B_2(\xi, v) = 0$ for some $\xi \neq 0$. Denote $q(\mu, v) = \mu^* Q v$, the bilinear form on $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ related to $Q$.

In [CLMS2], it has been proved that when $B_1 = B_2 = B$ and rank $B(\xi, \cdot) = constant$, $p > 2n/(n + 1)$, then for $u \in L^p_{loc}(\mathbb{R}^n)$, such that $B(D, u) \in W^{-1,p}_{loc}(\mathbb{R}^n)$ for some $r > p$, the quadratic form $q(u, u) \in H^{-2}_{loc}(\mathbb{R}^n)$. Here $D = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ denotes the differential operator.

Our aim in this paper is to show that the bilinear form $q(u, v)$ belongs to $H^r(\mathbb{R}^n)$ whenever $u \in L^p(\mathbb{R}^n)$, $v \in L^q(\mathbb{R}^n)$ where $1/p + 1/q = 1/r < 1 + 1/n$, $B_1(D, u) = 0$, $B_2(D, v) = 0$, under the condition that rank $B_1(\xi, \cdot)$ and rank $B_2(\xi, \cdot)$ are constant. Then the local case will be the consequence of this result and Hodge decomposition. We also prove that, without constant rank condition, the above result is still true provided that $p = q = 2$. We have not obtained a full local analogue of this result, though we have proved some partial results by applying an idea of Zhou [Zy].

The main tools we use are the theory of paracommutators, which were established by Janson and Peetre [JP], Li [Li] and Peng [P]. In Section 2, we give a brief review of the main results in paracommutator theory which we shall use; Section 3 is devoted to the main results of this paper, under the constant rank condition; in Section 4 we study the range of those bilinear forms that were considered in Section 3; Section 5 deals with the non-constant rank case; we give an alternative approach in dealing with these bilinear forms in Section 6. This will be further developed in another paper [LMcZ].

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2. PARACOMMUTATORS

Paracommutators were first introduced and studied by Janson and Peetre in [JP]. Further work has been done by Li [Li] and Peng [P]. They are operators $T_{\vec{\alpha}}(A)$ which can be expressed using the Fourier transform by

$$
\widehat{T_{\vec{\alpha}}(A)}(\vec{\xi}) = (2\pi)^{-n} \int \hat{b}(\vec{\xi} - \vec{\eta}) A(\vec{\xi}, \vec{\eta}) \hat{f}(\vec{\eta}) d\vec{\eta}, \quad \vec{\xi} \in \mathbb{R}^n,
$$

where $A$ is a fixed function on $\mathbb{R}^n \times \mathbb{R}^n$ for each operator and $b$ is called the symbol of $T_{\vec{\alpha}}(A)$. This is a generalization of commutators between singular integral operators and multiplier operators. For example, when $A(\vec{\xi}, \vec{\eta}) = (\vec{\xi}/|\vec{\xi}|)(-\vec{\eta}/|\vec{\eta}|)$, then $T_{\vec{\alpha}}(A) = [b, R_j]$, the commutator of the Riesz transform $R_j$ with the multiplier $b$. 

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To state some known results about paracommutators, we need more notations.

Let $M_p(\mathbb{R}^n)$ be the set of Fourier multipliers of $L^p(\mathbb{R}^n)$ ($p > 1$), and define $M_p(\mathbb{R}^n \times \mathbb{R}^n)$ in the following way: $m \in M_p(\mathbb{R}^n \times \mathbb{R}^n)$ if

$$m(\zeta, \eta) = \int_X \alpha(\zeta, x) \beta(\eta, x) \, d\mu(x)$$

for some $\sigma$-finite measure space $(X, \mu)$ and measurable functions $\alpha, \beta$ on $\mathbb{R}^n \times X$ such that

$$\int_X \|\alpha(\cdot, x)\|_{M_p(\mathbb{R}^n)} \|\beta(\cdot, x)\|_{M_p(\mathbb{R}^n)} \, d\mu(x) < \infty$$

and denote

$$\|m\|_{M_p(\mathbb{R}^n \times \mathbb{R}^n)} = \inf \int_X \|\alpha(\cdot, x)\|_{M_p(\mathbb{R}^n)} \|\beta(\cdot, x)\|_{M_p(\mathbb{R}^n)} \, d\mu(x),$$

where the inf is taken over all such $(X, \mu)$ and $\alpha, \beta$.

It is easy to see that $M_p(\mathbb{R}^n) \otimes M_p(\mathbb{R}^n) \subset M_p(\mathbb{R}^n \times \mathbb{R}^n)$.

For $\gamma > 0$, we say that $A$ satisfies condition $A_3(\gamma)$ if there exist $\delta > 0$ and $C > 0$ such that for all $\xi_0 \neq 0$, and all $r < \delta |\xi_0|$, we have

$$\|A(\xi_0, \eta)\|_{M_p(\mathbb{R}^n \times \mathbb{R}^n)} \leq C \left( \frac{r}{|\xi_0|} \right)^\gamma, \quad (A_3(\gamma))$$

where $\chi \in C_c^\infty(\mathbb{R}^n)$, $\text{supp} \chi \subset B(0, 2)$, $\chi(0) = 1$ on $B(0, 1)$.

For $U \times V \subset \mathbb{R}^n \times \mathbb{R}^n$, we say that $m \in M_p(U \times V)$ if $m(\xi, \eta) \chi_U(\xi) \chi_V(\eta) \in M_p(\mathbb{R}^n \times \mathbb{R}^n)$, where $\chi_U, \chi_V$ are characteristic functions of sets $U$ and $V$, respectively.

We say that $A$ satisfies condition $A_5$ if for every $\xi_0 \neq 0$, there exist $\delta > 0$ and $\eta_0 \in \mathbb{R}^n$, such that $A(\xi, \eta)^{-1} \in M_p(U \times V)$, with $U = \{\xi : |\xi| / |\xi_0| < \delta, |\xi_\perp| > |\xi_0|\}$ and $V = B(\eta_0, \delta |\xi_0|)$.

We have the following theorem.

**Theorem 2.1.** Suppose $1 < p < \infty$ and $A$ is homogeneous of degree 0, $A \in M_p(\mathbb{R}^n \times \mathbb{R}^n)$, and $A$ satisfies $A_3(\gamma)$. Then for $0 \leq s < \gamma$, the operator $I^{-s}T_{\lambda}(A)$ is bounded on $L^p(\mathbb{R}^n)$ for $b \in I'(\text{BMO})$ and $\|I^{-s}T_{\lambda}(A)\|_{op} \leq C \|b\|_{\text{BMO}}$ with $C$ independent of $s$. Here $I^s$ and $I^{-s}$ denote fractional integration and differentiation. Furthermore, if there is no $\xi \neq 0$ such that $A(\xi + \eta, \eta) = 0$ for a.e. $\eta$, then for $b \in I'(\text{BMO})$, $I^{-s}T_{\lambda}(A) = 0$ on $L^p(\mathbb{R}^n)$ implies $b = 0$. If $A$ also satisfies $A_5$, and $I^{-s}T_{\lambda}(A)$ is bounded on $L^p(\mathbb{R}^n)$, then $b \in I'(\text{BMO})$ and $\|b\|_{\text{BMO}} \leq C \|I^{-s}T_{\lambda}(A)\|_{op}$. 

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The case when $p = 2$ was proved by Janson and Peetre [JP]. When $s = 0$, $1 < p < \infty$, the result was obtained by Li [Li], but the proof can be easily applied to the case when $0 < s < \gamma$. Actually more general conditions on $A$ where considered in [JP] and [Li].

Remark 2.2. For a weight $w \in A_p$, $M_p(\mathbb{R}^n, w)$ denotes the space of bounded Fourier multipliers on $L^p(w)$, and $M_p(\mathbb{R}^n \times \mathbb{R}^n, w)$ is defined in the same way as $M_p(\mathbb{R}^n \times \mathbb{R}^n)$ with $M_p(\mathbb{R}^n)$ being replaced by $M_p(\mathbb{R}^n, w)$. Then Theorem 2.1 remains true if we replace every space by the corresponding weighted space.

3. COMPENSATED COMPACTNESS OF BILINEAR FORMS IN HARDY SPACES

Hardy spaces play an important role in harmonic analysis. They are defined as follows. Let $h \in C_0^\infty(\mathbb{R}^n)$, $h \geq 0$, $\text{Supp } h \subset B(0, 1)$, $h_t(x) = 1/tnh(x/t)$. For $0 < r < \infty$, define

$$H^r(\mathbb{R}^n) = \{ f \in S'(\mathbb{R}^n) : \sup_{t > 0} |h_t * f| \in L^r(\mathbb{R}^n) \}.$$  

Let

$$B_1 : \mathbb{R}^n \times \mathbb{R}^{N_1} \rightarrow \mathbb{R}^{m_1}$$

$$B_2 : \mathbb{R}^n \times \mathbb{R}^{N_2} \rightarrow \mathbb{R}^{m_2}$$

be two vector-valued bilinear forms. Therefore for every non-zero $\xi \in \mathbb{R}^n$, $B_j(\xi, \cdot)$ are linear maps from $\mathbb{R}^{N_j}$ to $\mathbb{R}^{m_j}$, $j = 1, 2$. We suppose in this section that rank $B_j(\xi, \cdot)$ are constant for $\xi \neq 0$.

Let $Q : \mathbb{R}^{m_2} \rightarrow \mathbb{R}^{m_1}$ be a linear map which satisfies $\mu^T Q \nu = 0$ whenever $B_1(\xi, \mu) = 0$, $B_2(\xi, \nu) = 0$ for some $\xi \neq 0$. Denote $q(\mu, \nu) = \mu^T Q \nu$, the bilinear form on $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ related to $Q$. Then we have the following theorem.

**Theorem 3.1.** Suppose $1 < p < \infty$, $1 < q < \infty$, $1/p + 1/q = 1$. Assume that rank $B_j(\xi, \cdot)$ are constant for $\xi \neq 0$. Then for $u \in L^p(\mathbb{R}^n)^{N_1}$, $v \in L^q(\mathbb{R}^n)^{N_2}$, such that $B_j(D, u) = 0$, $B_j(D, v) = 0$ in the distribution sense, we have $q(u, v) \in H^1(\mathbb{R}^n)$, where $D = (\partial/\partial x_1, \partial/\partial x_2, ..., \partial/\partial x_n)$.

**Example.** The best known special case of this theorem is the div-curl problem. In this case, $n = N_1 = N_2 = m_2 = 3$, $m_1 = 1$, $Q = I$, $B_1(\xi, \mu) = \xi \cdot \mu$, $B_2(\xi, \nu) = \xi \times \nu$. We have $B_1(\xi, \mu) = 0$, $B_2(\xi, \nu) = 0$ for some $\xi \neq 0$ if $\mu \cdot \nu = 0$. Therefore, according to Theorem 3.1, if $u \in L^p(\mathbb{R}^3)^3$, $\text{div } u = 0$, $v \in L^q(\mathbb{R}^3)^3$, $\text{curl } v = 0$, then $u \cdot v \in H^1(\mathbb{R}^3)$. This result has been proved in [CLMS2].
Proof of Theorem 3.1. For \( j = 1, 2 \), and \( \xi \neq 0 \), let \( H^j = \{ \xi \in \mathbb{R}^n : B_j(\xi, 2) = 0 \} \), and let \( \pi^j : \mathbb{R}^n \to H^j \) be the orthogonal projection. Then \( \pi^j \) are homogeneous of degree 0 in \( \xi \), which, because of the constant rank assumption, depend smoothly on \( \xi \). Note that \( \| \pi^j \| = 1 \).

Define \( P_j : L^p(\mathbb{R}^n)^N \to L^p(\mathbb{R}^n)^N \) by \( P_j u(\xi) = \pi^j \hat{u}(\xi) \). Notice that the functions \( P_j u \) are real valued, and that \( B_j(D, u) = 0 \) if and only if \( P_j u = u \).

Consider the bilinear form \( L(u, v) = q(P_j u, P_j v) \). To prove the theorem, it is enough to prove that \( L(u, v) \in H^p(\mathbb{R}^n) \) for all \( u \in L^p(\mathbb{R}^n)^N, v \in L^q(\mathbb{R}^n)^N \). For \( b \in \mathcal{A}(\mathbb{R}^n) \), by the Plancheral formula, we get

\[
\langle L(u, v), b \rangle = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{u}(\xi - \eta) \pi^j(\xi - \eta) Q^j\hat{b}(-\xi) \hat{v}(\eta) d\eta.
\]

Since \( \text{VMO}(\mathbb{R}^n)^* = H^p(\mathbb{R}^n) \), to prove that \( L(u, v) \in H^p(\mathbb{R}^n) \), we only have to prove that the operator \( T_b(A) \) defined by

\[
T_b(A) u(\xi) = \int_{\mathbb{R}^n} \hat{u}(\eta) \pi^j(\xi - \eta) Q^j \hat{b}(\xi - \eta) d\eta
\]

is bounded from \( L^p(\mathbb{R}^n)^N \) to \( L^q(\mathbb{R}^n)^N \) for \( b \in \text{VMO}(\mathbb{R}^n) \) with the operator bound being bounded by \( \| b \|_{\text{BMO}} \).

It is easy to see that \( T_b(A) \) is a paracommutator with \( A(\xi, \eta) = \pi^j Q^j \), which is homogeneous of degree 0 and belongs to \( M_j(\mathbb{R}^n \times \mathbb{R}^n) \). We want to prove that \( A \) satisfies condition \( A_3(j) \). Since \( q(\lambda_1, \lambda_2) = 0 \) if \( B_j(\eta, \lambda_1) = 0, B_j(\eta, \lambda_2) = 0 \), we have \( \pi^j Q^j = 0 \). Thus \( A(\xi, \eta) = \pi^j Q^j \). Now \( \pi^j \in M_j(\mathbb{R}^n) \). So, once we prove that \( \pi^j \) satisfies \( A_3(j) \), then it follows that \( A \) satisfies \( A_3(j) \) for all \( 1 < p < \infty \). By applying Theorem 2.1, we get that \( T_b(A) \) is bounded on \( L^p(\mathbb{R}^n) \) for \( b \in \text{VMO}(\mathbb{R}^n) \), and \( \| T_b(A) \|_{\text{op}} \leq C \| b \|_{\text{BMO}} \). We complete the proof.

So what remains is for us to prove the following lemma [Li].

Lemma 3.2. If \( m \) is a smooth function on \( \mathbb{R}^n \setminus \{ 0 \} \), homogeneous of degree 0, then \( m(\xi) - m(\eta) \) satisfies \( A_3(j) \) for all \( 1 < p < \infty \).

Proof. We need to prove that there exist \( C > 0 \) and \( \delta > 0 \) such that for all \( \xi_0 \neq 0 \),

\[
C \left( \frac{\xi_0 - \xi}{r} \right) (m(\xi) - m(\eta)) \chi \left( \frac{\eta - \xi_0}{r} \right) \leq C \frac{r}{|\xi_0|}, \quad r < \delta |\xi_0|,
\]

where \( \chi \) is as in the definition of \( A_3(j) \).
First we prove that if $D^* f \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, then
\[ |f|_{M_f(\mathbb{R}^n \times \mathbb{R}^n)} \leq C \sum_{|\alpha| \leq n+1} \|D^* f\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}. \tag{3.2} \]

Since
\[ f(\xi, \eta) = (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} \hat{f}(x, y) e^{ix\xi} e^{iy\eta} \, dx \, dy \]
and $\|e^{ix\xi}\|_{M_f(\mathbb{R}^n)} \leq 1$, $\|e^{iy\eta}\|_{M_f(\mathbb{R}^n)} \leq 1$, we have
\[ \|f\|_{M_f(\mathbb{R}^n \times \mathbb{R}^n)} \leq C \int_{\mathbb{R}^n \times \mathbb{R}^n} |\hat{f}(x, y)| \, dx \, dy \leq C \sum_{|\alpha| \leq n+1} \|D^* f\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}. \]

Just simply localizing (3.2) we can get: if $f \in C^{n+1}(B(\xi_0, 2r) \times B(\xi_0, 2r))$, then
\[ \left\| \left( \frac{\xi - \xi_0}{r} \right) f(\xi, \eta) \left( \frac{\eta - \eta_0}{r} \right) \right\|_{M_f(\mathbb{R}^n \times \mathbb{R}^n)} \leq C \sum_{|\alpha| \leq n+1} r^{|\alpha|} \sup_{\xi, \eta \in B(\xi_0, 2r)} |D^* f(\xi, \eta)|. \tag{3.3} \]

If furthermore, $f(\xi_0, \xi_0) = 0$ and $r \leq |\xi_0|$, then we have
\[ \left\| \left( \frac{\xi - \xi_0}{r} \right) f(\xi, \eta) \left( \frac{\eta - \xi_0}{r} \right) \right\|_{M_f(\mathbb{R}^n \times \mathbb{R}^n)} \leq C \frac{r}{|\xi_0|} \sum_{|\alpha| \leq n+1} |\xi_0|^{|\alpha|} \sup_{\xi, \eta \in B(\xi_0, 2r)} |D^* f(\xi, \eta)|. \tag{3.4} \]

This is because, when $|\alpha| \geq 1$, $r^{|\alpha|} \leq r \frac{|\xi_0|}{|\xi_0|}^{|\alpha|-1}$, and when $\alpha = 0$,
\[ |f(\xi_0, \eta)| = |f(\xi_0, \eta) - f(\xi_0, \xi_0)| \leq 2r \sup_{\xi, \eta \in B(\xi_0, 2r)} |Df(\xi, \eta)|. \]

Now (3.1) is proved by letting $f(\xi, \eta) = m(\xi) - m(\eta)$ in (3.4) and $\delta = 1/4$.

Remark 3.3. Theorem 3.1 was proved in [CLMS2] in the case when $B_1 = B_2$ and $p = q$, where the quadratic form $q(u, u)$ was considered instead of the bilinear form $q(u, v)$.
We can also consider the weighted case. Suppose for $1 < p < \infty$, $(w_1, w_2)$ is a pair of weights such that $w_1 \in A_p$ and

\[
\left( \int_{\mathbb{R}^n} |f(x)|^p w_2(x) \, dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}^n} |Mf(x)|^p w_1(x) \, dx \right)^{1/p}
\]

for all $f \in L^q(w_1)$. Then we have the following result.

**Theorem 3.4.** Suppose $1 < p < \infty$, $1 < q < \infty$, $1/p + 1/q = 1$ and $(w_1, w_2)$ satisfies the above conditions. Then for $u \in L^q(\mathbb{R}^n, w_1^{1-})$, $v \in L^q(\mathbb{R}^n, w_2^{1-})$ such that $B_1(D, u) = 0$, $B_2(D, v) = 0$ in the distribution sense, we have $q(u, v) \in H^1(\mathbb{R}^n)$.

**Proof.** By Remark 2.2 and a similar procedure to that used above, we only have to verify that $\pi_1^0$, $\pi_2^0$ are smooth and homogeneous of degree 0. To prove the second statement, we only have to show that $\pi_1^0 - \pi_2^0 \in A_3(1, w_1)$, which we can prove by making a minor change in the above proof that $\pi_1^0 - \pi_2^0 \in A_3(1)$.

We can prove the local version of Theorem 3.1 by using the following generalized Hodge decomposition theorem.

**Theorem 3.5.** If $u \in L^q(\mathbb{R}^n)^N$, $B(D, u) \in W^{-1/q}(\mathbb{R}^n)^n$ with $t > p > 1$, then $u = u_0 + u_1$ with $u_0 \in L^q(\mathbb{R}^n)^N$, $B(D, u_0) = 0$ and $u_1 \in L^q(\mathbb{R}^n)^N$.

**Proof.** As before $P$ is defined by $\hat{P}u(\xi) = \pi_2 \hat{u}(\xi)$. Then $u = Pu + (I - P) u = u_0 + u_1$. Obviously, $u_0 \in L^q(\mathbb{R}^n)^N$ and $B(D, u_0) = 0$. Let $H^1_\#$ be the orthogonal complement of $H_\#$. Then $I - \pi_2$ is the orthogonal projection onto it. Since for $\lambda \in H^1_\#$, $B(\xi, \lambda) = 0$ implies $\lambda = 0$, so $B(\xi, \cdot)$ is a one to one map from $H^1_\#$ to $\mathbb{R}^n$. There exists a smooth, matrix-valued function $D(\xi)$ of $\beta \neq 0$, homogeneous of degree $-1$, which satisfies $D(\xi) B(\xi, \mu) = \mu$ for all $\mu \in H^1_\#$. Thus the operator $S$ defined by the Fourier multiplier $D(\xi)$ is bounded from $W^{-1/q}(\mathbb{R}^n)^m$ to $L^q(\mathbb{R}^n)^N$. Therefore, $u_1 = S \cdot B(D, (I - P) u) = S(B(D, u)) \in L^q(\mathbb{R}^n)^N$.

**Theorem 3.6.** If $u \in L^q_{\text{loc}}(\mathbb{R}^n)^N$, $v \in L^q_{\text{loc}}(\mathbb{R}^n)^N$, $B_1(D, u) \in W^{-1/q}(\mathbb{R}^n)^m$, $B_2(D, v) \in W^{-1/q}(\mathbb{R}^n)^m$ with $t > p$, $s > q$, $1/p + 1/q = 1$, then $q(u, v) \in H^1_{\text{loc}}(\mathbb{R}^n)$.

**Proof.** By multiplying $u$ and $v$ by a smooth cut off function, we reduce the problem to that of proving $q(u, v) \in H^1_{\text{loc}}(\mathbb{R}^n)$ for $u \in L^q(\mathbb{R}^n)^N$, $v \in L^q(\mathbb{R}^n)^N$, $B_1(D, u) \in W^{-1/q}(\mathbb{R}^n)^m$, $B_2(D, v) \in W^{-1/q}(\mathbb{R}^n)^m$. By the Hodge decomposition,
Obviously, \( a_{ij} \) 

decomposition theorem for functions in \( H^b \) 

For any \( \eta \) implies \( \zeta = 0 \), then for \( 1/p + 1/q = 1 \), \( H^1(\mathbb{R}^n) \) is the smallest linear space containing 

\[ \{ q(u,v) : u \in L^q(\mathbb{R}^n)^N, v \in L^q(\mathbb{R}^n)^N, B_1(D,u) = 0, B_2(D,v) = 0 \}. \]

Proof. We only have to prove that if \( b \in \text{BMO} = (H^1(\mathbb{R}^n))^* \) and 

\[ \langle q(u,v), b \rangle = 0 \] 

for all \( u \in L^q(\mathbb{R}^n)^N, v \in L^q(\mathbb{R}^n)^N \) such that \( B_1(D,u) = 0 \) and \( B_2(D,v) = 0 \), then \( b = 0 \). But \( \langle q(u,v), b \rangle = 0 \) implies \( \langle L(u,v), b \rangle = 0 \) for all \( u \in L^q(\mathbb{R}^n)^N, v \in L^q(\mathbb{R}^n)^N \) where \( L(u,v) \) is the bilinear form we introduced in Section 3. Thus this implies \( T_\beta(A) = 0 \) on \( L^q(\mathbb{R}^n)^N \). Therefore by Theorem 2.1, \( b = 0 \).

Under stronger conditions on \( Q \) we can prove the following stronger decomposition theorem for functions in \( H^1(\mathbb{R}^n) \).

Theorem 4.1. Let \( \pi_1^qQ\pi_1^q = (a_{ij}(\xi, \eta))_{N \times N} \) be the matrix-valued function. Suppose one of the \( a_{ij}(\xi, \eta) \) satisfies that for any \( \xi \neq 0 \), \( \eta \neq 0 \) such that \( a_{ij}(\xi_0, \eta_0) \neq 0 \). For \( 1/p + 1/q = 1 \), let 

\[ F = \{ q(u,v) : \|u\|_p \leq 1, \|v\|_q \leq 1, B_1(D,u) = 0, B_2(D,v) = 0 \}. \]

Then any \( w \in H^1(\mathbb{R}^n) \) can be decomposed into \( w = \sum \lambda_k w_k \) with \( w_k \in F \) and \( \sum |\lambda_k| < \infty \).

Remark 4.3. The div-curl case was proved in [CLMS2].

Proof of Theorem 4.2. We first prove that such \( a_{ij}(\xi, \eta) \) satisfies A5. Obviously, \( a_{ij}(\xi, \eta) = \sum \beta_i(\xi) \beta_j(\eta) \) with \( \beta_i, \beta_j \) smooth, homogeneous of degree 0. For any \( \xi \neq 0 \), let \( \eta_0 \) be the one such that \( a_{ij}(\xi_0, \eta_0) \neq 0 \). Then for \( U, V \) defined as in A5, we have

\[ u = u_0 + u_1, \quad u_0 \in L^p(\mathbb{R}^n)^N, \quad B_1(D,u_0) = 0, \quad u_1 \in L^q(\mathbb{R}^n)^N, \]

\[ v = v_0 + v_1, \quad v_0 \in L^q(\mathbb{R}^n)^N, \quad B_2(D,v_0) = 0, \quad v_1 \in L^q(\mathbb{R}^n)^N. \]

Thus \( q(u,v) = q(u_0,v_0) + q(u_1,v_0) + q(u_0,v_1) + q(u_1,v_1) \). By Theorem 3.4, \( q(u_0,v_0) \in H^1(\mathbb{R}^n) \); and \( q(u_0,v_1) + q(u_1,v_0) + q(u_1,v_1) \in L^{1+}(\mathbb{R}^n) \) for some \( \varepsilon > 0 \). Therefore \( q(u,v) \in H^1_{\text{loc}}(\mathbb{R}^n) \).
Thus if we choose $\delta < 1/2 |a_0(\xi_0, \eta_0)|$, then

$$\frac{1}{|a_0(\xi_0, \eta_0)|} M(U \times V) \ll + \infty.$$ 

Such $a_0(\xi, \eta)$ satisfies the conditions in Theorem 2.1. Applying Theorem 2.1 we get

$$\|b\|_{BMO} \lesssim \sup_{[u], |v| = 1} |\langle T_d(a_0) u, v \rangle| = \sup_{[u], |v| = 1} |\langle T_d(A) u, v \rangle|,$$

where $u_i = (0, \ldots, u_i, \ldots, 0)$, $v_j = (0, \ldots, v_j, \ldots, 0)$ and $A(\xi, \eta) = \pi^1 Q \pi^2$. Therefore we proved

$$\|b\|_{BMO} \lesssim C \sup_{[u], |v| = 1} |\langle T_d(A) u, v \rangle| = C \sup_{[u], |v| = 1} |\langle q(u, v), b \rangle|.$$

Combining with Theorem 3.1 we get,

$$\|b\|_{BMO} \sim \sup_{[u], |v| = 1} |\langle q(u, v), b \rangle|.$$

The decomposition result follows by applying the following two lemmata, proofs of which were given in [CLMS2].

**Lemma 4.4.** Let $F$ be a bounded subset of a normed vector space $X$. We assume that $\bar{F}$ (closure of $F$ for the norm of $X$) contains the unit ball (centered at 0) of $X$. Then any $x$ in that ball can be written as

$$x = \sum_{j=0}^{\infty} \frac{1}{2^j} y_j$$

where $y_j \in F$ for all $j \geq 0$.
Lemma 4.5. Let $F$ be a bounded symmetric $(x \in F \Rightarrow -x \in F)$ subset of a normed vector space $X$. Then the closed convex hull $\tilde{F}$ of $F$ contains a ball centered at $0$ if and only if, for any $l \in X^*$, $\|l\|_{X^*}$ and sup$_{x \in X} \langle x, l \rangle$ are two equivalent norms.

5. NON-CONSTANT RANK CASE

In the previous sections we suppose that rank $B_j(\xi, \cdot)$, $j = 1, 2$ are constant. Actually we can drop this assumption in case $p = q = 2$, i.e., we can prove the following theorem.

Theorem 5.1. For $u \in L^2(\mathbb{R}^n \times V_1)$, $v \in L^2(\mathbb{R}^n \times V_2)$, with $B_1(D, u) = 0$, $B_2(D, v) = 0$ in the distribution sense, $q(u, v) \in H^1(\mathbb{R}^n)$.

Proof. When rank $B_1(\xi)$ and rank $B_2(\xi)$ are not constant, $\pi^1$ and $\pi^2$ are no longer necessarily smooth, but still bounded. Thus, $\pi^1\xi$ and $\pi^2\xi$ are in $M_2(\mathbb{R}^n)$, and therefore $A(\xi, \eta) = \pi^1Q\pi^2 \in M_2(\mathbb{R}^n \times \mathbb{R}^n)$. We want to prove that $A$ satisfies $A(3\gamma)$ for some $\gamma > 0$, i.e.

Lemma 5.2. There exist $\gamma > 0$ and $C > 0$ such that whenever $|\xi_0| = 1$, $r < 1$, $B = B(\xi_0, r)$, then

$$\|A(\xi_0, \eta)\|_{M_2(\mathbb{R}^n \times \mathbb{R}^n)} \leq Cr^\gamma.$$

Our proof of this lemma depends on the following result of Lojasiewicz [Lo].

Theorem 5.3. Let $f$ be a real analytic function defined on an open set $\Omega \subset \mathbb{R}^n$, $V_f = \{x \in \Omega : f(x) = 0\}$, and let $K$ be a compact subset of $\Omega$. Then for all $x \in K$, $|f(x)| \geq C\delta(d(x, V_f))^N$ for some positive constants $C$ and $N \geq 1$, where $d(x, V_f)$ is the distance from $x$ to $V_f$.

Proof of Lemma 5.2. Without losing generality we can suppose that $B_1 = B_2 = B$ and $Q$ is symmetric, otherwise just let

$$\tilde{Q} = \begin{pmatrix} 0 & Q \\ Q^* & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}.$$

then we have

$$\pi_1 = \begin{pmatrix} \pi_1^1 \\ 0 \\ 0 \\ \pi_1^2 \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} 0 \\ \pi_2^1 \end{pmatrix}, \quad \pi_1^1 \tilde{Q} \pi_1^2 = \begin{pmatrix} 0 & \pi_1^1 Q \pi_2^2 \\ (\pi_1^1 Q \pi_2^2)^* & 0 \end{pmatrix}.\]
Suppose rank $B$ takes the values $n_1 < \cdots < n_k + 1$. Let $\mathbb{S}^{n-1}$ be the unit sphere, and

$$S_j = \{ \xi \in \mathbb{S}^{n-1} : \text{rank } B(\xi, \cdot) \leq n_j \}, \quad 1 \leq j \leq k + 1.$$ 

It is clear that $S_j$ are closed subsets of $\mathbb{S}^{n-1}$ and

$$S_1 \subset S_2 \subset \cdots \subset S_k \subset S_{k+1} = \mathbb{S}^{n-1}.$$ 

First we want to prove that for $\xi_0 \in S_1$,

$$\|(I - \pi_{\xi_0}) \pi_\xi\| \leq C |\xi - \xi_0| \tag{1}$$

and for $\xi_0 \in S_{j+1} \setminus S_j$, $j = 1, \ldots,$

$$\|(I - \pi_{\xi_0}) \pi_\xi\| \leq \frac{C}{d(\xi_0, S_j)^{n_j}} |\xi - \xi_0|, \tag{2}$$

where $C$ and $N_j$ are independent of $\xi$ and $\xi_0$.

Since $B(\xi_0, \pi_\mu) = 0$ for all $\mu \in \mathbb{R}^N$, we have $|B(\xi_0, \pi_\mu)| = |B(\xi_0 - \xi, \pi_\mu)| \leq C |\xi - \xi_0| |\mu|$. On the other hand, if we let $\lambda_0(\xi_0)$ be the smallest non-zero eigenvalue of $B(\xi_0, \cdot)$, then $|B(\xi_0, \pi_\mu)| \geq \sqrt{\lambda_0(\xi_0)} \|(I - \pi_{\xi_0}) \pi_\mu\|$ for all $\mu \in \mathbb{R}^N$. Thus we get

$$\|(I - \pi_{\xi_0}) \pi_\xi\| \leq \frac{C}{\sqrt{\lambda_0(\xi_0)}} |\xi - \xi_0|. \tag{3}$$

To estimate $\lambda_0(\xi_0)$, we consider

$$\det(I - B(\xi_0, \cdot)B(\xi_0, \cdot)) = \lambda^N + C(\xi_0) \lambda^{N-1} + \cdots + C_N(\xi_0),$$

where $C_l(\xi_0)$, $l = 1, \ldots, N$, are polynomials of $\xi_0$. If $\xi_0 \in S_1$, we have $B(\xi_0, \cdot) = n_1$ and this implies $C_{n_1}(\xi_0) \neq 0$ and $C_l(\xi_0) = 0$, if $l > n_1$. Therefore

$$\lambda_0(\xi_0) \geq C |C_{n_1}(\xi_0)| \geq C \quad \text{if} \quad \xi_0 \in S_1,$$

because $S_1$ is a closed subset of $\mathbb{S}^{n-1}$. This, together with (3), implies (1).

If $\xi_0 \in S_{j+1} \setminus S_j$, $j = 1, \ldots, k$, we have $B(\xi_0, \cdot) = n_{j+1}$. This implies $C_{n_{j+1}}(\xi_0) \neq 0$ and $C_l(\xi_0) = 0$, if $l > n_{j+1}$. Especially, for $C_{n_{j+1}}(\xi_0)$ we have

$$C_{n_{j+1}}(\xi_0) \begin{cases} = 0, & \text{if } \xi_0 \in S_j, \\ \neq 0, & \text{if } \xi_0 \in \mathbb{S}^{n-1} \setminus S_j. \end{cases}$$
By Theorem 5.3, we have
\[ \lambda_0(\xi_0) \geq C |C_{\eta_0}(\xi_0)| \geq Cd(\xi_0, S_j)^{2N_j} \quad \text{for some } 2N_j \geq 1. \]
This together with (3) gives (2).

Now we are ready to prove Lemma 5.2. We first notice that
\[ \pi_x Q \pi_{\eta} = \pi_x (I - \pi_{\eta_0}) Q \pi_{\eta} + \pi_x \pi_{\eta_0} Q (I - \pi_{\eta_0}) \pi_{\eta_0}. \quad (4) \]
If \( \xi_0 \in S_j \), by (1), we have
\[ \| \pi_x Q \pi_{\eta} \|_M \leq Cr. \]
Suppose we have shown that for \( \xi_0 \in S_j, j \geq 2, \)
\[ \| \pi_x Q \pi_{\eta} \|_M \leq Cr. \quad (5) \]
Then for \( \xi_0 \in S_{j+1} \setminus S_j \), by (2) and (4), we have
\[ \| \pi_x Q \pi_{\eta} \|_M \leq \frac{Cr}{d(\xi_0, S_j)^{N_j}}. \]
Thus, for \( r < d(\xi_0, S_j)^{2N_j} \),
\[ \| \pi_x Q \pi_{\eta} \|_M \leq Cr^{1/2}; \]
for \( d(\xi_0, S_j)^{2N_j} < r < 1 \), take \( \xi_1 \in S_j \) such that \( d(\xi_0, \xi_1) = d(\xi_0, S_j) \). Let \( B_1 = B(\xi_1, r + d(\xi_0, S_j)) \), then \( B \subset B_1 \). Therefore by assumption (5) and the fact that \( 2N_j \geq 1 \), we get
\[ \| \pi_x Q \pi_{\eta} \|_M \leq C(r + d(\xi_0, S_j))^{\gamma_j} \]
\[ \leq C(r + r^{1/2N_j}) \leq Cr^{1/2N_j}. \]
Thus we proved that for \( \xi_0 \in S_{j+1} \),
\[ \| \pi_x Q \pi_{\eta} \|_M \leq Cr^{\gamma_{j+1}} \]
with \( \gamma_{j+1} = \min \{1/2, \gamma_j/2N_j \} \). By induction, we obtain that for \( \xi_0 \in \mathbb{S}^n \),
\[ \| \pi_x Q \pi_{\eta} \|_M \leq Cr^{\gamma} \]
for some \( \gamma > 0 \). This completes the proof of Lemma 5.2.

Since \( A = \pi_{\eta_0} Q \pi_\eta \) satisfies the conditions in Theorem 2.1 with \( p = 2 \), we have that \( T_{\lambda}(A) \) is bounded on \( L^2(\mathbb{R}^n) \) when \( b \in \text{BMO} \). Therefore as in the
proof of Theorem 3.1, we can show that for $u \in L^2(\mathbb{R}^n)^N$, $v \in L^2(\mathbb{R}^n)^N$, with $B_1(D, u) = 0$, $B_2(D, v) = 0$ in the distribution sense, we have $q(u, v) \in H^1(\mathbb{R}^n)$.

We also can get the same results as in Section 4 about the range of compensated quantities of non-constant rank case. The proofs remain the same.

6. ANOTHER APPROACH

In this section we give another approach to study compensated compactness through a special decomposition. More generally, suppose $Q(\xi)$, $B_1(\xi)$, $B_2(\xi)$ are smooth matrix-valued functions, which together with their derivatives, are bounded by powers of $|\xi|$ at infinity. Let $Q(D)$, $B_1(D)$, $B_2(D)$ be the corresponding pseudo-differential operators.

**Theorem 6.1.** Suppose there exist matrix-valued functions $A_1(\xi)$, $A_2(\xi)$, such that

$$Q(\xi) = B_1^*(\xi) A_1^*(\xi) + A_2^*(\xi) B_2^*(\xi) \quad \forall \xi \neq 0,$$

and $A_j(\xi) B_j(\xi)$ is smooth for $j = 1, 2$, and homogeneous of degree 0. Then, $u^* Q(D) v \in H^1(\mathbb{R}^n)$ if $u \in L^p(\mathbb{R}^n)^N$, $v \in L^q(\mathbb{R}^n)^N$; $B_1(-D) u = 0$, $B_2(D) v = 0$ with $1/p + 1/q = 1$. Furthermore, $A_j(\xi) B_j(\xi)$ is not the constant function. Let

$$F = \{ u^* Q(D) v : B_1(-D) u = 0, B_2(D) v = 0, \|u\|_p \leq 1, \|v\|_q \leq 1 \}.$$ 

Then any $w \in H^1(\mathbb{R}^n)$ can be decomposed as $w = \sum k \lambda_k w_k$ with $\sum |\lambda_k| < \infty,$ $w_k \in F$.

**Proof.** Suppose $B_1(-D) u = 0$, $B_2(D) v = 0$. Then

$$\hat{u}^* Q(D) \hat{v}(\xi) = \int_{\mathbb{R}^n} \hat{u}^*(\xi - \eta) Q(\eta) \hat{v}(\eta) \, d\eta$$

$$= \int_{\mathbb{R}^n} \hat{u}^*(\xi - \eta) [B_1^*(\eta) A_1^*(\eta) + A_2^*(\eta) B_2^*(\eta)] \hat{v}(\eta) \, d\eta$$

$$= \int_{\mathbb{R}^n} \hat{u}^*(\xi - \eta) B_1^*(\eta) A_1^*(\eta) \hat{v}(\eta) \, d\eta$$

$$= \int_{\mathbb{R}^n} \hat{u}^*(\xi - \eta) [B_1^*(\eta) A_1^*(\eta) - B_1^*(\eta - \xi) A_1^*(\eta - \xi)] \hat{v}(\eta) \, d\eta$$

and therefore $\langle u^* Q(D) v, b \rangle = \langle [K, b] u, v \rangle$ with

$$\hat{K}(\xi) = B_1^*(\xi) A_1^*(\xi) \hat{u}(\xi).$$
Since $A_{l}(\xi)B_{l}(\xi)$ is homogeneous of degree zero and smooth, by the theory of paracommutators, we have the desired boundedness of $[K, b]$.

To prove the second part, as before, we only have to show that $\|b\|_{BMO} \leq C \|[K, b]\|$. This can be obtained by observing that $A(\xi, \eta) = B_{l}^{*}(\eta) A_{l}^{*}(\eta) - B_{l}(\xi) A_{l}^{*}(\xi)$ satisfies condition 4 stated in Section 2 when $B_{l}(\xi) A_{l}^{*}(\xi)$ is a non-constant function, smooth, homogeneous of degree 0. We also have the following more general theorem.

**Theorem 6.2.** Suppose there exist matrix-valued functions $A_{l}(\xi), A_{2}(\xi)$, such that

$$Q(\xi) = B_{l}(\xi) A_{l}^{*}(\xi) + A_{2}(\xi) B_{2}(\xi) \quad \forall \xi \neq 0,$$

and $A_{l}(\xi) B_{l}(\xi)$ is smooth and homogeneous of degree 0. Then

$$u^{*}Q(D) v \in H^{1}(\mathbb{R}^{n}) + L^{1+\alpha}(\mathbb{R}^{n}) + L^{1+\alpha}(\mathbb{R}^{n})$$

if $u \in L^{q}(\mathbb{R}^{n})^{N}, v \in L^{q}(\mathbb{R}^{n})^{N}$, $A_{l}(-D) B_{l}(-D) u \in L^{q}(\mathbb{R}^{n}), A_{2}(D) B_{2}(D) v \in L^{q}(\mathbb{R}^{n})$ with $t > p$, $s > q$. $1/p + 1/q = 1$, $(1 + \epsilon) = 1/p + 1/q$, $1/(1 + \epsilon) = 1/t + 1/q$.

**Proof of Theorem 6.2.** As in the proof of Theorem 6.1, we have

$$u^{*}Q(D) v(\xi) = \int_{\mathbb{R}^{n}} \hat{u}(\xi - \eta) Q(\eta) \hat{v}(\eta) \, d\eta$$

$$= \int_{\mathbb{R}^{n}} \hat{u}(\xi - \eta) [B_{l}^{*}(\eta) A_{l}^{*}(\eta) + A_{2}(\eta) B_{2}(\eta) \hat{v}(\eta) \, d\eta$$

$$= \int_{\mathbb{R}^{n}} \hat{u}(\xi - \eta) A_{l}(\eta) B_{l}(\eta) \hat{v}(\eta) \, d\eta$$

$$+ \int_{\mathbb{R}^{n}} \hat{u}(\xi - \eta) B_{l}(\xi - \eta) A_{l}^{*}(\xi - \eta) \hat{v}(\eta) \, d\eta$$

$$+ \int_{\mathbb{R}^{n}} \hat{u}(\xi - \eta) [B_{l}^{*}(\eta) A_{l}^{*}(\eta) - B_{l}(\eta - \xi) A_{l}^{*}(\eta - \xi)] \hat{v}(\eta) \, d\eta.$$

Thus we have $u^{*}Q(D) v = u^{*}A_{l}(D) B_{l}(D) v + (A_{l}(-D) B_{l}(-D)) u^{*} v$ plus a term which we have proved belongs to $H^{1}(\mathbb{R}^{n})$.

In fact we can prove that $u^{*}Q(D) v \in H^{1}(\mathbb{R}^{n}) + L \log L(\mathbb{R}^{n})$ if $A_{l}(-D) B_{l}(-D) u \in L^{q}(\mathbb{R}^{n}), A_{2}(D) B_{2}(D) v \in L^{q}(\mathbb{R}^{n})^{N}$. See [LMcZ].

Now we can go back to Section 3 where $Q$ is a constant matrix and $B_{l}(\xi), B_{2}(\xi)$ depend linearly on $\xi$, and $L^{q}Q = 0$ whenever $B_{l}(\xi) \lambda = 0$, $B_{2}(\xi) \mu = 0$ for some $\xi \neq 0$. We have the following decomposition.
PROPOSITION 6.4. Under either condition (1) or (2) below, there exist \( A_1(\xi), A_2(\xi) \), which are smooth, homogeneous of degree \(-1\), such that
\[
Q = B'_1(\xi) A'_1(\xi) + A_2(\xi) B_2(\xi) \quad \forall \xi \neq 0.
\]

(1) \((N_1 - \text{rank } B_1(\xi))(N_2 - \text{rank } B_2(\xi)) = \text{constant}\);

(2) Suppose \( \text{rank } Q = N_1 \leq N_2 \), let \( Q^{-1} \) be the operator such that \( Q^{-1} Q = I \). \( B'_1(\xi) B_1(\xi) \) and \((Q^{-1})' B'_2(\xi) B_2(\xi) Q^{-1} \) are commutative.

Proof. Under condition (1), the decomposition is proved by Zhou [Zy]. He considers, for each \( \xi \neq 0 \), \( B'_1(\xi) A'_1(\xi) + A_2(\xi) B_2(\xi) = Q \) as a linear system, with unknowns \( A_1(\xi), A_2(\xi) \), whose rank is \( N_1 N_2 - (N_1 - \text{rank } B_1(\xi))(N_2 - \text{rank } B_2(\xi)) \). He also proved that \( N_1 N_2 - (N_1 - \text{rank } B_1(\xi))(N_2 - \text{rank } B_2(\xi)) \) is constant if and only if either rank \( B_1(\xi) \) and rank \( B_2(\xi) \) are constant, or \((N_1 - \text{rank } B_1(\xi))(N_2 - \text{rank } B_2(\xi)) = 0\).

Under condition (2), we can suppose \( Q = I \). Otherwise, just let \( \tilde{v} = Qv \), \( \tilde{B}_2(\xi) = B_2(\xi) Q^{-1} \). Then
\[
A_1(\xi) = (B'_1(\xi) B_1(\xi) + B'_2(\xi) B_2(\xi))^{-1} B'_1(\xi)
\]
\[
A_2(\xi) = (B'_1(\xi) B_1(\xi) + B'_2(\xi) B_2(\xi))^{-1} B'_2(\xi)
\]
is the solution. The invertability of \( B'_1(\xi) B_1(\xi) + B'_2(\xi) B_2(\xi) \) comes from the fact that \( \mu \cdot v = 0 \) if \( B_1(\xi) \mu = 0, B_2(\xi) v = 0 \).

Thus Theorem 3.1, Theorem 3.6 and Theorem 4.1 are easy consequences of Theorem 6.1, Theorem 6.2 and Proposition 6.4. Actually we can deal with some non-constant rank cases shown in the following examples.

EXAMPLE. It is proved in [CLMS2] that \( u(x, y) v(x, y) \in H^1_{loc}(\mathbb{R}^2) \) if
\[
u \in L^q(\mathbb{R}^2), \quad \frac{\partial u}{\partial x} \in W^{-1, q}(\mathbb{R}^2), \quad \nu \in L^q(\mathbb{R}^2), \quad \frac{\partial v}{\partial y} \in W^{-1, q}(\mathbb{R}^2).
\]
In fact, this is the case when \( Q = I, B_1(\xi) = \xi_1, B_2(\xi) = \xi_2 \), which satisfies (1) in Proposition 6.3 with non-constant rank. (Actually (2) is also satisfied).

Actually in this case, i.e., when \( Q \) is a constant matrix and \( B_1, B_2 \) linearly depend on \( \xi \), and satisfy the conditions in Proposition 6.4, we have the following result.

PROPOSITION 6.5. For any \( 1 < p < \infty, 1 < q < \infty, 1 \leq 1/p + 1/q < 1 + 1/n \), if \( u \in L^p_{loc}(\mathbb{R}^n), \nu \in L^q_{loc}(\mathbb{R}^n), B_1(D) u = 0, B_2(D) \nu = 0 \) in distribution sense, then
\[
\sup_{r > 0} |h_r * (u^* Qv)(x)| \leq CM(u^r)^{1/p}(x) M(v^r)^{1/q}(x).
\]
**Proof.** \( h \) can be taken such that \( h(x) = (\phi(x))^3 \) with \( \phi \geq 0 \), Supp \( \phi \subset B(0, 1) \). Denote \( \phi^*_t(y) = \phi((x-y)/t) \). Then

\[
h_t \ast (u^*Qv)(x) = \frac{1}{t^3} \langle (\phi^*_t)^3, u^*Qv \rangle + \frac{1}{t^3} \langle \phi^*_t, (\phi^*_t u)^* Q\phi^*_t v \rangle
\]

\[
= \frac{1}{t^3} \langle [K, \phi^*_t] u, \phi^*_t v \rangle + \frac{1}{t^3} \langle \phi^*_t, (\phi^*_t u, A_t(D) B_t(D)(\phi^*_t v) \rangle
\]

\[
+ \frac{1}{t^3} \langle \phi^*_t A_t(D) B_t(D)(\phi^*_t u), \phi^*_t v \rangle = 1 + II + III.
\]

By the boundedness of paracommutators,

\[
|I| \leq \frac{1}{t^2} \| I^{-t} \phi^*_t \|_{\text{BMO}} \| \phi^*_t u \|_p \| \phi^*_t v \|_q,
\]

where \( 1/p + 1/q = 1 + s/n \) with \( 0 \leq s < 1 \). Therefore, since \( \| I^{-t} \phi^*_t \|_{\text{BMO}} \leq \| I^{-t} \phi^*_t \|_{\infty} \approx C/t^s \),

\[
|I| \leq \frac{1}{t^2} C \left( \int_{|x-t|} |u|^p \left( \int_{|x-t|} |v|^q \right)^{1/q} \right)
\]

\[
= \frac{1}{|B(x,t)|} \left( \int_{|x-t|} |u|^p \right)^{1/p} \left( \frac{1}{|B(x,t)|} \int_{|x-t|} |v|^q \right)^{1/q}.
\]

For the second term, choose \( q_1 \) such that \( 1/p + 1/q_1 = 1 \). Then

\[
|II| \leq \frac{1}{t^2} \| (\phi^*_t)^2 u \|_p \| A_t(D) B_t(D)(\phi^*_t v) \|_{q_1}.
\]

Since \( A_t(D) \) is a pseudo-differential operator of degree \(-1\), we have, for \( 1/q_2 = 1/q_1 + 1/n \),

\[
\| A_t(D) B_t(D)(\phi^*_t v) \|_{q_2} \leq C \| B_t(D)(\phi^*_t v) \|_{q_2}
\]

\[
= C \left( \left( \int_{|x-t|} |v|^q \right)^{1/q} \right)
\]

\[
\leq C \| B_t(D) \phi^*_t \|_{n/(1-s)} \left( \int_{|x-t|} |v|^q \right)^{1/q}
\]

\[
\leq C \left( \left( \int_{|x-t|} |v|^q \right)^{1/q} \right)
\]

since \( 1/q_2 = 1/q + (1-s)/n \). Thus we proved

\[
|II| \leq C \left( \frac{1}{|B(x,t)|} \int_{|x-t|} |u|^p \right)^{1/p} \left( \frac{1}{|B(x,t)|} \int_{|x-t|} |v|^q \right)^{1/q}.
\]
Similarly, we can get the same estimate for $\mathrm{III}$. By taking supremum over $t > 0$, we complete the proof of Proposition 6.5.

An immediate corollary of this Proposition is that

**Theorem 6.6.** Under the same conditions as in Proposition 6.4, $u^Q(D) v \in H'(\mathbb{R}^n)$ if $u \in L^p(\mathbb{R}^n)$, $v \in L^q(\mathbb{R}^n)$, $B_2(-D) u = 0$, $B_2(D) v = 0$ with $1/r = 1/p + 1/q < 1 + 1/n$.

In fact, the bilinear quantity $u^Qv$ we have considered in the paper is defined as a distribution. What we have proved is that this quantity belongs to Hardy space $H'$ under certain conditions. On the other hand, the quantity $u^Qv$ makes sense pointwise and is in fact a measurable function in $L^r$. They are in general not the same. To distinguish the difference, let $(u^Qv)_d$ and $(u^Qv)_m$ be the distribution and pointwise function respectively. To investigate the relationship between these two quantities, we first quote the following statement in [CLMS].

**Proposition.** Suppose $\phi \in C_0^\infty(\mathbb{R}^n)$ satisfies $\int \phi = 1$. For $0 < r < 1$, there exists a linear, continuous map $P$ from $H'(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$, such that $P(f) = f$ if $f \in H'(\mathbb{R}^n) \cap L^r_{\mathrm{loc}}$, and $f * \phi$ converges a.e. to $P(f)$ (and in $L^r$) as $t$ goes to 0 for every $f \in H'(\mathbb{R}^n)$.

In [CLMS], it is proved that, among many examples, if $u \in L^p$, $v \in L^q$ for $1 < p < \infty$, $1 < q < \infty$, $1/p + 1/q < 1 + 1/n$, and $\text{div} u = 0$, $\text{curl} v = 0$, then $(u \cdot v)_d \in H'$ for $1/r = 1/p + 1/q$, and $P((u \cdot v)_d) = (u \cdot v)_m$.

We can prove the following more general result.

**Theorem 6.7.** Under the same conditions as in Theorem 6.6, we have $P((u^Qv)_d) = (u^Qv)_m$.

**Proof.** Similar as in the proof of Theorem 6.6, we can prove that

$$|h_t * (u^Qv)_d - h_t * u^Qh_t * v| \leq C \left( \frac{1}{|B(x, t)|} \int_{B(x, t)} |u(y)|^p \, dy \right)^{1/p} \times \left( \frac{1}{|B(x, t)|} \int_{B(x, t)} |v(y) - h_t * v(x)|^q \, dy \right)^{1/q} \quad (\ast)$$

Therefore, if we choose $h$ such that $\int h = 1$, then by letting $t$ tends to 0 in (\ast), we have $P((u^Qv)_d) = (u^Qv)_m$. 


REFERENCES