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# Schur–Weyl Reciprocity for the Hecke Algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$

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## INTRODUCTION

The purpose of this paper is to give a reciprocity between $U_q(h)$ and $\mathscr{H}_{n,r}$, the Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$ introduced by Ariki and Koike [1].

The Schur–Weyl reciprocity was originally discovered for $GL(m)$ and $\mathfrak{S}_n$ [13; 15, p. 130]. This is the first example of dual pairs and has been generalized to various pairs of groups and algebras. Jimbo [7, 8] proved a $q$-analogue of the original reciprocity, namely that between $U_q(gl_m)$ and the Hecke algebra $\mathscr{H}_n$ of type $A$. Ram [12] utilizes the reciprocity to obtain a character formula of $\mathscr{H}_n$.

Let $K = \mathbb{Q}(q, u_1, \dots, u_r)$ be the field of rational functions in variables $q, u_1, \dots, u_r$. We adopt $K$ as the base field for both the quantized universal enveloping algebra $U_q(gl_r)$ and the Hecke algebra $\mathscr{H}_n$.

We denote by $U_q(h)$ the $K$-subalgebra of $U_q(gl_r)$ generated by $q^{E_{ii}}$'s $(1 \le i \le r)$. In this paper, we show that the commutant of $U_q(h)$ in $\mathrm{End}((K^r)^{\otimes n})$ is isomorphic to a quotient of $\mathscr{H}_{n,r}$. We also determine the irreducible decomposition of $(K^r)^{\otimes n}$ under the action of $\mathscr{H}_{n,r}$. As a consequence, we obtain the reciprocity for $U_q(h)$ and $\mathscr{H}_{n,r}$.

Let us review the classical Schur–Weyl reciprocity in a modified sense, i.e., that between $U(\mathfrak{g})$ and $\mathfrak{S}_{n,r}$. Here, $U(\mathfrak{g})$ denotes the universal

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enveloping algebra of $\mathfrak{g} = gl_{m_1} \oplus \cdots \oplus gl_{m_r}$, and $\mathfrak{S}_{n,r}$ is the group consisting of $n \times n$ permutation matrices whose nonzero entries are $r$th roots of unity. The group $\mathfrak{S}_{n,r}$ is generated by $\mathfrak{S}_n$ and

$$s_1 = e^{2\pi\sqrt{-1}/r}E_{11} + E_{22} + \cdots + E_{nn}.$$

The vector representation of $\mathfrak{g}$ is $\mathbb{C}^m = \mathbb{C}^{m_1} \oplus \cdots \oplus \mathbb{C}^{m_r}$, which has the standard basis $v_j^i$ $(1 \le j \le r,\ 1 \le i \le m_j)$. On $(\mathbb{C}^m)^{\otimes n}$, $\mathfrak{S}_n$ acts by permuting components of the tensor product. The action of $gl_m$ on $(\mathbb{C}^m)^{\otimes n}$ is infinitesimally diagonal. We can extend the action of $\mathfrak{S}_n$ to that of $\mathfrak{S}_{n,r}$ by letting $s_1$ act on $(\mathbb{C}^m)^{\otimes n}$ by

$$s_1\left(v_{j_1}^{i_1} \otimes \cdots \otimes v_{j_n}^{i_n}\right) = e^{2\pi\sqrt{-1}j_1/r}v_{j_1}^{i_1} \otimes \cdots \otimes v_{j_n}^{i_n}.$$

Since $U(\mathfrak{g})$ is a subalgebra of $U(gl_m)$, we naturally have the action of $U(\mathfrak{g})$ on $(\mathbb{C}^m)^{\otimes n}$ through that of $U(gl_m)$ on it. Thereby, $(\mathbb{C}^m)^{\otimes n}$ is a $(U(\mathfrak{g}) \times \mathfrak{S}_{n,r})$-module. It is well known that both of the irreducible representations of $\mathfrak{S}_{n,r}$ and $U(\mathfrak{g})$ occurring in $(\mathbb{C}^m)^{\otimes n}$ are indexed by the set $\Lambda_{m_1,\ldots,m_r}(n)$ of $r$-tuples $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of Young diagrams with $\sum_{i=1}^r |\lambda^{(i)}| = n$, and $l(\lambda^{(i)}) \le m_i$ for $i = 1, \ldots, r$ [11]. The irreducible representation space corresponding to $\underline{\lambda} \in \Lambda_{m_1,\ldots,m_r}(n)$ is denoted by $W_{\underline{\lambda}}$ for $U(\mathfrak{g})$ and $V_{\underline{\lambda}}$ for $\mathfrak{S}_{n,r}$, respectively. Then we actually have

$$(\mathbb{C}^m)^{\otimes n} = \bigoplus_{\underline{\lambda} \in \Lambda_{m_1,\ldots,m_r}(n)} W_{\underline{\lambda}} \otimes V_{\underline{\lambda}}$$

as a $(U(\mathfrak{g}) \times \mathfrak{S}_{n,r})$-module. As a consequence, each image of $U(\mathfrak{g})$ and of the group ring $\mathbb{C}\mathfrak{S}_{n,r}$ in $\mathrm{End}_{\mathbb{C}}((\mathbb{C}^m)^{\otimes n})$ is the full centralizer algebra of the other. The same situation also appears in a natural setting for finite fields, which will be explained in the Appendix.

We will give a $q$-analogue of the above story for the special case that $m_i = 1$ for $i = 1, \ldots, r$. It is an interesting problem to establish a $q$-analogue for the general case.

## 1. PRELIMINARIES

Let $U_q(\mathfrak{g}) = U_q(gl_r)$ be the quantized universal enveloping algebra of $\mathfrak{g} = gl_r$ over $K = \mathbb{Q}(q, u_1, \ldots, u_r)$, defined by the following generators and relations [8]:

$$\text{Generators:} \qquad \begin{cases} q^{\pm\varepsilon_i} & (1 \le i \le r), \\ e_i & (1 \le i < r), \\ f_i & (1 \le i < r). \end{cases}$$

Relations:
$$q^{\varepsilon_i}q^{-\varepsilon_i} = q^{-\varepsilon_i}q^{\varepsilon_i} = 1,\ q^{\varepsilon_i}q^{\varepsilon_j} = q^{\varepsilon_j}q^{\varepsilon_i},$$
$$q^{\varepsilon_i}e_jq^{-\varepsilon_i} = \begin{cases} q^{-1}e_j & (j = i-1) \\ qe_j & (j = i) \\ e_j & (\text{otherwise}), \end{cases}$$
$$q^{\varepsilon_i}f_jq^{-\varepsilon_i} = \begin{cases} qf_j & (j = i-1) \\ q^{-1}f_j & (j = i) \\ f_j & (\text{otherwise}), \end{cases}$$
$$e_if_j - f_je_i = \delta_{ij}\frac{q^{\varepsilon_i - \varepsilon_{i+1}} - q^{-\varepsilon_i + \varepsilon_{i+1}}}{q - q^{-1}},$$
$$e_{i\pm 1}e_i^2 - (q + q^{-1})e_ie_{i\pm 1}e_i + e_i^2e_{i\pm 1} = 0,$$
$$f_{i\pm 1}f_i^2 - (q + q^{-1})f_if_{i\pm 1}f_i + f_i^2f_{i\pm 1} = 0,$$
$$e_ie_j = e_je_i,\ f_if_j = f_jf_i \qquad (i > j+1).$$

It is well-known that $U_q(\mathfrak{g})$ has a Hopf algebra structure with coproduct $\Delta: U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ defined by
$$\Delta(q^{\pm\varepsilon_i}) = q^{\pm\varepsilon_i} \otimes q^{\pm\varepsilon_i},$$
$$\Delta(e_i) = e_i \otimes 1 + q^{\varepsilon_i - \varepsilon_{i+1}} \otimes e_i,$$
$$\Delta(f_i) = f_i \otimes q^{-\varepsilon_i + \varepsilon_{i+1}} + 1 \otimes f_i.$$

On the vector space $V = K^r$, with the standard basis $v_i$ $(1 \le i \le r)$, an action $\rho$ of $U_q(\mathfrak{g})$ is defined by
$$\rho(e_i)v_j = \begin{cases} v_{j-1} & (j = i+1) \\ 0 & (j \ne i+1), \end{cases}$$
$$\rho(f_i)v_j = \begin{cases} v_{j+1} & (j = i) \\ 0 & (j \ne i), \end{cases}$$
$$\rho(q^{\varepsilon_i})v_j = \begin{cases} qv_j & (j = i) \\ v_j & (j \ne i). \end{cases}$$

This is called the natural representation of $U_q(\mathfrak{g})$. Put $\Delta^{(2)} = \Delta$ and $\Delta^{(k)} = (\Delta^{(k-1)} \otimes \mathrm{id}) \circ \Delta$ for $k \ge 3$. By using $\Delta^{(n)}$ so obtained, one can make $V^{\otimes n}$ into a $U_q(\mathfrak{g})$-module:
$$\rho(x)(v_{i_1} \otimes \cdots \otimes v_{i_n}) = \Delta^{(n)}(x)(v_{i_1} \otimes \cdots \otimes v_{i_n}) \qquad \text{for } x \in U_q(\mathfrak{g}).$$

This action is also denoted by $\rho$. We let $U_q(h)$ be the $K$-subalgebra of $U_q(\mathfrak{g})$ generated by $q^{\pm \varepsilon_i}$ $(1 \le i \le r)$. If $\rho$ is restricted to $U_q(h)$, we denote it by $\tilde{\rho}$.

Denote by $\mathscr{H}_n$ the Hecke algebra of the symmetric group $\mathfrak{S}_n$. More precisely, $\mathscr{H}_n$ is the $K$-algebra defined by the following generators and relations:

Generators: $g_2, \dots, g_n$.

Relations:

$$(g_i - q)(g_i + q^{-1}) = 0 \quad (2 \le i \le n),$$
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad (2 \le i < n),$$
$$g_i g_j = g_j g_i \quad (i > j+1).$$

The algebra $\mathscr{H}_n$ also acts on $V^{\otimes n}$ by

$$\sigma(g_k) = \mathrm{id}_V^{\otimes(k-2)} \otimes \check{R} \otimes \mathrm{id}_V^{\otimes(n-k)},$$

where $\check{R} \in \mathrm{End}_K(V \otimes V)$ is defined by

$$\check{R}(v_i \otimes v_j) = \begin{cases} q v_i \otimes v_j & (i = j) \\ v_j \otimes v_i & (i > j) \\ v_j \otimes v_i + (q - q^{-1}) v_i \otimes v_j & (i < j). \end{cases}$$

We will denote $T_k = \rho(g_k)$ for $2 \le k \le n$. A $q$-analogue of the Schur–Weyl reciprocity due to Jimbo asserts that each of $\rho(U_q(\mathfrak{g}))$ and $\sigma(\mathscr{H}_n)$ is the full centralizer algebra of the other in $\mathrm{End}_K(V^{\otimes n})$.

We now recall the definition and properties of the Hecke algebra $\mathscr{H}_{n,r}$ for a positive integer $r$. For further discussions about $\mathscr{H}_{n,r}$, readers may refer to [1]. The Hecke algebra $\mathscr{H}_{n,r}$ is the $K$-algebra defined by generators and relations as follows:

Generators: $g_1, g_2, \dots, g_n$.

Relations:

$$(g_1 - u_1) \cdots (g_1 - u_r) = 0,$$
$$(g_i - q)(g_i + q^{-1}) = 0 \quad (2 \le i \le n),$$
$$g_1 g_2 g_1 g_2 = g_2 g_1 g_2 g_1,$$
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad (2 \le i < n),$$
$$g_i g_j = g_j g_i \quad (i > j+1).$$

Note that our $\mathscr{H}_{n,r}$ is isomorphic to Ariki and Koike's when $q$ is replaced by $q^2$ in [1].

We define $t_j$ $(j = 1, \ldots, n)$ recursively by $t_1 = g_1$, $t_j = g_j t_{j-1} g_j$ $(j \geq 2)$, and $\mathcal{T}_{n,r}$ to be the $K$-subalgebra of $\mathcal{H}_{n,r}$ generated by these elements.

For an $r$-tuple of Young diagrams $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ with size $\sum_{i=1}^{r} |\lambda^{(i)}| = n$, a tableau $\underline{S} = (S^{(1)}, \ldots, S^{(r)})$ of shape $\underline{\lambda}$ is said to be standard if each $j$ $(1 \leq j \leq n)$ occurs exactly once and each $S^{(i)}$ $(1 \leq i \leq r)$ is such a tableau that entries in each column are increasing from top to bottom and in each row from left to right. If $i$ is located in the intersection of the $l$th row and the $m$th column of $S^{(p)}$, then we write $\tau(\underline{S}; i) = p$ and $c(\underline{S}; i) = m - l$. For each standard tableau $\underline{S}$ we associate a character of $\mathcal{T}_{n,r}$ by

$$\varphi_{\underline{S}}(t_i) = u_{\tau(\underline{S}; i)} q^{2c(\underline{S}; i)} \qquad (1 \leq i \leq r).$$

PROPOSITION 1.1 [1]. (1) *The algebra $\mathcal{T}_{n,r}$ is commutative and semi-simple.*

(2) *The complete set of irreducible representations of $\mathcal{T}_{n,r}$ is*

$$\{\varphi_{\underline{S}} | \underline{S} \text{ is standard.}\}.$$

(3) *Irreducible representations of $\mathcal{H}_{n,r}$ are parametrized by r-tuples $\underline{\lambda}$ of size n.*

We denote by $V_{\underline{\lambda}}$ the irreducible $\mathcal{H}_{n,r}$-module corresponding to $\underline{\lambda}$.

PROPOSITION 1.2 [1]. *Let $W$ be an $\mathcal{H}_{n,r}$-module. If $\varphi_{\underline{S}}$ occurs in $W$ considered as a $\mathcal{T}_{n,r}$-module, then $W$ contains $V_{\underline{\lambda}}$ as an irreducible component, where $\underline{\lambda}$ is the shape of $\underline{S}$.*

## 2. THE ACTION OF $\mathcal{H}_{n,r}$ ON $V^{\otimes n}$

We will denote the basis element $v_{i_1} \otimes \cdots \otimes v_{i_n}$ of $V^{\otimes n}$ by $(i_1, \ldots, i_n)$. Define the endomorphisms $\theta$ and $\varpi$ on $V^{\otimes n}$ by

$$\theta(i_1, \ldots, i_n) = (i_2, \ldots, i_n, i_1),$$

$$\varpi(i_1, \ldots, i_n) = u_{i_1} q^{\mu^{(i_1)} - 1}(i_1, \ldots, i_n),$$

respectively, where $\mu^{(i)} = \#\{j; 1 \leq j \leq n,\ i_j = i\}$, the multiplicity of $i$ in the multiset $\{i_1, \ldots, i_n\}$.

THEOREM 2.1. *The action of $g_i$ on $V^{\otimes n}$ defined by*

$$\tilde{\sigma}(g_1) = T_2^{-1} \cdots T_n^{-1} \theta \varpi,$$

$$\tilde{\sigma}(g_i) = T_i \qquad (2 \leq i \leq n),$$

*gives a representation of $\mathcal{H}_{n,r}$.*

The rest of this section is devoted to proving Theorem 2.1. We make $T_1 = \tilde{\sigma}(g_1)$. Jimbo's result [8] shows that the endomorphisms $T_i$ $(2 \le i \le n)$ satisfy the relations:

$$(T_i - q)(T_i + q^{-1}) = 0 \qquad (2 \le i \le n),$$
$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad (2 \le i < n),$$
$$T_i T_j = T_j T_i \qquad (i > j+1).$$

Hence, we only have to show

$$(T_1 - u_1) \cdots (T_1 - u_r) = 0,$$
$$T_1 T_2 T_1 T_2 = T_2 T_1 T_2 T_1,$$
$$T_1 T_j = T_j T_1 \qquad (j \ge 3).$$

We prove these relations in the following three lemmas, each of which corresponds to each of the above relations respectively.

LEMMA 2.2. *For $1 \le j \le r$ and $1 \le k \le n$, let $W_{j,k}$ be the subspace of $V^{\otimes n}$ spanned by $\{(i_1, \ldots, i_n) \in V^{\otimes n};\ i_k \ge j\}$ and put $W_{r+1,k} = (0)$.*

(1) *For $(i_1, \ldots, i_n) \in W_{j,k}$, we have $(i_1, \ldots, i_k, i_{k-1}, \ldots, i_n) \in W_{j,k-1}$, and*

$$T_k^{-1}(i_1, \ldots, i_n) \equiv q^{-\delta(k)}(i_1, \ldots, i_k, i_{k-1}, \ldots, i_n) \qquad \operatorname{mod} W_{j+1,k-1},$$

*where $\delta(k)$ is 1 or 0 according to whether $i_{k-1} = i_k$ or not. In particular, we have $T_k^{-1} W_{j,k} \subset W_{j,k-1}$.*

(2) *For $(i_1, \ldots, i_n) \in W_{j,1}$, we have $(i_2, \ldots, i_{k-1}, i_1, i_k, \ldots, i_n) \in W_{j,k}$, and*

$$T_k^{-1} \cdots T_n^{-1}(i_2, \ldots, i_n, i_1)$$
$$\equiv q^{-\mu_{i_1}(k)}(i_2, \ldots, i_{k-1}, i_1, i_k, \ldots, i_n) \operatorname{mod} W_{j+1,k-1},$$

*where $\mu^{(i)}(k) = \#\{j;\ k \le j \le n,\ i_j = i\}$.*

(3) $(T_1 - u_j) W_{j,1} \subset W_{j+1,1}$ *for* $1 \le j \le r$.

(4) $(T_1 - u_1) \cdots (T_1 - u_r) = 0$.

*Proof.* (1) It is a direct consequence of the definition of $T_k^{-1}$.

(2) Use the descending induction on $k$. The case $k = n$ is nothing but the case $k = n$ in (1). If

$$T_k^{-1} \cdots T_n^{-1}(i_2, \ldots, i_n, i_1) \equiv q^{-\mu_{i_1}(k)}(i_2, \ldots, i_{k-1}, i_1, i_k, \ldots, i_n)$$
$$\operatorname{mod} W_{j+1,k-1},$$

then we have

$$T_{k-1}^{-1}T_k^{-1} \cdots T_n^{-1}(i_2,\dots,i_n,i_1)$$
$$\equiv q^{-\mu_{i_1}(k-1)}(i_2,\dots,i_{k-2},i_1,i_{k-1},\dots,i_n)\operatorname{mod}\left(W_{j+1,k-2} + T_{k-1}^{-1}W_{j+1,k-1}\right).$$

Since $T_{k-1}^{-1}W_{j+1,k-1} \subset W_{j+1,k-2}$ by (1), the induction proceeds.

(3) By putting $k = 2$ in (2), we have

$$T_2^{-1} \cdots T_n^{-1}\theta\varpi(i_1,\dots,i_n) \equiv u_{i_1}(i_1,\dots,i_n) \qquad \operatorname{mod} W_{j+1,1}.$$

If $i_1 > j$, then $(i_1,\dots,i_n) \equiv 0 \operatorname{mod} W_{j+1,1}$. Thus we have $T_1(i_1,\dots,i_n) \in W_{j,1}$ and

$$T_1(i_1,\dots,i_n) \equiv u_j(i_1,\dots,i_n) \qquad \operatorname{mod} W_{j+1,1}.$$

Therefore $(T_1 - u_j)W_{j,1} \subset W_{j+1,1}$.

(4) We have $(T_1 - u_r)\cdots(T_1 - u_1)W_{1,1} \subset (T_1 - u_r)\cdots(T_1 - u_k)W_{k,1}$ for $1 \le k \le r$, which means $(T_1 - u_1)\cdots(T_1 - u_r) = 0$. ∎

LEMMA 2.3. (1) *$\theta T_j = T_{j-1}\theta$ and $\varpi T_j = T_j\varpi$ for $j \ge 3$.*

(2) *$\theta^2 T_2 = T_n\theta^2$ and $\theta^{-1}\varpi\theta\varpi T_2 = T_2\theta^{-1}\varpi\theta\varpi$.*

(3) *$T_1T_2T_1 = (T_2^{-1} \cdots T_n^{-1})(T_2^{-1} \cdots T_{n-1}^{-1})(\theta\varpi)^2$.*

(4) *We have*

$$\left(T_2^{-1} \cdots T_n^{-1}\right)\left(T_2^{-1} \cdots T_{n-1}^{-1}\right)T_n$$
$$= \left(T_2^{-1} \cdots T_k^{-1}\right)\left(T_2^{-1} \cdots T_{k-1}^{-1}\right)T_k\left(T_{k+1}^{-1} \cdots T_n^{-1}\right)\left(T_k^{-1} \cdots T_{n-1}^{-1}\right)$$

*for $k \ge 3$.*

(5) *$T_1T_2T_1T_2 = T_2T_1T_2T_1$.*

*Proof.* (1) and (2) are direct consequences of the definition of $\theta$ and $\varpi$.

(3) By (1), we have

$$\begin{aligned} T_1T_2T_1 &= \left(T_2^{-1} \cdots T_n^{-1}\right)\theta\varpi\left(T_3^{-1} \cdots T_n^{-1}\right)\theta\varpi \\ &= \left(T_2^{-1} \cdots T_n^{-1}\right)\theta\left(T_3^{-1} \cdots T_n^{-1}\right)\varpi\theta\varpi \\ &= \left(T_2^{-1} \cdots T_n^{-1}\right)\left(T_2^{-1} \cdots T_{n-1}^{-1}\right)(\theta\varpi)^2. \end{aligned}$$

(4) Use the descending induction on $k$. It is obvious when $k = n$. For $k \geq 3$, we have

$$\begin{aligned}
&(T_2^{-1} \cdots T_k^{-1})(T_2^{-1} \cdots T_{k-1}^{-1})T_k(T_{k+1}^{-1} \cdots T_n^{-1})(T_k^{-1} \cdots T_{n-1}^{-1}) \\
&\quad = (T_2^{-1} \cdots T_{k-1}^{-1})(T_2^{-1} \cdots T_{k-2}^{-1}) \\
&\qquad \times (T_k^{-1}T_{k-1}^{-1}T_k)(T_{k+1}^{-1} \cdots T_n^{-1})(T_k^{-1} \cdots T_{n-1}^{-1}) \\
&\quad = (T_2^{-1} \cdots T_{k-1}^{-1})(T_2^{-1} \cdots T_{k-2}^{-1}) \\
&\qquad \times (T_{k-1}T_k^{-1}T_{k-1}^{-1})(T_{k+1}^{-1} \cdots T_n^{-1})(T_k^{-1} \cdots T_{n-1}^{-1}) \\
&\quad = (T_2^{-1} \cdots T_{k-1}^{-1})(T_2^{-1} \cdots T_{k-2}^{-1})T_{k-1}(T_k^{-1} \cdots T_n^{-1}) \\
&\qquad \times (T_{k-1}^{-1}T_k^{-1} \cdots T_{n-1}^{-1})
\end{aligned}$$

and the induction proceeds.

(5) By putting $k = 2$ in (4), we have

$$(T_2^{-1} \cdots T_n^{-1})(T_2^{-1} \cdots T_{n-1}^{-1})T_n = (T_3^{-1} \cdots T_n^{-1})(T_2^{-1} \cdots T_{n-1}^{-1}).$$

Therefore, by (2) and (3), we have

$$\begin{aligned}
T_1T_2T_1T_2 &= (T_2^{-1} \cdots T_n^{-1})(T_2^{-1} \cdots T_{n-1}^{-1})(\theta\varpi)^2T_2 \\
&= (T_2^{-1} \cdots T_n^{-1})(T_2^{-1} \cdots T_{n-1}^{-1})T_n(\theta\varpi)^2 \\
&= (T_3^{-1} \cdots T_n^{-1})(T_2^{-1} \cdots T_{n-1}^{-1})(\theta\varpi)^2 \\
&= T_2T_1T_2T_1. \quad \blacksquare
\end{aligned}$$

LEMMA 2.4. *$T_1T_j = T_jT_1$ for $j \geq 3$.*

*Proof.* Lemma 2.3 (1) shows that

$$\begin{aligned}
T_1T_j &= T_2^{-1} \cdots T_n^{-1}\theta\varpi T_j \\
&= T_2^{-1} \cdots T_n^{-1}T_{j-1}\theta\varpi \\
&= T_2^{-1} \cdots T_{j-1}^{-1}T_j^{-1}T_{j-1}T_{j+1}^{-1} \cdots T_n^{-1}\theta\varpi \\
&= T_2^{-1} \cdots T_{j-2}^{-1}T_jT_{j-1}^{-1}T_j^{-1} \cdots T_n^{-1}\theta\varpi \\
&= T_jT_2^{-1} \cdots T_n^{-1}\theta\varpi \\
&= T_jT_1. \quad \blacksquare
\end{aligned}$$

These complete the proof of Theorem 2.1.

## 3. SCHUR–WEYL RECIPROCITY FOR $(U_q(h), \mathscr{H}_{n,r})$ ON $V^{\otimes n}$

We first give the irreducible decomposition of the representation $\tilde{\sigma}$ of $\mathscr{H}_{n,r}$ on $V^{\otimes n}$. Let $\Lambda_{\underline{1}}$ be the set of $r$-tuples of Young diagrams $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of size $\sum_{i=1}^r |\lambda^{(i)}| = n$ such that each component $\lambda^{(i)}$ has length $l(\lambda^{(i)}) \le 1$. We can think of $\lambda^{(i)}$ as a nonnegative integer.

THEOREM 3.1. *The irreducible decomposition of $V^{\otimes n}$ under the action $\tilde{\sigma}$ of $\mathscr{H}_{n,r}$ is given by*

$$V^{\otimes n} = \bigoplus_{\underline{\lambda} \in \Lambda_{\underline{1}}} V_{\underline{\lambda}}.$$

*Proof.* For each $\underline{\lambda} \in \Lambda_{\underline{1}}$, we have $\dim V_{\underline{\lambda}} = n!/\prod_{i=1}^r \lambda^{(i)}!$, and hence the dimension of the right-hand side is equal to

$$\sum_{\underline{\lambda} \in \Lambda_{\underline{1}}} \frac{n!}{\prod_{i=1}^r \lambda^{(i)}!} = r^n,$$

which coincides with $\dim V^{\otimes n}$. Therefore, it suffices to prove that $V_{\underline{\lambda}} \subset V^{\otimes n}$ for each $\underline{\lambda} \in \Lambda_{\underline{1}}$. Because of Proposition 1.2, we only have to show that for each $\underline{\lambda} \in \Lambda_{\underline{1}}$ there exists a simultaneous eigenvector for $\mathscr{T}_{n,r}$ in $V^{\otimes n}$ with eigenvalues $\varphi_{\underline{S}}$, where $\underline{S}$ is a certain standard tableau of shape $\underline{\lambda}$.

Put $p_k = \lambda^{(r)} + \cdots + \lambda^{(k)}$ for $1 \le k \le r$, and $p_{r+1} = 0$. Define $v_{\underline{S}} = (i_1, \dots, i_n)$ by $i_j = k$ if $p_{k+1} + 1 \le j \le p_k$.

We show that

$$\tilde{\sigma}(t_j) v_{\underline{S}} = u_k q^{2(j - p_{k+1} - 1)} v_{\underline{S}} \tag{3.1}$$

if $p_{k+1} + 1 \le j \le p_k$. By the descending induction on $l$, we see that

$$T_{l+1} \cdots T_{p_{k+1}+1} v_{\underline{S}} = \left(i_1, \dots, i_{l-1}, k, i_l, \dots, \widehat{i_{p_{k+1}+1}}, \dots, i_n\right) \tag{3.2}$$

for $1 \le l \le p_{k+1}$. Since $i_{p_{k+1}} = k + 1 > k = i_{p_{k+1}+1}$, (3.2) holds for $l = p_{k+1}$. Assume (3.2) for $l$. Then it is seen that

$$T_l \cdots T_{p_{k+1}+1} v_{\underline{S}} = \left(i_1, \dots, i_{l-2}, k, i_{l-1}, i_l, \dots, \widehat{i_{p_{k+1}+1}}, \dots, i_n\right)$$

since $i_{l-1} > k$, and the induction proceeds. Putting $l = 1$, we have

$$T_2 \cdots T_{p_{k+1}+1} v_{\underline{S}} = \left(k, i_1, i_2, \dots, \widehat{i_{p_{k+1}+1}}, \dots, i_n\right).$$

Since

$$\tilde{\sigma}\left(t_{p_{k+1}+1}\right) = T_{p_{k+1}+2}^{-1} \cdots T_n^{-1} \theta \varpi T_2 \cdots T_{p_{k+1}+1},$$

we have

$$\tilde{\sigma}(t_{p_{k+1}+1})v_{\underline{S}} = u_k q^{\lambda^{(k)}-1} T_{p_{k+1}+2}^{-1} \cdots T_n^{-1}(i_1, \ldots, \widehat{i_{p_{k+1}+1}}, \ldots, i_n, k).$$

By a similar argument as that for (3.2), we can show that

$$T_l^{-1} \cdots T_n^{-1}(i_1, \ldots, \widehat{i_{p_{k+1}+1}}, \ldots, i_n, k)$$
$$= (i_1, \ldots, \widehat{i_{p_{k+1}+1}}, \ldots, i_l, k, i_{l+1}, \ldots, i_n)$$

for $p_k \leq l \leq n-1$. Hence we have

$$\tilde{\sigma}(t_{p_{k+1}+1})v_{\underline{S}}$$
$$= u_k q^{\lambda^{(k)}-1} T_{p_{k+1}+2}^{-1} \cdots T_{p_k}^{-1}(i_1, \ldots, \widehat{i_{p_{k+1}+1}}, \ldots, i_{p_k}, k, i_{p_k+1}, \ldots, i_n).$$

Since $i_{p_{k+1}+1} = \cdots = i_p = k$, the right-hand side is equal to

$$u_k q^{\lambda^{(k)}-1} \cdot q^{-(p_k - p_{k+1}-1)} v_{\underline{S}} = u_k v_{\underline{S}}.$$

This shows that (3.1) holds for the case $j = p_{k+1}+1$. For the case $p_{k+1} + 2 \leq j \leq p_k$, we only have to observe that $T_j v_{\underline{S}} = q v_{\underline{S}}$ to see (3.1). ∎

THEOREM 3.2. (1) *Each of* $\tilde{\rho}(U_q(h))$ *and* $\tilde{\sigma}(\mathscr{H}_{n,r})$ *is the full centralizer algebra of the other in* $\mathrm{End}_K(V^{\otimes n})$.

(2) *For* $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \in \Lambda_{\underline{1}}$,

$$V_{\underline{\lambda}} = \{v \in V^{\otimes n} \mid \tilde{\rho}(q^{\varepsilon_i})v = q^{\lambda^{(i)}}v \text{ for } 1 \leq i \leq r\}.$$

*Proof.* We first observe that $\tilde{\rho}(q^{\varepsilon_i})$ commutes with $T_1, \ldots, T_n$ for $1 \leq i \leq r$, which easily follows from

$$\tilde{\rho}(q^{\varepsilon_i})(i_1, \ldots, i_n) = q^{\mu^{(i)}}(i_1, \ldots, i_n),$$

and the fact that $T_1 \cdots T_n$ preserve the weight $\underline{\mu} \in \Lambda_{\underline{1}}$ of the vector $(i_1, \ldots, i_n)$.

Let $\phi_{\underline{\mu}}$ be the character of $U_q(h)$ defined by

$$\phi_{\underline{\mu}}(q^{\varepsilon_i}) = q^{\mu^{(i)}} \qquad (1 \leq i \leq r).$$

Then we have the isotypical decomposition of $V^{\otimes n}$ as $U_q(h)$-modules;

$$V^{\otimes n} = \sum_{\underline{\mu}} (V^{\otimes n})_{\underline{\mu}}, \tag{3.3}$$

where

$$(V^{\otimes n})_{\underline{\mu}} = \{v \in V^{\otimes n} \mid \tilde{\rho}(q^{\varepsilon_i})v = q^{\mu^{(i)}}v \text{ for } 1 \leq i \leq r\}.$$

It is apparent that the characters $\phi_\mu$ and $\phi_{\mu'}$ are inequivalent for distinct weights $\underline{\mu}$ and $\underline{\mu}'$. Therefore, dimension of $\tilde{\rho}(U_q(h))$ is equal to $|\Lambda_1|$.

On the other hand, by Schur's lemma, the dimension of the commutant $\tilde{\sigma}(\mathscr{H}_{n,r})'$ of $\tilde{\sigma}(\mathscr{H}_{n,r})$ is equal to $|\Lambda_1|$. Since $\tilde{\rho}(U_q(h)) \subset \tilde{\sigma}(\mathscr{H}_{n,r})'$, we can conclude that they coincide.

Let us recall that for any semisimple algebra acting on a module, say $M$, its image in $\mathrm{End}_K(M)$ is isomorphic to the direct product of matrix algebras which correspond to irreducible representations occurring in $M$. Thereby we know that

$$\tilde{\sigma}(\mathscr{H}_{n,r}) \simeq \bigoplus_{\underline{\lambda} \in \Lambda_1} \mathrm{End}_K(V_{\underline{\lambda}}).$$

Hence $\dim \tilde{\sigma}(\mathscr{H}_{n,r}) = \sum_{\underline{\lambda} \in \Lambda_1} \binom{n}{\lambda^{(1)}, \ldots, \lambda^{(r)}}^2$.

On the other hand, the decomposition (3.3) shows that the dimension of the commutant $\tilde{\rho}(U_q(h))'$ of $\tilde{\rho}(U_q(h))$ is equal to

$$\sum_{\underline{\mu} \in \Lambda_1} \binom{n}{\mu^{(1)}, \ldots, \mu^{(r)}}^2.$$

Therefore, we have $\tilde{\rho}(U_q(h))' = \tilde{\sigma}(\mathscr{H}_{n,r})$.

Since the space $(V^{\otimes n})_{\underline{\lambda}}$ has the element $(r, \ldots, r, r-1, \ldots, 1)$ in which each $k$ repeats $\lambda^{(k)}$ times, the space contains $V_{\underline{\lambda}}$. Combining Theorem 3.1 and Eq. (3.3), we see $V_{\underline{\lambda}} = (V^{\otimes n})_{\underline{\lambda}}$ for any $\underline{\lambda} \in \Lambda_1$ as desired. ∎

## 4. THE ACTION OF CERTAIN LATTICES OF $U_q(h)$ ON $V^{\otimes n}$

If we put aside the representation theory, the story we have considered in previous sections can be discussed not only over fields but also over $A = \mathbb{Z}[q, q^{-1}, u_1, \ldots, u_r]$.

Let us denote by $V_A^{\otimes n}$ the $A$-lattice spanned by the basis elements $(i_1, \ldots, i_n)$. The Hecke algebra $\mathscr{H}_{n,r}$ has a natural $A$-lattice which is the $A$-subalgebra generated by $g_1, \ldots, g_n$. It is obvious that $\tilde{\sigma}(\mathscr{H}_{n,r}(A))$ preserves $V_A^{\otimes n}$.

There are several natural ways to choose an $A$-lattice in $U_q(h)$. We consider two lattices: one is the $A$-subalgebra generated by $q^{\varepsilon_i}$'s and the other is analogous to the Cartan part of the Kostant $\mathbb{Z}$-form introduced by Lusztig [10, 11]. These are mapped to $A$-subalgebras in $\mathrm{End}_A(V_A)$ by $\tilde{\rho}$, which are $A$-free of finite rank. In the following, we give $A$-free bases of these, as well as proving that these preserve $V_A^{\otimes n}$.

Let $\mathscr{S}$ be a set of dominant weights in an alcove as follows.

$$\mathscr{S} = \{\underline{\nu} = (\nu_1, \dots, \nu_{r-1}) | n \geq \nu_1 \geq \cdots \geq \nu_{r-1} \geq 0\}.$$

Put $\nu_{r-1} = p_r$, $\nu_{i-1} - \nu_i = p_i$ $(2 \leq i < r)$, and $n - \nu_1 = p_r$. Note that this bijection $\underline{\nu} \leftrightarrow \underline{p} = (p_1, \dots, p_r)$ shows that the cardinality of $\mathscr{S}$ is equal to the dimension of $\tilde{\rho}(U_q(h))$.

We define a polynomial of $(r+1)$-variables as

$$\begin{aligned} F_{\underline{\nu}}(X_0, \dots, X_r) = &(X_0 - X_1)(X_0 - qX_1) \cdots (X_0 - q^{p_1 - 1}X_1) \\ &\times (X_1 - X_2) \cdots (X_1 - q^{p_2 - 1}X_2) \\ &\times (X_{r-1} - X_r) \cdots (X_{r-1} - q^{p_r - 1}X_r), \end{aligned}$$

which is homogeneous of degree $n$. In the following, we are mainly concerned with $F_{\underline{\nu}}(q^n X_r, X_1, \dots, X_r)$.

PROPOSITION 4.1. (1) *If*

$$F_{\underline{\nu}}(q^n X_r, X_1, \dots, X_r) = \sum_{\underline{\mu} \in \Lambda_1} a_{\underline{\mu}\underline{\nu}} X_1^{\mu^{(1)}} \cdots X_r^{\mu^{(r)}},$$

*then, for any* $\underline{\nu}', \underline{\nu}$ *in* $\mathscr{S}$,

$$\sum_{\underline{\mu} \in \Lambda_1} q^{(\underline{\nu}', \underline{\mu})} a_{\underline{\mu}\underline{\nu}} = \delta_{\underline{\nu}', \underline{\nu}} F_{\underline{\nu}}(q^n, q^{\nu_1}, \dots, q^{\nu_{r-1}}, 1),$$

*where* $q^{(\underline{\nu}', \underline{\mu})}$ *stands for* $q^{\nu'_1 \mu^{(1)} + \cdots + \nu'_{r-1} \mu^{(r-1)}}$.

(2) *For any* $\underline{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{Z}^r$, *we have*

$$\begin{aligned} &\tilde{\rho}(q^{\alpha_1 \varepsilon_1 + \cdots + \alpha_r \varepsilon_r}) \\ &\quad = \sum_{\underline{\nu} \in T} \frac{F_{\underline{\nu}}(q^{n+\alpha_r}, q^{\alpha_1}, \dots, q^{\alpha_{r-1}}, q^{\alpha_r})}{F_{\underline{\nu}}(q^n, q^{\nu_1}, \dots, q^{\nu_{r-1}}, 1)} \tilde{\rho}(q^{\nu_1 \varepsilon_1 + \cdots + \nu_{r-1} \varepsilon_{r-1}}). \end{aligned}$$

(3)

$$\frac{F_{\underline{\nu}}(q^{n+\alpha_r}, q^{\alpha_1}, \dots, q^{\alpha_{r-1}}, q^{\alpha_r})}{F_{\underline{\nu}}(q^n, q^{\nu_1}, \dots, q^{\nu_{r-1}}, 1)} \in \mathbb{Z}[q, q^{-1}].$$

(4) *The* $A$*-algebra generated by* $\tilde{\rho}(q^{\varepsilon_i})$ $(1 \leq i \leq r)$ *has an* $A$*-free basis* $\{\tilde{\rho}(q^{\nu_1 \varepsilon_1 + \cdots + \nu_{r-1} \varepsilon_{r-1}}) | \underline{\nu} \in \mathscr{S}\}$.

*Proof.* (1) We will evaluate the left-hand side, which is equal to $F_{\underline{\nu}}(q^n, q^{\nu'_1}, \ldots, q^{\nu'_{r-1}}, 1)$. Assume that it is nonzero. Then we have

$$\begin{aligned} \nu'_{r-1} &\neq 0, 1, \ldots, p_r - 1, \\ \nu'_{r-2} &\neq \nu'_{r-1}, \nu'_{r-1} + 1, \ldots, \nu'_{r-1} + p_{r-1} - 1, \ldots, \\ \nu'_1 &\neq \nu'_2, \nu'_2 + 1, \ldots, \nu'_2 + p_2 - 1, \\ n &\neq \nu'_1, \nu'_1 + 1, \ldots, \nu'_1 + p_1 - 1, \end{aligned}$$

which leads to

$$\nu'_{r-1} \geq p_r, \nu'_{r-2} - \nu'_{r-1} \geq p_{r-1}, \ldots, \nu'_1 - \nu'_2 \geq p_2,$$
$$n - \nu'_1 \geq p_1.$$

Since the sum of these on both sides is $n$, these inequalities must be equalities. It deduces $\underline{\nu}' = \underline{\nu}$, which proves (1).

(2) A conclusion of (1) is that the matrix $(q^{\nu_1\mu^{(1)} + \cdots + \nu_{r-1}\mu^{(r-1)}})$, whose rows are indexed by $\nu$ and columns by $\mu$, is nonsingular, which means that $\{\tilde{\rho}(q^{\nu_1\varepsilon_1 + \cdots + \nu_{r-1}\varepsilon_{r-1}}) | \underline{\nu} \in \mathcal{S}\}$ is a $K$-basis of $\tilde{\rho}(U_q(h))$. Hence we can write

$$\tilde{\rho}(q^{\alpha_1\varepsilon_1 + \cdots + \alpha_r\varepsilon_r}) = \sum_{\underline{\nu} \in \mathcal{S}} b_{\underline{\alpha\nu}} \tilde{\rho}(q^{\nu_1\varepsilon_1 + \cdots + \nu_{r-1}\varepsilon_{r-1}})$$

for any fixed $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$. If we apply both sides to a weight vector of $V^{\otimes n}$ with weight $\underline{\mu}$, we have

$$q^{\alpha_1\mu^{(1)} + \cdots + \alpha_r\mu^{(r)}} = \sum_{\underline{\nu} \in \mathcal{S}} b_{\underline{\alpha\nu}} q^{\nu_1\mu^{(1)} + \cdots + \nu_{r-1}\mu^{(r-1)}}.$$

Hence,

$$\begin{aligned} F_{\underline{\nu}}(q^{n+\alpha_r}, q^{\alpha_1}, \ldots, q^{\alpha_r}) &= \sum_{\underline{\mu}} a_{\underline{\mu\nu}} q^{\alpha_1\mu^{(1)} + \cdots + \alpha_r\mu^{(r)}} \\ &= \sum_{\underline{\nu}, \underline{\nu}'} a_{\underline{\mu\nu}} b_{\underline{\alpha\nu'}} q^{\nu'_1\mu^{(1)} + \cdots + \nu'_{r-1}\mu^{(r-1)}} \\ &= \sum_{\underline{\nu}'} b_{\underline{\alpha\nu'}} \delta_{\underline{\nu}'\underline{\nu}} F_{\underline{\nu}}(q^n, q^{\nu_1}, \ldots, q^{\nu_{r-1}}, 1) \\ &= b_{\underline{\alpha\nu}} F_{\underline{\nu}}(q^n, q^{\nu_1}, \ldots, q^{\nu_{r-1}}, 1), \end{aligned}$$

by which we have the required coefficients.

(3) It is obvious since the quotient

$$\frac{F_{\underline{\nu}}(q^{n+\alpha_r}, q^{\alpha_1}, \ldots, q^{\alpha_{r-1}}, q^{\alpha_r})}{F_{\underline{\nu}}(q^{n}, q^{\nu_1}, \ldots, q^{\nu_{r-1}}, 1)}$$

is a product of $q$-binomial coefficients up to a power of $q$.

(4) This is a direct consequence of (2) and (3). ∎

The next $A$-lattice we consider is the $A$-subalgebra of $U_q(h)$ generated by $q^{\varepsilon_i}$'s and

$$\begin{bmatrix} q^{\pi} \\ N \end{bmatrix} = \prod_{s=1}^{N} \frac{q^{\pi-s+1} - q^{-\pi+s-1}}{q^{s} - q^{-s}},$$

where $N \geq 0$ and $\pi \in P = \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$. If $\pi$ is replaced by an integer $l$, it is a $q$-binomial coefficient, which we denote by $\begin{bmatrix} l \\ N \end{bmatrix}$.

For comparison, we recall Lusztig's Kostant $\mathbb{Z}$-form for $U_q(sl_r)$ [11]. It is the $\mathbb{Z}[q, q^{-1}]$-algebra generated by $e_i^{(N)} = e_i^N/[N]!$, $f_i^{(N)} = f_i^N/[N]!$, where $[N]! = \prod_{s=1}^{N}((q^s - q^{-s})/(q - q^{-1}))$. The Cartan part of this algebra has a $\mathbb{Z}[q, q^{-1}]$-free basis

$$\left\{ \prod_{i=1}^{r-1} \left( K_i^{\delta_i} \begin{bmatrix} K_i \\ t_i \end{bmatrix} \right) \middle| t_i \geq 0,\ \delta_i = 0, 1 \right\},$$

where $K_i = q^{\varepsilon_i - \varepsilon_{i+1}}$ $(1 \leq i < n)$ and

$$\begin{bmatrix} K_i \\ N \end{bmatrix} = \prod_{s=1}^{N} \frac{q^{-s+1}K_i - q^{s-1}K_i^{-1}}{q^{s} - q^{-s}}.$$

PROPOSITION 4.2. (1) *Put*

$$\begin{bmatrix} q^{\varepsilon} \\ \underline{\nu} \end{bmatrix} = \prod_{i=1}^{r} \begin{bmatrix} q^{\varepsilon_i} \\ \nu^{(i)} \end{bmatrix}$$

*for* $\underline{\nu} \in \Lambda_1$. *Then we have*

$$\tilde{\rho}\left( \begin{bmatrix} q^{\varepsilon} \\ \underline{\nu} \end{bmatrix} \right)(i_1, \ldots, i_n) = \delta_{\underline{\mu}, \underline{\nu}}(i_1, \ldots, i_n),$$

*where* $\mu$ *is the weight of* $(i_1, \ldots, i_n)$.

(2) *For* $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$, $\underline{N} = (N_1, \ldots, N_r) \in \mathbb{Z}^r$, *and* $\pi_i \in P$, *we have*

$$\tilde{\rho}\left( \prod_{i=1}^{r} \left( q^{\alpha_i \varepsilon_i} \begin{bmatrix} q^{\pi_i} \\ N_i \end{bmatrix} \right) \right) = \sum_{\underline{\nu} \in \Lambda_1} \left( \prod_{i=1}^{r} q^{\alpha_i \nu^{(i)}} \begin{bmatrix} (\pi_i, \underline{\nu}) \\ N_i \end{bmatrix} \right) \tilde{\rho}\left( \begin{bmatrix} q^{\varepsilon} \\ \underline{\nu} \end{bmatrix} \right).$$

(3) *The $A$-algebra generated by* $\tilde{\rho}\left(\begin{bmatrix} q^{\pi} \\ N \end{bmatrix}\right)$ *and* $\tilde{\rho}(q^{\pi})$ $(\pi \in P,\ N \geq 0)$ *has an $A$-free basis* $\{\tilde{\rho}\left(\begin{bmatrix} q^{\pi} \\ \underline{\nu} \end{bmatrix}\right) | \underline{\nu} \in \Lambda_{\underline{1}}\}$.

*Proof.* (1) It is enough to evaluate $\begin{bmatrix} \nu_i \\ \mu_i \end{bmatrix}$. If all of these are nonzero, then we must have $\mu_i \neq 0, 1, \ldots, \nu_i - 1$ for all $i$. It is nothing but the condition $\mu_i \geq \nu_i$, and we conclude that $\underline{\mu} = \underline{\nu}$, in which case the value is 1.

(2) is a direct consequence of (1). Then (3) easily follows since $q$-binomial coefficients are Laurent polynomials in $q$. ∎

## APPENDIX

We explain a finite field version of the classical Schur–Weyl reciprocity in our setting.

Let $p$ be an odd prime such that $r \geq n$, $p \geq r + n$, and $r$ divides $p - 1$. Set $q = p^r$ and let $\mathbb{F}_q$ be the field of $q$ elements. The general linear group over the field $\mathbb{F}_q$, which we denote by $G = GL(r, q)$, admits the Frobenius actions

$$F_1((g_{ij})) = (g_{ij}^p) \text{ and } F_w((g_{ij})) = w^{-1}(g_{ij}^p)w \qquad \text{for } g = (g_{ij}),$$

where $w = E_{r,1} + \sum_{i=1}^{r-1} E_{i,i+1}$. We denote by $G^{F_w}$ the group of Frobenius fixed points with respect to $F_w$, which is isomorphic to $GL(r, p)$.

We are dealing with the following $G$-module:

$$E = V \otimes_{\mathbb{F}_q} V^{F_w} \otimes_{\mathbb{F}_q} \cdots \otimes_{\mathbb{F}_q} V^{F_w^{n-1}},$$

where $V = \mathbb{F}_q^r$ is the natural representation of $G$ and

$$\hat{\rho}(g)(v_{i_1} \otimes \cdots \otimes v_{i_n}) = g v_{i_1} \otimes F_w(g) v_{i_2} \otimes \cdots \otimes F_w^{n-1}(g) v_{i_n}.$$

Steinberg's tensor product theorem [14] says that $E$ is isomorphic to the irreducible module with highest weight $(1 + \cdots + p^{r-1})\varepsilon_1$ as a $SL(n, q)$-module. ($\varepsilon_1$ is the highest weight of $V$.)

Note that the set of irreducible representations of $SL(r, \overline{\mathbb{F}_q})$ with highest weights lying in an alcove of the weight lattice exhausts all irreducible representations of $SL(r, q)$ [4].

As a $G^{F_w}$-module, $E$ is nothing but $V^{\otimes n}$, which admits $\mathfrak{S}_n$-action as before. We also denote by $\rho$ and $\sigma$ the action of $G^{F_w}$ and $\mathfrak{S}_n$ on $V^{\otimes n}$ respectively.

LEMMA. *Each of $\sigma(\mathbb{F}_q \mathfrak{S}_n)$ and $\rho(\mathbb{F}_q G^{F_w})$ is the full centralizer algebra of the other.*

*Proof.* Let us review results in [3]. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a Young diagram of size $n$. Then, by definition, the Weyl module $W_\lambda$ is a cyclic module generated by a highest weight vector $\Phi$ in $V^{\otimes n}$. (For its definition, see [3, 4.2].) This module can be defined over $\mathbb{F}_q$. Besides, the collection of $\mathfrak{S}_n$ translations of $\Phi$ spans the Specht module $S_\lambda$. Since all Young diagrams are $p$-regular under our assumption, these Specht modules are irreducible.

Corollary 2 of [3, p. 232] states that $W_\lambda$ is absolutely irreducible if and only if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $\lambda_1 - \lambda_n + n \leq p$. Hence, under our assumption, $W_\lambda$ is irreducible, and the space of highest weights in the $W_\lambda$-isotypical component of $V^{\otimes n}$ apparently contains translation of $\Phi$ under $\mathfrak{S}_n$-action. This shows that $V^{\otimes n}$ contains the direct sum of $W_\lambda \otimes S_\lambda$'s. By counting dimensions, we have

$$V^{\otimes n} = \bigoplus_{\lambda} W_\lambda \otimes S_\lambda.$$

Thereby, $V^{\otimes n}$ is a multiplicity-free semi-simple $(G^{F_w} \times \mathfrak{S}_n)$-module, which proves that each of $\sigma(\mathbb{F}_q \mathfrak{S}_n)$ and $\rho(\mathbb{F}_q G^{F_w})$ is the full centralizer algebra of the other. ∎

Let $F_1$ be the natural Frobenius action on $V$. If we set $s_1 = F_1 \otimes \mathrm{id}_V \otimes \cdots \otimes \mathrm{id}_V$, then one can extend the action of $\mathfrak{S}_n$ on $V^{\otimes n}$ to that of $\mathfrak{S}_{n,r}$. We denote this action of $\mathfrak{S}_{n,r}$ by $\tilde{\sigma}$.

Put $T_0 = G^{F_1} \cap G^{F_w}$. Each element of $T_0$ is of the form

$$h\left(\operatorname{diag}\left(\mathbb{F}_p^\times, \ldots, \mathbb{F}_p^\times\right)\right)h^{-1},$$

where $h = \sum \gamma^{(i-1)(j-1)} E_{ij}$, $\gamma$ is an element in $\mathbb{F}_p^\times$ whose order is precisely $r$. Thus $T_0$ is a split torus of $G^{F_w}$. By restricting $\rho$ to $T_0$, we have the action of $T_0$ on $V^{\otimes n}$, which we denote by $\tilde{\rho}$.

PROPOSITION. *Each of $\tilde{\sigma}(\mathbb{F}_q \mathfrak{S}_{n,r})$ and $\rho(\mathbb{F}_q T_0)$ is the full centralizer algebra of the other. Hence, corresponding to the inclusion $\hat{\rho}(G) \supset \rho(G^{F_w}) \supset \tilde{\rho}(T_0)$, we have their full centralizer algebras $\mathbb{F}_q \subset \sigma(\mathbb{F}_q \mathfrak{S}_n) \subset \tilde{\sigma}(\mathbb{F}_q \mathfrak{S}_{n,r})$ on $E$.*

*Proof.* As we have remarked in Section 4, each irreducible component of $V_A^{\otimes n}$ is stable under $(\mathscr{H}_{n,r}(A))$-action. On the other hand, we know that its reduction modulo $u_i = q^{i-1}$ and $q = e^{2\pi\sqrt{-1}/r}$ remains irreducible by a result proved in [2]. Hence we have

$$E = \bigoplus_{\underline{\lambda} \in \Lambda_1} V_{\underline{\lambda}}. \tag{A.1}$$

If we choose columns of the matrix $h$ as a basis for $V$, these are simultaneous eigenvectors for $T_0$ corresponding to distinct weights of $T_0$. Hence $E$ is decomposed as a $T_0$-module into $\oplus_{\mu \in \Lambda_1}(E)_\mu$, such that each weight space is nonzero and $(\mathbb{F}_q \mathfrak{S}_{n,r})$-stable, since one can easily check that $F_1$ and $\tilde{\rho}(T_0)$ commute on $V$.

Comparing this weight decomposition with (A.1), we have that these weight spaces are an irreducible $(\mathbb{F}_q \mathfrak{S}_{n,r})$-module. Therefore, $V^{\otimes n}$ is a multiplicity-free semi-simple $(T_0 \times \mathfrak{S}_{n,r})$-module, by which we conclude the required result. ∎

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