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On the q-Commutations in $U_q(\mathfrak{n})$ at Roots of One

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Let ε be a root of one and $\mathfrak{g}$ a semisimple Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{h} + \mathfrak{n}^-$. Let U_ε^+ (resp. $U_\varepsilon^{res\,+}$) be the nonrestricted (resp. restricted) quantum enveloping algebra of $\mathfrak{n}$. We prove that $\operatorname{Fract} U_\varepsilon^+$ is a quantum Weyl field. We then give a description of the ε-center of U_ε^+. Let $U_\varepsilon^{fin\,+}$ be the finite part of $U_\varepsilon^{res\,+}$. Via the Drinfeld correspondence, the $U_\varepsilon^{fin\,+}$-covariant space of a Weyl module is ε-central. In case $\mathfrak{g} = \mathfrak{sl}_n$, this enables us to describe this space in terms of semistandard Young tableaux.

0. INTRODUCTION

Let q be a nonzero complex number. A $\mathbb{C}$-algebra defined by generators X_i, $1 \le i \le m$, and relations $X_i X_j = q^{a_{i,j}} X_j X_i$, $1 \le i < j \le m$, $a_{i,j} \in \mathbb{Z}$, will be called "algebra of regular functions on an affine quantum space." Its skew field of fractions will be called a quantum Weyl field. The X_i, $1 \le i \le m$, will be called a system of q-commuting generators.

Let $\mathfrak{g}$ be a semisimple Lie $\mathbb{C}$-algebra of rank n. Let Δ be the root system associated with the choice of a Cartan subalgebra $\mathfrak{h}$, and let Δ^+ be the set of positive roots. We fix a decomposition of the longest element w_0 of the Weyl group.

Let q be an indeterminate and $U_q(\mathfrak{g})$ be the simply connected quantized enveloping algebra, defined as in [16, 3.2.9]. Let ε be an lth root of one, $l \neq 2$, and $l \neq 3$ if $\mathfrak{g}$ has a component of type G_2. We define U_ε (resp. U_ε^{res}) to be the nonrestricted (resp. restricted) form as in [11]. As in the classical case, let U_ε^+ (resp. $U_\varepsilon^{res\,+}$) be its "nilpotent" subalgebra. Let E_α, $\alpha \in \Delta^+$, be the root vectors of U_ε^+, and $\operatorname{Gr} U_\varepsilon^+$ be its associated graded algebra (see Section 1.5).

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Generalizing results of [1], [17], [10], we prove in Section 3.2 that $\operatorname{Fract} U_\varepsilon^+$ is a quantum Weyl field, i.e., it contains an algebra A_{w_0} of regular functions on an affine quantum space, such that $\operatorname{Fract} A_{w_0} = \operatorname{Fract} U_\varepsilon^+$. The method we use specifies the description of the E_α's in term of a system of ε-commuting generators of A_{w_0}. Moreover, the algebra A_{w_0} is isomorphic to $\operatorname{Gr} U_\varepsilon^+$. The two main tools of the proof are the following. The first one is the notion of a roots package (see Definition 2.2), which arises in Weyl modules (see Lemmas 2.1 and 2.2). The second tool is the Drinfeld correspondence, which can be made precise by the universal $\mathscr{R}$-matrix (see Section 1.6).

An element in U_ε^+ will be called ε-central if it ε-commutes with the E_α's. As in [1] for the generic case, the realization of U_ε^+ as a quantum Weyl field is of great help in obtaining its ε-center. As for U_ε (see [13]), Corollary 4.2 asserts that the center Z of U_ε^+ is generated by the specialization Z_ε of the generic center (see [7]) and the algebra Z_0 generated by the E_α^l, $\alpha \in \Delta^+$. For this result, we first study the center of $\operatorname{Gr} U_\varepsilon^+$ and then lift our results by the (abstract) isomorphism $\operatorname{Gr} U_\varepsilon^+ \simeq A_{w_0} \subset \operatorname{Fract} U_\varepsilon^+$.

In Section 5, we give an application of the ε-center to some covariants of Weyl modules. Let U_ε^{fin+} be the subalgebra of U_ε^{res+} generated by E_α, $\alpha \in \Delta^+$. Let λ be a dominant weight and $V_\varepsilon(\lambda)$ be a dominant weight and $V_\varepsilon(\lambda)$ be the U_ε^{res+}-Weyl module as in Section 1.7. Then, as in the generic case, its dual $V_\varepsilon(\lambda)^*$, viewed as a U_ε^{fin+}-module, can be naturally embedded in U_ε^+, endowed with the adjoint action of U_ε^{fin+}, via the Drinfeld isomorphism. Inside U_ε^+ the U_ε^{fin+}-invariants of $V_\varepsilon(\lambda)^*$ are ε-central elements (cf. Proposition 5.1). For $\mathfrak{g} = \mathfrak{sl}_n$, this provides a complete description of the U_ε^{fin+}-invariants of $V_\varepsilon(\lambda)^*$ in terms of semistandard Young tableaux (cf. Theorem 5.3).

1. PRELIMINARIES AND NOTATIONS

1.1. Let $\mathfrak{g}$ be a semisimple Lie $\mathbb{C}$-algebra of rank n. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}$ be the triangular decomposition and $\{\alpha_i\}_i$ be a basis of the root system Δ resulting from this decomposition. We note $\mathfrak{b} = \mathfrak{n} + \mathfrak{h}$ and $\mathfrak{b}^- + \mathfrak{h}$, the two opposed Borel subalgebras. Let P be the weight lattice generated by the fundamental weights ϖ_i, $1 \le i \le n$, and $P^+ := \sum_i \mathbb{N}\varpi_i$ the monoid of integral dominant weights. Let W be the Weyl group, generated by the reflections corresponding to the simple roots $s_i := s_{\alpha_i}$. Let w_0 be the longest element of W. We denote by $(\ ,\ )$ the W-invariant form on P. For each root we set $d_\alpha = (\alpha, \alpha)/2$, and $\alpha\check{} = (1/d_\alpha)\alpha$. We have $(\alpha_j\check{}, \varpi_i) = \delta_{ij}$.

1.2. Let q be indeterminate and $U_q(\mathfrak{g})$ be the simply connected quantized enveloping algebra, defined as in [16, 3.2.9]. Let U_q^+ (resp. U_q^-) be the subalgebra generated by the canonical generators $E_i = E_{\alpha_i}$ (resp. $F_i = F_{\alpha_i}$) of positive (resp. negative) weights. For all λ in P, let $\tau(\lambda)$ be the corresponding element in the algebra U^0 of the torus of $U_q(\mathfrak{g})$. We have the triangular decomposition $U_q(\mathfrak{g}) = U_q^- \otimes U^0 \otimes U_q^+$. We set

$$U_q(\mathfrak{b}) = U_q^+ \otimes U^0, \quad U_q(\mathfrak{b}^-) = U_q^- \otimes U^0. \tag{1.2.1}$$

$U_q(\mathfrak{g})$ is endowed with a structure of Hopf algebra with comultiplication Δ and antipode S.

We define in $U_q(\mathfrak{g})$ the left and right adjoint actions by

$$\operatorname{ad} v.u = v_{(1)}uS(v_{(2)}), \quad u.\widetilde{\operatorname{ad}}\, v = S(v_{(1)})uv_{(2)},$$

where $u, v \in U_q(\mathfrak{g})$ and $\Delta(v) = v_{(1)} \otimes v_{(2)}$ with the Sweedler notation. In particular, if u is an element in $U_q(\mathfrak{g})$ of weight μ, then

$$\operatorname{ad} E_i(u) = E_i u - q^{(\alpha_i, \mu)} u E_i, \qquad 1 \le i \le n. \tag{1.2.2}$$

If n is a nonnegative integer and α a positive root, we set

$$[n]_\alpha = \frac{1 - q^{nd_\alpha}}{1 - q^{d_\alpha}}, \quad [n]_\alpha ! = [n]_\alpha [n-1]_\alpha \cdots [1]_\alpha .$$

1.3. The dual space $U_q(\mathfrak{g})^*$ is endowed with the natural left and right regular actions of $U_q(\mathfrak{g})$: $u.c(v) = c(uv)$, $c.u(v) = c(vu)$, $u, v \in U_q(\mathfrak{g})$, $c \in U_q(\mathfrak{g})^*$. For all λ in P^+, let $V_q(\lambda)$ be the simple $U_q(\mathfrak{g})$-module with highest weight λ. For any integral dominant weight λ, we fix a weight basis $\{v_\mu\}$ of $V_q(\lambda)$ (this is a misleading notation because there is generally more than one vector of weight μ). We denote by $\{v_\mu^*\}$ its dual basis. $V_q(\lambda)^*$ is endowed with a natural right $U_q(\mathfrak{g})$-module structure. For all ξ in $V_q(\lambda)^*$ and v in $V_q(\lambda)$, let $c_{\xi, v}^\lambda$ in $U_q(\mathfrak{g})^*$ be given by $c_{\xi, v}^\lambda(u) = \xi(uv)$, $u \in U_q(\mathfrak{g})$. Then, we have $u.c_{\xi, v}^\lambda = c_{\xi, uv}^\lambda$ and $c_{\xi, v}^\lambda.u = c_{\xi u, v}^\lambda$. If ξ (resp. v) has weight ν (resp. μ), we set (if no confusion occurs) $c_{\nu, \mu}^\lambda = c_{\xi, v}^\lambda$. For any integral dominant weight λ, let $C(\lambda)$ (resp. $C^+(\lambda)$) be the space generated by the $c_{\xi, v}^\lambda$ (resp. $c_{\xi, \lambda}^\lambda$) $\xi \in V_q(\lambda)^*$, $v \in V_q(\lambda)$. We set $R = \bigoplus_{\lambda \in P^+} C(\lambda)$, $R^+ = \bigoplus_{\lambda \in P^+} C^+(\lambda)$. R^+ and R are subalgebras of the Hopf dual of $U_q(\mathfrak{g})$. R^+ is a P^+-graded algebra, and the $C^+(\lambda)$'s are its graded components. R (resp. R^+) is a $U_q(\mathfrak{g})$ (resp. U_q^+) module for the left and right regular actions. Moreover, the left and right adjoint actions of $U_q(\mathfrak{g})$ provide right and left adjoint actions of $U_q(\mathfrak{g})$, (resp. U_q^+) on R (resp. R^+).

1.4. [16, Lemma 7.1.9] asserts that the natural morphism $U_q(\mathfrak{g}) \to R^*$ is an embedding. Let λ be in P^+. By [16, 14.11, Remark] the $U_q(\mathfrak{g})$-module structure on $V_q(\lambda)$ extends to a R^*-module structure.

For all w in W, we define the corresponding element N_w in the quantum Weyl group (cf. [16, 10.2.1], [18]). Set $N_i := N_{s_{\alpha_i}}$. Recall the following facts:

(i) N_w is an invertible element of R^* [16, Lemma 10.2.2].

(ii) The automorphism $T_w\colon a \mapsto N_w a N_w^{-1}$ restricts to an automorphism of $U_q(\mathfrak{g})$ and identifies with the Lusztig automorphism [19] associated with w [16, Theorem 10.2.6, 10.5.2].

(iii) Set $T_i := T_{s_{\alpha_i}}$. If $w = s_{i_1} \cdots s_{i_k}$ is a reduced decomposition of w, then $T_w = T_{i_1} \cdots T_{i_k}$ [19].

(iv) Let M be a finite-dimensional $U_q(\mathfrak{g})$-module, let i be in $[1, n]$, and let v be an E_i-invariant element of M with weight μ. Then, up to a multiplicative power of q, $N_i.v = (F_i^m/[m]_{\alpha_i}!)v$, where $m = (\mu, \alpha_i^\vee)$ [16, 10.2.2(9)].

1.5. We fix a decomposition of the longest element of the Weyl group $w_0 = s_{i_1} \dots s_{i_N}$, where $N = \dim \mathfrak{n}$. Set $y_0 = \mathrm{Id}$, $y_l = s_{i_1} s_{i_2} \cdots s_{i_l}$, $1 \le l \le N$, $\beta_l = y_{l-1}(\alpha_{i_l})$, $1 \le l \le N$. We endow an order into the set Δ^+ of positive roots: $\beta_N < \cdots \beta_2 < \beta_1$. We now introduce the following elements in U_q^+ (resp. U_q^-): $E_{\beta_l} = T_{y_{l-1}}(E_{i_l})$, $1 \le l \le N$ (resp. $F_{\beta_l} = T_{y_{l-1}}(F_{i_l})$). Fix w in W and a reduced decomposition $\mathbf{s}_w$ of w. We may assume without loss of generality that the decomposition of w_0 verifies $\mathbf{s}_{w_0} = \mathbf{s}_w \mathbf{s}_{w'}$, $w' \in W$. We shall say that the reduced decomposition $\mathbf{s}_w$ of w defines the elements E_{β_l}, F_{β_l}, $1 \le l \le l(w)$. For each positive root α, and all nonnegative integers n, set $E_\alpha^{(n)} = (1/[n]_{d_\alpha}!)E_\alpha^n$, $F_\alpha^{(n)} = (1/[n]_{d_\alpha}!)F_\alpha^n$. For each $\psi \in \mathbb{N}^N$, let E^ψ, $E^{(\psi)}$, F^ψ, $F^{(\psi)}$ be the ordered products $E^\psi := \prod_{l=N}^1 E_{\beta_l}^{\psi_l}$, $E^{(\psi)} := \prod_{l=N}^l E_{\beta_l}^{(\psi_l)}$, $F^\psi := \prod_{l=N}^1 F_{\beta_l}^{\psi_l}$, and $F^{(\psi)} := \prod_{l=N}^1 F_{\beta_l}^{(\psi_l)}$. The components of ψ in $\mathbb{N}^N$ will be indexed by the positive roots $\psi = (\psi_{\beta_1}, \psi_{\beta_2}, \dots, \psi_{\beta_N}) = (\psi_1, \psi_2, \dots, \psi_N)$. The set $\mathbb{N}^N$ is endowed with a natural lexicographic ordering.

We know (cf. [19]) that these elements generate a Poincaré–Birkoff–Witt basis of U_q^+. Moreover, the order on Δ^+ defines a natural filtration on U_q^+. The associated graded $\mathrm{Gr}\, U_q^+$ is generated by the $\mathrm{Gr}\, E_\beta$, $\beta \in \Delta^+$, and the relations ([20], [12, Lemma 1.7])

$$\mathrm{Gr}\, E_\beta \,\mathrm{Gr}\, E_{\beta'} = q^{(\beta, \beta')} \mathrm{Gr}\, E_\beta \,\mathrm{Gr}\, E_{\beta'}, \qquad \beta > \beta'. \tag{1.5.1}$$

For each w in W, set

$$\mathfrak{n}_w := \mathfrak{n} \cap w(\mathfrak{n}^-), \quad U_q(\mathfrak{n}_w) := U_q^+ \cap T_w\big(U_q(\mathfrak{b}^-)\big).$$

LEMMA. *For* $1 \le i \le n$, $w \in W$, $w = s_{i_1} \cdots s_{i_k}$, *we have*

(i) $w\alpha_i \in \Delta^+$ *iff* $T_w E_i \in U_q^+$.

(ii) $-w\alpha_i \in \Delta^+$ *iff* $T_w E_i \in U_q(\mathfrak{b}^-)$.

(iii) $U_q(\mathfrak{n}_w)$ *is the algebra generated by the* E_{β_j}, $1 \le j \le k$, *defined by the above decomposition of* w.

Proof. (i) is given by [19], [12, 1.6]. The hypothesis of (ii) implies that there exists p, $1 \le p \le k$, such that $s_{i_{p+1}} \cdots s_{i_k}(\alpha_i) = \alpha_{i_p}$. From (i), and the triangular decomposition, $T_w E_i = T_{i_1} \cdots T_{i_p}(E_{i_p}) = T_{i_1} \cdots T_{i_{p-1}}(-F_{i_p}\tau(\alpha_{i_p})) \in U_q(\mathfrak{b}^-)$.

Let us prove (iii). By the definition, $U_q(\mathfrak{n}_w)$ is an algebra. Let B_w be the space generated by the polynomial basis provided by the E_{β_j}, $1 \le j \le k$. As a consequence of (ii), $B_w \subset U_q(\mathfrak{n}_w)$. Moreover, the reduced decomposition of w may be completed into a reduced decomposition of w_0. Thus, we can complete our polynomial basis into a polynomial basis of U_q^+. By (ii) and the triangular decomposition, we obtain the reverse inclusion. ∎

1.6. We know that $U_q(\mathfrak{g})$ is an almost cocommutative Hopf algebra (cf. [14]). Let $\mathscr{R} = \mathscr{R}_{(1)} \otimes \mathscr{R}_{(2)}$ be the universal $\mathscr{R}$-matrix of $U_q(\mathfrak{g})$. We recall the expression of the $\mathscr{R}$-matrix as an ordered product [18, 3.3]:

$$\mathscr{R} = \Bigl(\prod_{\alpha \in \Delta^+} \exp_\alpha\bigl((1 - q_\alpha^{-2})E_\alpha \otimes F_\alpha\bigr)\Bigr) q^{H_i \otimes H_i}, \tag{1.6.1}$$

where (H_i) is an orthonormal basis in $\mathfrak{h}$ and $\exp_\alpha(x) := \sum_{n \ge 0} x^n/[n]_\alpha!$.

Let φ be the restriction homomorphism from $U_q(\mathfrak{g})^*$ to $U_q(\mathfrak{b}^-)^*$ and $J^- = \operatorname{Ker}\varphi \cap R$. The following is well known (cf. [4], [21], [5], [6]).

THEOREM. *We have*

(i) *There exists an injective algebra antihomomorphism* $\beta^+ \colon R/J^- \to U_q(\mathfrak{b})$, *given by* $c \mapsto (\mathrm{Id} \otimes c)(\mathscr{R})$, $c \in R/J^-$.

(ii) *There exists an injective* $\operatorname{ad} U_q(\mathfrak{g})$-*homomorphism* $\zeta\colon R \to U_q(\mathfrak{g})$.

(iii) *The natural projection* $\pi\colon R \to R/J^-$ *restricts into an embedding* $R^+ \hookrightarrow R/J^-$ *and for* $x \in C^+(\lambda)$, $\zeta(x) = \tau(-\lambda)\beta^+(x) \subset \tau(-2\lambda)U_q^+$.

1.7. Let $\mathscr{A}$ be the algebra $\mathbb{C}[q]$ localized at the multiplicative set generated by $(q-1)$, $(q-\omega)$, where ω is a kth root of one, for $k = 2$, and $k = 3$ if $\mathfrak{g}$ has a component of type G_2. We define the $\mathscr{A}$-algebra $U_{\mathscr{A}}$ generated by the E_i, F_i, $1 \le i \le n$, $\tau(\lambda)$, $\lambda \in P^+$. The specialization of $U_{\mathscr{A}}$ at a root of one ε ($\varepsilon^k \ne 1$) will be denoted U_ε. We define in a similar way the $\mathscr{A}$-forms $U_{\mathscr{A}}(\mathfrak{b}^+)$, $U_{\mathscr{A}}(\mathfrak{b}^-)$, $U_{\mathscr{A}}^+$, $U_{\mathscr{A}}^-$, $U_{\mathscr{A}}^0$ and the specializations $U_\varepsilon(\mathfrak{b})$, U_ε^+, etc. As before, $\mathscr{A}$-forms and ε-specialization have a natural PBW basis [12]. Let $U_{\mathscr{A}}^{res+}$ be the $\mathscr{A}$-algebra generated by the $E_\alpha^{(n)}$, $\alpha \in \Delta^+$, $\mathfrak{n} \in \mathbb{N}$.

We define in a similar way the algebra $U_{\mathscr{A}}^{res-}$ and $U_{\mathscr{A}}^{res}$ (see [11, 9.3.1]). For all λ in P^+, we set $V_{\mathscr{A}}(\lambda) := U_{\mathscr{A}}^{res-} v_\lambda \subset V_q(\lambda)$. By [11, Proposition 10.1.4], $V_{\mathscr{A}}(\lambda)$ is a $U_{\mathscr{A}}^{res}$-module and $\mathbb{C}(q) \otimes_{\mathscr{A}} V_{\mathscr{A}}(\lambda) = V_q(\lambda)$; moreover, it is a free $\mathscr{A}$-module. By 1.4 (iv), the extremal vectors $v_{w\lambda} := N_w v_\lambda$, $w \in W$, belong to $V_{\mathscr{A}}(\lambda)$. Hence, as T_w restricts to $U_{\mathscr{A}}^{res}$, it follows that N_w acts on $V_{\mathscr{A}}(\lambda)$. We define as above the specializations U_ε^{res+}, U_ε^{res-}, $V_\varepsilon(\lambda)$.

2. INTEGRAL MODULES AND ROOT PACKAGES

2.1. A decomposition of w_0 defines a partition of Δ^+ in the following way.

Definition and Notation. Fix $w_0 = s_{i_1} \cdots s_{i_N}$. For $1 \le j \le n$, we call root packages the sets $\Delta_j^+ := \{\beta_l, i_l = j\}$. For m, $1 \le m \le k := \operatorname{Card}\Delta_j^+$, we define $\alpha_{j,m}$ to be the mth element in the decreasing sequence of the roots of Δ_j^+: $\alpha_{j,1} > \alpha_{j,2} > \cdots > \alpha_{j,m} > \cdots > \alpha_{j,k}$. Set $\Delta_{j,m}^+ := \{\alpha_{j,t}, 1 \le t \le m\}$.

Note that the partition of Δ^+ into root packages depends on the choice of the reduced decomposition. In the case $\mathfrak{g} = \mathfrak{sl}_4$, the decompositions $s_1s_2s_3s_1s_2s_1$ and $s_2s_1s_3s_2s_1s_3$ of w_0 define different partitions of Δ^+ that are, respectively,

$$\{\{\alpha_1, \alpha_2, \alpha_3\}, \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3\}, \{\alpha_1 + \alpha_2 + \alpha_3\}\},$$
$$\{\{\alpha_2, \alpha_1 + \alpha_2 + \alpha_3\}, \{\alpha_1 + \alpha_2, \alpha_3\}, \{\alpha_2 + \alpha_3, \alpha_1\}\}.$$

Let M be a $\mathscr{A}$-module, $v, w \in M$. In the sequel, $v \equiv w$ means that v equals w up to a (multiplicative) invertible element of $\mathscr{A}$.

Lemma. *With the notation above, suppose $\beta_{i_p} = \alpha_{j,k}$, $\beta_{i_q} = \alpha_{j,k+1}$. Let $\psi^{(j,k)} \in \mathbb{N}^N$ such that $\psi_l^{(j,k)} = 1$ if $\beta_l \in \Delta_{j,k}^+$, and $\psi_l^{(j,k)} = 0$ otherwise. Let $\phi^{(j,k)} \in \mathbb{N}^N$ such that $\phi_l^{(j,k)} = (\alpha_j, \alpha_l{}^\vee)$ if $p < l < q$ and $\phi_l^{(j,k)} = 0$ otherwise. Let y_l be as in Section* 1.5. *Then*

(i) *Set $\psi := \psi^{(j,k+1)}$ and $\phi := \psi^{(j,k-1)} + \phi^{(j,k)}$. In $V_{\mathscr{A}}(\varpi_j)$, one has $F^{(\psi)} v_{\varpi_j} \equiv F^{(\phi)} v_{\varpi_j} \equiv v_{y_q \varpi_j}$.*

(ii) *If $\alpha \le \alpha_{j,k+1}$, then $v^*_{y_p\varpi_j} S(E_\alpha) = \delta_{\alpha, \alpha_{j,k+1}} v^*_{y_q\varpi_j}$.*

(iii) $y_p\varpi_j - \alpha_{j,k+1} = y_q\varpi_j$.

Proof. Let us prove that $F^{(\psi)} v_{\varpi_j} \equiv v_{y_q\varpi_j}$. We have to prove that $F^{(\psi^{(j,k)})} v_{\varpi_j} \equiv v_{y_p\varpi_j} \Rightarrow F^{(\psi^{(j,k+1)})} v_{\varpi_j} \equiv v_{y_q\varpi_j}$. The result then follows by induction. Write $y_p = w_1 s_j w_2 s_j \cdots w_k s_j$, and $y_q = w_1 s_j w_2 s_j \cdots w_{k+1} s_j$, where the w_l have no s_j in their decomposition.

$$\begin{aligned} F^{(\psi^{(j,k+1)})}v_{\varpi_j} &= F_{\alpha_{(j,k+1)}}F^{(\psi^{(j,k)})}v_{\varpi_j} \equiv F_{\alpha_{j,k+1}}v_{y_p\varpi_j} \\ &= N_{w_1s_jw_2s_j\cdots w_{k+1}}F_jN^{-1}_{w_1s_jw_2s_j\cdots w_{k+1}}v_{y_p\varpi_j} \equiv N_{w_1s_jw_2s_j\cdots w_{k+1}}F_jN^{-1}_{\omega_{k+1}}v_{\varpi_j} \\ &= N_{w_1s_jw_2s_j\cdots w_{k+1}}F_jv_{\varpi_j} \equiv N_{w_1s_jw_2s_j\cdots w_{k+1}}v_{s_j\varpi_j} \equiv v_{y_q\varpi_j}. \end{aligned}$$

We now prove that $F^{(\phi)}v_{\varpi_j} \equiv v_{y_q\varpi_j}$. By the previous assertion, it is enough to prove that $F^{(\phi^{(j,k)})}v_{w_1s_jw_2s_j\cdots w_{k-1}s_j\varpi_j} \equiv v_{y_q\varpi_j}$. For $p < l < q$, set $z_l = s_{i_{p+1}} \cdots s_{i_l}$. We prove that $F_{\beta_l}^{-(\alpha_j,\alpha_l\check{})}v_{y_pz_{l-1}s_j\varpi_j} \equiv v_{y_pz_ls_j\varpi_j}$. The result will follow by induction.

$$\begin{aligned} F_{\beta_l}^{-(\alpha_j,\alpha_l\check{})}v_{y_pz_{l-1}s_j\varpi_j} &= N_{y_pz_{l-1}}F_{i_l}^{-(\alpha_j,\alpha_l\check{})}N^{-1}_{y_pz_{l-1}}v_{y_pz_{l-1}s_j\varpi_j} \\ &\equiv N_{y_pz_{l-1}}F_{i_l}^{-(\alpha_j,\alpha_l\check{})}v_{s_j\varpi_j} \equiv N_{y_pz_{l-1}}v_{s_{i_l}s_j\varpi_j} \equiv v_{y_pz_ls_j\varpi_j}. \end{aligned}$$

Let $\alpha < \alpha_{j,k+1}$. Then, acting by $T_{y_p}^{-1}$ gives $F_\alpha v_{y_p\varpi_j} = 0$. Hence, by (i), $\alpha \le \alpha_{j,k+1} \Rightarrow F_\alpha v_{y_p\varpi_j} = \delta_{\alpha,\alpha_{j,k+1}}v_{y_q\varpi_j}$. This is (ii) via the Chevalley automorphism, and it gives (iii). ∎

EXAMPLE. Let $\mathfrak{g}$ be of type B_2, with the notation of [2, Planche II]. Fix the following decomposition: $w_0 = s_1s_2s_1s_2$. Then the order in Δ^+ is given by

$$\alpha_1 > \alpha_1 + \alpha_2 > \alpha_1 + 2\alpha_2 > \alpha_2.$$

So,

$$\begin{aligned} \Delta_1^+ &= \{\alpha_{1,1} = \alpha_1, \alpha_{1,2} = \alpha_1 + 2\alpha_2\}, \\ \Delta_2^+ &= \{\alpha_{2,1} = \alpha_1 + \alpha_2, \alpha_{2,2} = \alpha_2\}. \end{aligned}$$

(i) gives $\alpha_1 + (\alpha_1 + 2\alpha_2) = 2(\alpha_1 + \alpha_2)$ and $(\alpha_1 + \alpha_2) + \alpha_2 = (\alpha_1 + 2\alpha_2)$.

(ii) gives that $F_{\alpha_1+2\alpha_2}F_{\alpha_1}v_{\varpi_1}$ and $F^{(2)}_{\alpha_1+\alpha_2}v_{\varpi_1}$ are extremal vectors of weight $s_1s_2s_1\varpi_1$ in $V_{\mathscr{A}}(\varpi_1)$. Similarly, $F_{\alpha_2}F_{\alpha_1+\alpha_2}v_{\varpi_2}$ and $F_{\alpha_1+2\alpha_2}v_{\varpi_2}$ are extremal vectors of weight $s_1s_2s_1s_2\varpi_1$ in $V_{\mathscr{A}}(\varpi_2)$.

2.2. The integers j, k, p, are fixed as above. An element $\Lambda \in \mathbb{N}^N$ is said to satisfy $\mathscr{P}$ iff $F^{(\Lambda)}v_{\varpi_j} = v_{y_p\varpi_j}$, up to a nonzero scalar. Set $\chi = \psi^{(j,k)}$. By the previous lemma, χ verifies $\mathscr{P}$. Moreover,

LEMMA. *Let Λ be in $\mathbb{N}^N$ and suppose that Λ verifies $\mathscr{P}$. Then*

(i) $\chi \ge \Lambda$.

(ii) $\beta \notin [\alpha_{j,k}, \alpha_{j,1}]$ *implies that* $\Lambda_\beta = 0$.

(iii) $\Lambda_{\alpha_{j,k}} \le 1$.

Proof. First, remark that if $\alpha_{j,1} < \beta$, then $F_\beta v_{\varpi_j} = 0$. Indeed, the α_j component of F_β is zero, by construction of F_β. Now, let β be such that $\alpha_{j,k+1} < \beta < \alpha_{j,k}$. Then, by the previous remark, we have $F_\beta F^{(\chi)} v_{\varpi_j} = F_\beta v_{y_p \varpi_j} \equiv N_{y_p}^{-1} T_{y_p}^{-1}(F_\beta) v_{\varpi_j} = 0$ (for the last equality, we can repeat the argument above, with $T_{y_p}^{-1}(F_\beta)$ instead of F_β). This provides (i) by induction.

We now prove (ii). By the previous assertion, we have $\beta > \alpha_{j,1} \Rightarrow \Lambda_\beta = 0$. Hence, we can decompose $\Lambda = \Lambda^0 + \Lambda^1$, where the support of Λ^0 (resp. Λ^1) is in $[\beta_N, \alpha_{j,k}[$ (resp. $[\alpha_{j,k}, \alpha_{j,1}]$). Remark that, by Lemma 1.5, $F := T_{y_p}^{-1}(F^{(\Lambda_0)})$ has a (strictly) negative weight. Suppose now $F^{(\Lambda_0)} F^{(\Lambda_1)} v_{\varpi_j} = F^{(\Lambda)} v_{\varpi_j} \equiv v_{y_p \varpi_j}$; then acting on both sides by $N_{y_p}^{-1}$, we obtain that $v_{\varpi_j} \in F.V_{\mathscr{A}}(\varpi_j)$. Hence a contradiction and (ii) holds.

Suppose $\Lambda_{\alpha_{j,k}} > 1$. Let Λ' be in $\mathbb{N}^N$ such that $\Lambda'_\beta = \Lambda_\beta - 1$ if $\beta = \alpha_{j,k}$ and $\Lambda' = \Lambda_\beta$ otherwise. Then $F^{(\Lambda')} v_{\varpi_j}$ is nonzero of weight $y_p \varpi_j + \alpha_{j,k}$. Let r be such that $y_r \alpha_j = \alpha_{j,k-1}$ if $k \neq 1$ and $r = 0$ if $k = 1$. Then, $r < p$ and Lemma 2.1 (iii), applied to $k-1$, given $y_p \varpi_j = y_r \varpi_j - \alpha_{j,k}$. Hence, $F^{(\Lambda')} v_{\varpi_j} = v_{y_r \varpi_j}$, up to a nonzero scalar. By the hypothesis $v_{y_r \varpi_j} \in F_{\alpha_{j,k}} V_{\mathscr{A}}(\varpi_j) = T_{y_p}(\alpha_j) V_{\mathscr{A}}(\varpi_j)$. As above, acting by $T_{y_r}^{-1}$ on both sides yields a contradiction. So, (iii) holds. ∎

3. FUNCTIONS ON AFFINE QUANTUM SPACE AND QUANTUM WEYL FIELDS

In the sequel, we fix a decomposition of ω_0. For each λ in P^+, we fix a basis $\{v_\mu\}$ of the $\mathscr{A}$-module $V_{\mathscr{A}}(\lambda)$. If $\{v_\mu^*\}$ is the dual basis, we then have the matrix coefficient $c^\lambda_{v_\nu^*, v_\mu}$. For each symbol $Y_{\alpha_{j,k}}$, we will frequently use the notation $Y_{j,k}$.

3.1. An algebra defined by generators X_i, $1 \leq i \leq m$, and relations $X_i X_j = q^{a_{i,j}} X_j X_i$, $1 \leq i < j \leq m$, $a_{i,j} \in \mathbb{Z}$, will be called an "algebra of regular functions on an affine quantum space." Its skew field of fractions will be called a quantum Weyl field. The X_i, $1 \leq i \leq m$, will be called a system of q-commuting generators (SQCG). The (antisymmetric) matrix $A := (a_{i,j})$ will be called a matrix associated with the SQCG.

EXAMPLE 1. By 1.5 $\{\operatorname{Gr} E_\beta,\ \beta \in \Delta^+\}$ is a system of q-commuting generators for $\operatorname{Gr} U_q^+$. Let $A(\mathfrak{n})$ be the associated matrix. Then $A(\mathfrak{n})$ is a $N \times N$ antisymmetric matrix with the lower triangular part given by $A(\mathfrak{n})_{\beta,\beta'} = (\beta, \beta')$.

DEFINITION. Let $\beta \in \Delta^+$, $\beta = \beta_{i_p} = \alpha_{j,k}$. We define as in [10, 2.3] the following elements c_β $(= c_{j,k})$ of $\operatorname{Fract} R^+$:

$$c_\beta = c^{\varpi_{i_p}}_{y_p\varpi_{i_p}, \varpi_{i_p}}, \quad a_\beta = c_{j,k-1}^{-1} c_{j,k},$$

with the convention $c_{j,0} = c^{\varpi_{i_p}}_{K\varpi_{i_p}, \varpi_{i_p}}$.

EXAMPLE 2. We shall prove in Corollary 3.2 and (3.3.2) (see also [10, 2.3, 3.1] for another proof) that the $\{a_\beta, \ \beta \in \Delta^+\}$ is the SQCG of an algebra of regular functions on an affine quantum space whose associated matrix is the transpose of $A(\mathfrak{n})$.

3.2. We still denote by c_β and a_β the elements of $\operatorname{Fract} U_{\mathscr{A}}(\mathfrak{b}^+)$ corresponding to c_β, $a_\beta \in \operatorname{Fract} R^+$ via the Drinfeld antihomomorphism β^+ (cf. 1.6) (in particular, $c_{j,0} = \tau(-\varpi_{i_p})$). These elements satisfy the following properties.

PROPOSITION. *The c_β (resp. a_β) are q-commuting elements in the $\mathscr{A}$-form $U_{\mathscr{A}}(\mathfrak{b}^+)$ (resp. $\operatorname{Fract} U^+_{\mathscr{A}}$). For all $\beta = \beta_p$ in Δ^+, we have $c_\beta = (E_\beta P + Q)\tau(-\varpi_l)$, where P and Q are polynomials in E_α, $\alpha > \beta$. Moreover, the ε-specialization of P is nonzero.*

Proof. The fact that the c_β and the a_β are q-commuting elements is a consequence of [10, Proposition 2.3]. By Theorem 1.6 (iii), $c_{j,k} = c_\beta \in \beta^+(C^+(\varpi_j)) \subset \tau(-\varpi_j)U_q(\mathfrak{g})$. This also holds for $c_{j,k-1}$; hence $a_\beta \in \operatorname{Fract} U_q^+$. Set $X_\beta = X_{j,k} = \tau(\varpi_j)c_\beta \in U_q^+$. By 1.6, we have

$$X_\beta := \sum_{\psi \in \mathbb{N}^N} A_\psi E^\psi, \qquad \text{where } A_\psi = v'_{y_p\varpi_j}\left(F^{(\psi)} v_{\varpi_j}\right) \prod \left(1 - q_\alpha^{-2}\right)^{\psi_\alpha}. \tag{3.2.1}$$

By the Weyl character formula, $v'_{y_p\varpi_j}(F^{(\psi)}v_{\varpi_j})$ is nonzero iff $F^{(\psi)}v_{\varpi_j} = v_{y_p\varpi_j}$, up to a nonzero scalar. Hence, X_β is a sum of monomials $A_\psi E^\psi$, where ψ verifies $\mathscr{P}$ and $A_\psi \in U^+_{\mathscr{A}}$. By Lemma 2.2 (ii), (iii), we have $X_\beta = (E_\beta P + Q)$, where P and Q are polynomials in E_α, $\alpha > \beta$. By Lemma 2.2 (i), $A_\chi E^\chi$ is one of these monomials. By construction, it can be factorized on the left by E_β, and by Lemma 2.1 (i). A_χ is invertible in $\mathscr{A}$. Hence, the ε-specialization of P is nonzero. ∎

Let $\beta = \alpha_{j,k}$. We retain the notation of the proof above, and we set $X_\beta = X_{j,k} = \tau(\varpi_j)c_\beta \in U^+_{\mathscr{A}}$. Then, (3.2.1) and Lemma 2.2 (i) give (up to a unit in $\mathscr{A}$)

$$\operatorname{Gr} X_{j,k} = \prod_{m \le k} \operatorname{Gr} E_{\alpha_{j,m}}. \tag{3.2.2}$$

In the following corollary, we fix an element w in W and its reduced decomposition $\mathbf{s}_w$. We assume as in 1.5 that the decomposition of w_0 verifies $\mathbf{s}_{w_0} = \mathbf{s}_w \mathbf{s}_{w'}$, $w' \in W$. We have

COROLLARY. *Let A_w be the algebra generated by the a_β, where β runs over the weights of $\mathfrak{n}_w$. Then, A_w is an algebra of regular functions on an affine quantum space, with GK-dimension $l(w)$.* $\operatorname{Fract} U_{\mathscr{A}}(\mathfrak{n}_w) = \operatorname{Fract} A_w$; *in particular*, $\operatorname{Fract} U_{\mathscr{A}}^+ = \operatorname{Fract} A_{w_0}$, *and the associated matrix of A_{w_0} is $A(\mathfrak{n})$. This remains true after ε-specialization.*

Proof. A_w is generated by the a_β, which q-commute. From the assumption preceding the corollary, if $\beta \in \mathfrak{n}_w$ and $\alpha > \beta$, then $\alpha \in \mathfrak{n}_w$. Then, the equality $\operatorname{Fract} U_{\mathscr{A}}(\mathfrak{n}_w) = \operatorname{Fract} A_w$ follows from Lemma 1.5 (iii) and Proposition 3.2. Recalling that the E^ψ, $\psi \in \mathbb{N}^N$ form a PBW-basis of $U_{\mathscr{A}}^+$, we deduce that the only relations in A_w are the q-commutations. ∎

3.3. In this section, we make more precise the q-commutations inside $U_{\mathscr{A}}(\mathfrak{b}^+)$. By [16, Proposition 9.1.5],

$$c_\beta c_{\beta'} = q^{(s(\beta), s(\beta')) - (s(\beta), \varpi_j') - (\varpi_j, s(\beta'))} c_{\beta'} c_\beta, \qquad \beta = \alpha_{j,k} > \beta' = \alpha_{j',k'}, \tag{3.3.1}$$

where $s(\beta)$ is the weight of c_β. By (3.2.2), we have $s(\alpha_{j,k+1}) = s(\alpha_{j,k}) + \alpha_{j,k+1}$ (*). It follows that

$$a_\beta a_{\beta'} = q^{-(\beta, \beta')} a_{\beta'} a_\beta, \quad X_\beta X_{\beta'} = q^{(s(\beta), s(\beta') - 2\varpi_{j'})} X_{\beta'}, X_\beta,$$
$$\beta > \beta' = \alpha_{j',k'}. \tag{3.3.2}$$

Recall that for all in P, $\tau(\lambda)$ is a group-like element, and so $\operatorname{ad} \tau(\lambda)$ is an automorphism of $U_q(\mathfrak{g})$.

LEMMA. *Suppose $\alpha_{j,k}$ is not almost minimal. Let $\sigma = \operatorname{ad} \tau(\alpha_{j,k+1})$ and let D be a σ-derivation on $U_q(\mathfrak{g})$, i.e., $D(uv) = D(u)v + \sigma(u)D(v)$, $u, v \in U_q(\mathfrak{g})$. Suppose $D(c_{j,k}) = c_{j,k+1}$. Then, $D(c_{j,k}^n) = [n]_{\alpha_{j,k+1}} c_{j,k}^{n-1} c_{j,k+1}$, up to a power of q.*

Proof. By (3.2.2) and (1.5.1), we have

$$X_{j,k} X_{j,k+1} = q^{(s(\alpha_{j,k}), \alpha_{j,k+1})} X_{j,k+1} X_{j,k} \ (**).$$

Comparing with (3.3.2), we obtain $(s(\alpha_{j,k}), s(\alpha_{j,k})) = -2(s(\alpha_{j,k}), \varpi_j)$. Hence, $(s(\alpha_{j,k+1}), s(\alpha_{j,k+1})) = -2(s(\alpha_{j,k+1}), \varpi_j)$. Using (*) on the left-hand side gives

$$\big(\alpha_{j,k+1}, s(\alpha_{j,k})\big) = (\alpha_{j,k+1}, \varpi_j) - \tfrac{1}{2}(\alpha_{j,k+1}, \alpha_{j,k+1}). \tag{3.3.3}$$

Using (3.3.3) and (**), we obtain $\sigma(c_{j,k})c_{j,k+1} = q^{(\alpha_{j,k+1}, s(\alpha_{j,k}))}c_{j,k}c_{j,k+1} = q^m c_{j,k+1}c_{j,k}$, with $m = (\alpha_{j,k+1}, s(\alpha_{j,k})) - (\alpha_{j,k+1}, \varpi_j) - \frac{1}{2}(\alpha_{j,k+1}, \alpha_{j,k+1})$. Using (3.3.3) again, we have $m = (\alpha_{j,k+1}, \alpha_{j,k+1})$. We have proved that $\sigma(c_{j,k})c_{j,k+1} = q^{(\alpha_{j,k+1}, \alpha_{j,k+1})}c_{j,k+1}c_{j,k}$. This implies the lemma by induction on n.

4. CENTER OF U_ε^+

Recall that by convention, we denote by the same symbol an element and its ε-specialization (when it exists). For example, we set $X_\beta = X_{j,k} = \tau(\varpi_j)c_\beta \in U_\varepsilon^+$, and these elements exist by Proposition 3.2.

4.1. We suppose in the sequel that l is odd and is not divisible by 3 if $\mathfrak{g}$ has a component of type G_2. Let ε be an lth root of one.

Let $\beta = \alpha_{j,k} \in \Delta_j^+$. We say that β is an almost minimal root if β is minimal in Δ_j^+, i.e., $\beta = \alpha_{j,m}$, $m = \operatorname{Card}\Delta_j^+$. Set $X_{j,0} = 1$ and $X_{j,k+1} = 0$ if $\alpha_{j,k}$ is almost minimal. If β is almost minimal, then let $z_j := X_\beta = X_{j,k}$. For $\lambda \in P^+$, $\lambda = \sum n_j\varpi_j$, set $z_\lambda = \prod z_j^{n_j}$.

Definition. An ε-central element a in $\operatorname{Fract} U_\varepsilon^+$ is an element that ε-commutes with the generators E_i, i.e., there exists a weight η (unique modulo lP) such that $aE_i = \varepsilon^{(\alpha_i, \eta)}E_i a$. The class of η modulo lP will be called the commutation weight of a.

Lemma. *Let $\beta = \alpha_{j,k}$, $\lambda \in P^+$.*

(i) *if β is not almost minimal and $\gamma = \alpha_{j,k+1}$, $\alpha \leq \gamma$, then* $\operatorname{ad} E_\alpha(c_\beta) = \delta_{\gamma,\alpha}c_\gamma$ *in R^+.*

(ii) *z_λ has weight $\lambda - w_0\lambda$ and commutation weight* $(\operatorname{Id} + w_0)(\lambda)$ *modulo lP.*

(iii) $\operatorname{Gr} z_j = \prod_{\alpha\in\Delta_j^+} \operatorname{Gr} E_\alpha$.

Proof. (i) is a consequence of Lemma 2.1 (ii) and the definition of c_β (cf. 3.1).

(ii) is provided by [8, 2.2.1].

(iii) is a particular case of (3.2.2). ∎

Remark. If $u = z_0 z_\mu$, $z_0 \in Z_0$, has weight λ and commutation weight λ', then, by (ii), $\mu = \frac{1}{2}(\lambda + \lambda')$ modulo lP. The elements z_λ and z_μ have the same commutation weight iff $\lambda = \mu + lP + \operatorname{Ker}(\operatorname{Id} + w_0)$.

4.2. The group H generated by $-w_0$ acts naturally on $[1, n]$ by $h(\alpha_i) = \alpha_{h(i)}$, $h \in H$. Set $-w_0(i) = i^*$. Let I be a subset of $[1, n]$ containing exactly one representative for each class modulo H and $\bar{I}$ its complementary subset. Let Z_ε be the algebra generated by the $z_i z_{i*}$, if $i \neq i*$, and the z_i, if $i = i*$. Then, by (ii) of the previous lemma, Z_ε is central in U_ε^+. Remark that Z_ε may be seen as the specialization of the generic center (see [7]). As ε is an lth root of one, we know [12] that E_β^l, $\beta \in \Delta^+$, is also central in U_ε^+. Let Z_0 be the algebra generated by those elements.

Let $\overline{P}$ be the lattice $P/lP + \mathrm{Ker}(\mathrm{Id} + w_0)$. As in [8, 2.2], each class of elements in $\overline{P}$ has a representative in P^+. For all λ in P^+, let C^λ be the line in U_ε^+ generated by z_λ. For all $\overline{\pi}$ in $\overline{P}$, set $C^{\overline{\pi}} = \bigoplus_{\overline{\lambda} = \overline{\pi}} C^\lambda$ (this is a direct sum by Lemma 4.1 (iii)). The following theorem describes the ε-central elements in U_ε^+.

THEOREM. *Let C_ε be the set of ε-central elements in U_ε^+. Then $C_\varepsilon = \bigcup_{\overline{\lambda} \in \overline{P}} Z_0 C^{\overline{\lambda}}$. In particular, $C_\varepsilon \subset \bigoplus_{\mu \in P^+} Z_0 z_\mu$.*

Proof.

Step 1. Let $a \in C_\varepsilon$ be of the form $\prod X_\beta^{n_\beta}$, $0 \leq n_\beta < l$; then $a \in \bigcup_{\lambda \in P^+} C^\lambda$.

Proof. We have to show that $n_\beta = 0$ for β not almost minimal. We prove this assertion by (reverse) induction on β. Suppose that $\alpha_{j,k}$ is not almost minimal and $n_\beta = 0$ for $\beta > \alpha_{j,k}$. Let $\gamma = \alpha_{j,k+1}$ and $\gamma^>$ be the smallest root greater than γ. By Lemma 4.1 (i) and Theorem 1.6, β in Δ_i^+, $\beta > \alpha_{j,k}$, implies that $\tau(-2\varpi_i)X_\beta$ is ad $E_{\gamma^>}$-invariant. Hence, by [18, 2.4.1] (see also [10, 2.1]), ad E_γ acts as a twisted derivation on the algebra generated by those $\tau(-2\varpi_i)X_\beta$. To be more precise, on this algebra, ad E_γ is a derivation twisted by the automorphism ad $\tau(\gamma)$. Let $\tau = \prod_{1 \leq i \leq n,\, \beta \in \Delta^+} \tau(-2n_\beta \varpi_i)$. On the one hand, by the (quantum) Leibniz formula, Lemma 4.1 (i) and Lemma 3.3 give ad $E_\gamma(\tau a) = [n_{j,k}]_\gamma \tau a X_{j,k}^{-1} X_{j,k+1}$. On the other hand, as a is ε-central, we have ad $E_{j,k+1}(\tau a) = E_{j,k+1}\tau a$, up to a multiplicative scalar (which may be zero). This gives $[n_{j,k}]_\gamma X_{j,k+1} = E_{j,k+1} X_{j,k}$, up to a scalar. By (3.2.1) and Lemma 2.1 (i), the left-hand side contains $E^{(\phi)}$ in its PBW-decomposition, and the right-hand side does not. Hence, $[n_{j,k}]_\gamma = 0$. Thus, $n_{j,k} = 0$, and the assertion follows inductively. ∎

Step 2. Each monomial b in $\mathrm{Gr}\, U_\varepsilon^+$ such that $b\, \mathrm{Gr}\, E_\beta = \varepsilon^{(\beta,\nu)} \mathrm{Gr}\, E_\beta b$, $\nu \in P$, lifts to $\bigcup_{\overline{\lambda} \in \overline{P}} Z_0 C^{\overline{\lambda}}$.

Proof. Let $b = \prod \mathrm{Gr}\, E_\beta^{n_\beta}$. As $\mathrm{Gr}\, E_\beta^l$ lifts to Z_0, we can restrict to the case $0 \leq n_\beta < l$. Let $c := \prod a_\beta^{n_\beta} \in \mathrm{Fract}\, U_\varepsilon^+$. By Corollary 3.2, the hypothesis on b implies that $c a_\beta = \varepsilon^{(\beta,\nu)} a_\beta c$. This easily implies by Proposition 3.2 that $c E_\beta = \varepsilon^{(\beta,\nu)} E_\beta c$. In particular, c is ε-central in $\mathrm{Fract}\, U_\varepsilon^+$. Remark now that $c = \prod X_{j,k}^{n_{j,k} - n_{j,k+1}}$, with the convention $n_{j,k+1} = 0$ if $n_{j,k}$ is almost minimal. Define $r_{\alpha_{j,k}} = r_{j,k}$ by $r_{j,k} \equiv n_{j,k} - n_{j,k+1}$ $[l]$ and $0 \leq r_{j,k} < l$. Remark that $|n_{j,k} - n_{j,k+1}| < l$. Hence, multiplying c by an (unique) element z_0 in Z_0, we obtain that $\prod X_{j,k}^{r_{j,k}}$ is ε-central in U_ε^+. So, Step 1 gives $z_0 c \in \bigcup_{\lambda \in P^+} C^\lambda$. In particular, $r_\beta = 0$, for β not almost minimal. Thus, $z_0 = 1$. $\mathrm{Gr}(c) = b$ by (3.2.2), and c is the required element. ∎

We can now finish the proof of the theorem. Let $a \in C_\varepsilon$ and $b := \mathrm{Gr}\, a$. Then, b verifies the hypothesis of Step 2. Let c_0 in $\bigcup_{\bar\lambda \in \bar P} Z_0 C^{\bar\lambda}$ such that $\mathrm{Gr}\, c_0 = b$. a and c_0 have the same commutation weight, $c_1 := a - c_0$ is in C_ε and has a lower degree. Hence, the theorem follows by induction. ∎

Remark that if $w_0 = -\mathrm{Id}$, then C_ε is the center of U_ε^+. In general, we have

COROLLARY. *Let Z be the center of U_ε^+. Then,*

(i) $\mathrm{Gr}\, Z$ *is the center of* $\mathrm{Gr}\, U_\varepsilon^+$.

(ii) *Z is generated by Z_ε and Z_0, i.e., $Z = Z_0[Z_\varepsilon]$.*

(iii) *U_ε^+ is projective over its center, of rank $l^{N-\#I}$.*

Proof. (i) and (ii) are direct consequences of Step 2 and the previous theorem. Let S be the multiplicative set in $\mathrm{Gr}\, U_\varepsilon^+$ generated by the $\mathrm{Gr}\, E_\beta$, β not almost minimal or $\beta = \alpha_{i,k}$, $i \in \bar I$. By Lemma 4.1 (iii), $S^{-1}\mathrm{Gr}\, U_\varepsilon^+$ is free over $S^{-1}\mathrm{Gr}\, Z$, of rank $l^{N-\#I}$. This yields (iii). ∎

Remark 1. Recall that for this result, we assumed l odd. If l is even, then Z_0 has to be replaced by a larger algebra Z_0'. This algebra can easily be obtained for each simple Lie algebra $\mathfrak{g}$. In fact, every argument in our proofs remains true, except for the last assertion of the proof of Step 1.

Remark 2. We conjecture that the assertion (i) is true for $U_\varepsilon(\mathfrak{g})$, i.e., the center of $\mathrm{Gr}\, U_\varepsilon$ is the graded space of the center. This conjecture has been verified with Maple V for "small" classical Lie algebras $\mathfrak{g}$ and all exceptional Lie algebras.

4.3. The center of an algebra of functions on a quantum space is directly connected with the kernel of its associated matrix. So, we deduce the following result, which may be written in terms of root systems (and roots packages).

COROLLARY. *Fix a decomposition of w_0. Let $A(\mathfrak{n})$ be as in* [3.1, *Example* 1]. *The rank of $A(\mathfrak{n})$ is $N - \#I$. Let $B(\mathfrak{n})$ be the submatrix of $A(\mathfrak{n})$ obtained by omitting the almost minimal roots of Δ_i^+, $i \in I$. Then the determinant of $B(\mathfrak{n})$ is a power of* 2 (*and* 3 *if* $\mathfrak{g}$ *has a component of type G_2*).

5. U_ε^+-INVARIANTS IN $V_\varepsilon(\lambda)^*$

5.1. By [11, 9.3.6], U_ε^{res+} (cf. 1.7) is generated by the E_i, $E_i^{(l)}$, $1 \le i \le n$. By [11, 9.3.6], the E_i generate a finite-dimensional Hopf subalgebra U_ε^{fin+}. Define the $U_{\mathscr{A}}^{res+}$-module morphism (cf. Theorem 1.6):

$$\zeta_{\mathscr{A}}^\lambda\colon V_{\mathscr{A}}(\lambda)^* \to V_{\mathscr{A}}(\lambda)^* \otimes v_\lambda \hookrightarrow C^+(\lambda) \xrightarrow{\zeta} \tau(-2\lambda)U_q^+.$$

The following proposition gives a characterization of the U_ε^{res+}-invariants in $V_\varepsilon(\lambda)^*$.

PROPOSITION. *The image of $\zeta_{\mathscr{A}}^\lambda$ is contained in $\tau(-2\lambda)U_{\mathscr{A}}^+$ and, by specialization, $\zeta_{\mathscr{A}}^\lambda$ defines an injective morphism of U_ε^{res+}-module: $\zeta_\varepsilon^\lambda$: $V_\varepsilon(\lambda)^* \hookrightarrow \tau(-2\lambda)U_\varepsilon^+$. Moreover, setting $P_\lambda^+ := P^+ \cap (\lambda + lP)$, we have*

$$V_\varepsilon(\lambda)^{*U_\varepsilon^{fin+}} = \left(\zeta_\varepsilon^\lambda\right)^{-1}\left(\tau(-2\lambda)\sum_{\mu \in P_\lambda^+} Z_0 z_\mu\right).$$

Proof. Let $u^* \in V_{\mathscr{A}}(\lambda)^*$. By 1.6, $\zeta_{\mathscr{A}}^\lambda(u^*) = \tau(-2\lambda)\sum a_\psi E^\psi u^*(F^{(\psi)}v_\lambda)$, where the a_ψ are units in $\mathscr{A}$. If v^* belongs to a basis of $V_{\mathscr{A}}(\lambda)^*$, then $v^*(F^{(\psi)}v_\lambda) \in \mathscr{A}$. Hence, $\zeta_{\mathscr{A}}^\lambda(v^*) \in \tau(-2\lambda)U_{\mathscr{A}}^+$. Suppose $\zeta_{\mathscr{A}}^\lambda(u^*) \in (q-\varepsilon)\cdot\tau(-2\lambda)U_{\mathscr{A}}^+$; then $u^*(F^{(\psi)}v_\lambda) \in (q-\varepsilon)\mathscr{A}$ for all ψ. Thus, $u^* \in (q-\varepsilon)\cdot V_{\mathscr{A}}(\lambda)^*$. This proves the injectivity of the specialization of the U_ε^{res+}-module morphism $\zeta_\varepsilon^\lambda$. We now claim that $(\zeta_\varepsilon^\lambda)^{-1}(\tau(-2\lambda)\sum_{\mu\in P_\lambda^+} Z_0 z_\mu) \subset V_\varepsilon(\lambda)^{*U_\varepsilon^{fin+}}$. Indeed, $\tau(-2\lambda)\sum_{\mu\in P_\lambda^+} Z_0 z_\mu$ is ad E_i-invariant by (1.2.2) and Lemma 4.1 (ii).

We now prove the reverse inclusion. Suppose $u^* \in V_\varepsilon(\lambda)^{*U_\varepsilon^{fin+}}$ has weight ν. Then, $\tau(-2\lambda)a := \zeta_\varepsilon^\lambda(u^*)$ is ad E_i-invariant by Theorem 1.6, with weight $\lambda+\nu$. This implies by (1.2.2) that a is ε-central and that the commutation weight of a is equal to $\lambda-\nu$. By Theorem 4.2 and Remark 4.1, we have $a \in \sum_{\mu\in P_\lambda^+} Z_0 z_\mu$. This gives the claimed equality. ∎

5.2. Let $\tau(-2\lambda)K_\lambda^+$ be the image of $V_\varepsilon(\lambda)^*$ by the embedding $\zeta_\varepsilon^\lambda$. By [6, Proposition 4.1 (iii)], we have

$$K_\lambda^+ K_\mu^+ \subset K_{\lambda+\mu}^+. \tag{5.2.1}$$

By Proposition 5.1, the vector space $V_\varepsilon(\lambda)^{*U_\varepsilon^{fin+}}$ is isomorphic to the intersection of K_λ^+ and $\sum_{\mu\in P_\lambda^+} Z_0 z_\mu$. Both spaces can be made explicit, but the intersection is not quite clear. In return, the intersection of the associated graded spaces can be made explicit. We shall prove in 5.3 that, when $\mathfrak{g} = \mathfrak{sl}_n$, the associated graded space of the intersection is the intersection of the associated graded spaces. First, we give the following lemma.

LEMMA. *Let $\lambda \in P^+$ and let X be an extremal vector in K_λ^+, i.e., $\tau(-2\lambda)X$ corresponds via $\zeta_\varepsilon^\lambda$ to an extremal vector in $V_\varepsilon(\lambda)^*$. Then, $X^l \in \sum_{\mu \in lP^+} Z_0[z_\mu]$.*

Proof. It is enough to prove that X^l is central. Indeed, by Remark 4.1 and Theorem 4.2, a central element whose weight belongs to lQ^+ is in $\sum_{\mu \in (l/2)Q^+ \cap P^+} Z_0[z_\mu]$, which is a subset of $\sum_{\mu \in lP^+} Z_0[z_\mu]$ (recall that l is odd).

First, we prove this assertion for $\lambda = \varpi_j$, $1 \le j \le n$. Suppose that X corresponds to the extremal vector $v_{w\varpi_j}^*$, $w \in W$, with w minimal for the length. Thus we have a reduced decomposition $s_w = w_1 s_j \cdots w_k s_j$, where the w_l have no s_j in their decomposition. There exists an element w' of the Weyl group with reduced decomposition $s_{w'}$ such that $s_w s_{w'}$ is a reduced decomposition of w_0. Fix this decomposition of w_0 and set $\beta := w_1 s_j \cdots w_k(\alpha_j)$. Then, X is the X_β defined in the proof of 3.2. By (3.3.2) and Corollary 3.2, X^l commutes with the generators of A_{w_0}; thus it commutes with U_ε^+, and we have the claimed assertion.

Suppose that $\lambda = \sum_i \lambda_i \varpi_i$. Suppose also that X corresponds to the extremal vector $v_{w\lambda}^* \in V_\varepsilon(\lambda)^*$ and that X_j corresponds to $v_{w\varpi_j}^* \in V_\varepsilon(\varpi_j)^*$. Then, $\prod X_j^{l\lambda_j}$ and X^l are equal by (5.2.1), because they both correspond to $v_{wl\lambda}^* \in V_\varepsilon(l\lambda)^*$. The lemma follows. ∎

5.3. In this section, we suppose $\mathfrak{g}$ of type A_n (see [2, Table I]). Let ϵ_i, $1 \le i \le n+1$, be such that $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \le i \le n$, $\sum_{i=1}^{n+1} \epsilon_i = 0$. Fix the decomposition of w_0:

$$w_0 = (s_n \cdots s_1)(s_n \cdots s_2) \cdots (s_n).$$

This decomposition settles the following order on Δ^+ (cf. 1.5):

$$\begin{aligned} \alpha_n > \alpha_n + \alpha_{n-1} > \cdots > \alpha_n + \cdots + \alpha_1 > \alpha_{n-1} > \cdots \\ > \alpha_{n-1} + \cdots + \alpha_1 > \cdots > \alpha_2 + \alpha_1 > \alpha_1. \end{aligned}$$

The roots packages (cf. 2.1) are defined by

$$\alpha_{i,k} = \alpha_{i-k+1} + \cdots + \alpha_{n-k+1}, \qquad 1 \le k \le i \le n. \tag{5.3.1}$$

The principal symbols of the generators of the center are given by

$$z_i = \prod_{k=1}^{i} E_{\alpha_{i,k}}, \quad 1 \le i \le n, \qquad z_\lambda = \prod_i E_{\alpha_{i,k}}^{\lambda_i}. \tag{5.3.2}$$

Let $\lambda = \sum \lambda_i \varpi_i \in P^+$, and $a_i = \sum_{j=i}^n \lambda_j$. A Young diagram of shape λ is a left justified sequence of rows with a_i boxes in the ith row. A semistandard

Young tableau of shape λ is a filling of the boxes with numbers 1 to $n+1$ such that

(R): The entries are not decreasing in the rows.

(C): The entries are strictly increasing in the columns.

Let $\mathbb{T}$ (resp. $\mathbb{T}_\lambda$) be the set of semistandard Young tableaux (resp. semistandard Young tableaux of shape λ). We know that $\#\mathbb{T}_\lambda = \dim V_\varepsilon(\lambda)^*$. Moreover, there exists a basis B_λ of $V_\varepsilon(\lambda)^*$ and a one-to-one correspondence $\mathbb{T}_\lambda \to B_\lambda$ such that a semistandard tableau T maps to a vector of weight $-\sum_i k_i \epsilon_i$, where k_i is the number of i in T. We can define $\mathbb{T}_\lambda^\mu$ to be the set of semistandard tableaux of shape λ with weight μ. If $n \geq i_1 \geq \cdots \geq i_k \geq 1$, $T_m \in \mathbb{T}_{\varpi_{i_m}}$, and $\lambda := \sum_m \varpi_{i_m}$. Then, the T_m are single columns with i_m rows, and we naturally define the concatenation $T_1 * \cdots * T_k$, which belongs to $\mathbb{T}_\lambda$ if (R) and (C) are verified.

Let $\eta\colon \mathbb{T} \to \operatorname{Gr} U_\varepsilon^+$ such that $\eta(T) = \prod_{\beta \in \Delta^+} \operatorname{Gr} E_\beta^{n_\beta}$, where $n_{\alpha_i + \cdots + \alpha_j}$, $1 \leq i \leq j \leq n$, is the number of $j+1$ in the ith row of T. Let η_λ be its restriction to $\mathbb{T}_\lambda$.

EXAMPLE. Let $\mathfrak{g}$ be of type A_2. The following tableau T:

1	1	2	3	3
2	3			

is of shape $\lambda = 3\varpi_1 + 2\varpi_2$ and corresponds to a vector of weight $\mu = -2\epsilon_1 - 2\epsilon_2 - 3\epsilon_3$. Let $T_1, \ldots, T_5$ be its single columns from left to right. We have $T_1, T_2 \in \mathbb{T}_{\varpi_2}$, $T_3, T_4, T_5 \in \mathbb{T}_{\varpi_1}$. The concatenation $T_1 * \cdots * T_5$ gives T. $\eta(T) = \operatorname{Gr} E_{\alpha_1} \operatorname{Gr} E_{\alpha_1+\alpha_2}^2 \operatorname{Gr} E_{\alpha_2}$, $\eta(T_1) = 1$, $\eta(T_2) = \operatorname{Gr} E_{\alpha_2}$, $\eta(T_3) = \operatorname{Gr} E_{\alpha_1}$, $\eta(T_4) = \eta(T_5) = \operatorname{Gr} E_{\alpha_1+\alpha_2}$.

LEMMA. *For all λ in P^+, η_λ yields a one-to-one correspondence between $\mathbb{T}_\lambda$ and a basis of $\operatorname{Gr} K_\lambda^+$. Moreover, if $T \in \mathbb{T}_\lambda^\mu$, then the weight of $\eta(T)$ is $\mu + \lambda$.*

Proof. We have $\#\mathbb{T}_\lambda = \dim \operatorname{Gr} K_\lambda^+$. Moreover, the number m in the ith row belong to $[i, n+1]$; thus η_λ is an embedding. So, by a cardinality argument, it is enough to prove that η maps $\mathbb{T}_\lambda$ into $\operatorname{Gr} K_\lambda^+$.

We sketch a proof of this assertion for $\lambda = \varpi_i$. With the notation of 1.6, let $\{v_i\}$, be a basis of $V_q(\varpi_1)$ and $C := (c_{v_i^*, v_j})$ be the matrix with coefficients in $C(\varpi_1)|_{U_q(b)}$ corresponding to this basis. As in [5, 4.3], we can calculate $\beta^+(C)$ from the representation $V_q(\varpi_1)$. Taking the graded space, we have, for an appropriate basis,

$$\operatorname{Gr}\tau(\varpi_1)\beta^+(C) = \begin{pmatrix} 1 & & & & \\ \operatorname{Gr}E_1 & 1 & & (0) & \\ \operatorname{Gr}E_{12} & \operatorname{Gr}E_2 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \operatorname{Gr}E_{1\dots n} & \operatorname{Gr}E_{2\dots n} & \cdots & \operatorname{Gr}E_n & 1 \end{pmatrix},$$

up to a power of q.

Now, as in [5, 4.3], $K^+_{\varpi_i}$, $1 \le i \le n$, are generated by the quantum minors whose columns are the i first columns of $\tau(\varpi_1)\beta^+(C)$. Let $T \in \mathbb{T}_{\varpi_i}$ and $t_1 < t_2 \cdots < t_i$ be its components. It is easily verified that $\eta(T)$ is the graduate of the quantum minor $\Delta_{\{t_1,\dots,t_i\},\{1,\dots,i\}}$. This gives the result.

Now, for all λ, we can decompose $T \in \mathbb{T}_\lambda$ into single columns $T = T_1 * \cdots * T_k$. We have $\eta(T) = \prod \eta(T_m) \in \prod K^+_{\varpi_{i_m}} \subset K^+_\lambda$, by (5.2.1). The assertion on weights can be verified for T in $\mathbb{T}_{\varpi_i}$ as above and then generalized for all λ. ∎

THEOREM. *For $T \in \mathbb{T}$, $1 \le i \le n$, $1 \le k \le n-i+1$, let $m_{i,k} = m_{i,k}(T)$ be the number of $k+i$ on the ith row. Then, for $\lambda = \sum \lambda_i \varpi_i \in P^+$, the character formula of the U_ε^{fin+} invariants in $V_\varepsilon(\lambda)^*$ is given by*

$$\dim V_\varepsilon(\lambda)^{*\,U_\varepsilon^{fin+}}_\nu = \#\{T \in \mathbb{T}^\nu_\lambda \mid \forall i, k,\, m_{i,k} \equiv \lambda_{n-k+1}[l]\}.$$

Proof. First remark that, by the definition and by (5.3.1), $\eta(T) = \prod \operatorname{Gr} E_{\alpha_{i,k}}^{m_{i-k+1,n+i+1}}$. We prove the theorem by a double inequality.

$\le$: Fix a basis $\{k_\nu\}$ of $K_\lambda^{+U_\varepsilon^{fin+}} := \tau(2\lambda)\zeta_\varepsilon^\lambda(V_\varepsilon(\lambda)^{*U_\varepsilon^{fin+}})$ such that $\{\operatorname{Gr} k_\nu\}$ is a basis of $\operatorname{Gr} K_\lambda^{+U_\varepsilon^{fin+}}$. We suppose k_ν of weight $\nu+\lambda$, i.e., k_ν identifies with a vector of weight ν in $V_\varepsilon(\lambda)^{*U_\varepsilon^{fin+}}_\nu$, and $\operatorname{Gr} k_\nu = \prod \operatorname{Gr} E_{\alpha_{i,k}}^{a_{i,k}}$. Then, by Proposition 5.1, $\operatorname{Gr} k_\nu$ is in $\operatorname{Gr}(\sum_{\mu\in P^+_\lambda} Z_0 z_\mu)$, and this implies $a_{i,k} \equiv \lambda_i\ [l]$ by (5.3.2). By the first remark and the previous Lemma, $\eta_\lambda^{-1}(\operatorname{Gr} k_\nu)$ is in $\{T \in \mathbb{T}^\nu_\lambda \mid \forall i, k,\, m_{i,k} \equiv \lambda_{n-k+1}[l]\}$. So, $\le$ holds.

$\ge$: Let T be in the set defined above. By Proposition 5.1 and the previous Lemma, it is enough to prove that $\eta(T) \in \operatorname{Gr} K^+_\lambda$ can be lifted into $K^+_\lambda \cap \sum_{\mu\in P^+_\lambda} Z_0 z_\mu$. We now show this assertion.

Let $m_k := \operatorname{Inf}_i m_{i,k}$. From T and $i \in [1, n]$, we construct a new tableau as follows. On the jth row of T, $1 \le j \le i$, cancel the rightmost m_{n-i+1} boxes containing $n-i+j+1$; the (nonempty) boxes remaining on the right-hand side are moved to the left (see 5.4).

By [5.4, Claim 1], we obtain a semistandard tableau $T' \in \mathbb{T}_{\lambda - m_{n-i+1}\varpi_i}$. Write $T \xrightarrow{i} T'$. By (5.3.2) and the first remark of the proof, we have

$$\eta(T) = \eta(T')\operatorname{Gr} z_i^{m_{n-i+1}}. \tag{5.3.3}$$

Now, let $T = T_n \xrightarrow{n} T_{n-1} \cdots \xrightarrow{1} T_1 = \tilde{T}$. By hypothesis on T, $\tilde{T}$ is a semistandard tableau of shape $\tilde{\lambda} := \lambda - \sum_i m_{n-i+1}\varpi_i$, and $m_{i,k}(\tilde{T}) \equiv 0$ $[l]$. On the one hand, this implies, by [5.4, Claim 2], that $\tilde{T}$ is a concatenation of l-columns, where an l-column is the lth exponent (for concatenation) of a single column. On the other hand, the fundamental weights are minuscule; hence, by the previous lemma, the image by η of a single column (in $\mathbb{T}_{\varpi_i}$) can be lifted into an extremal vector (in $K^+_{\varpi_i}$). By Lemma 5.2, $\eta(\tilde{T})$ can be lifted into $K_{\tilde{\lambda}} \cap \sum_{\mu \in lP^+} Z_0[z_\mu]$. By (5.3.3), $\eta(T) = \eta(\tilde{T})\operatorname{Gr} z_{\lambda - \tilde{\lambda}}$, and the assertion follows. ∎

5.4. This section is an appendix for Section 5.3.

Let T be in $\mathbb{T}_\lambda$, $a_i = \sum_{j=i}^n \lambda_i$, and $t_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq a_i$, be the entry in the ith row, jth column of T. Fix i_0, $1 \leq i_0 \leq n$. Let m be an integer such that $1 \leq m \leq a_{i_0}$ and $k\colon [1, i_0] \to \mathbb{N}$, such that

(a) $1 \leq k(i) \leq k(i) + m - 1 \leq a_i$.

(b) k is decreasing.

(c) $t_{i_0+1, k(i_0)} = \emptyset$. Let T' be the tableau obtained from T by canceling $t_{i,j}$ $1 \leq i \leq i_0$, $k(j) \leq j \leq k(j) + m - 1$; the boxes remaining on the right-hand side are moved to the left. It is clear, for $1 \leq i \leq i_0$, $t'_{i,j} = t_{i,j}$ if $j < k(i)$ and $t'_{i,j} = t_{i,j+m-1}$, otherwise. Then,

Claim 1. T' is a semistandard Young tableau of shape $\lambda - m\varpi_i$.

Proof. Clearly, T' verifies (R), and by (b), (c) it verifies (C). The number of entries in the ith row of T' is $\lambda_i - m$ if $1 \leq i \leq i_0$ and λ_i, otherwise. This gives λ of shape $\lambda - m\varpi_i$. ∎

Example. In this example, $\mathfrak{g}$ is of type A_5, $\lambda = 2\varpi_1 + \varpi_2 + 2\varpi_3 + 3\varpi_4$. In the following tableau T, we have underlined the entries to cancel:

1	1	2	3	$\underline{3}$	$\underline{4}$	4	5
2	3	3	$\underline{4}$	$\underline{5}$	5		
3	4	5	$\underline{5}$	$\underline{6}$			
5	5	6					

The cancellation corresponds to $m = 2$, $i_0 = 3$, $k(1) = 5$, $k(2) = k(3) = 4$. We then obtain T':

1	1	2	3	4	5
2	3	3	5		
3	4	5			
5	5	6			

Claim 2. With the notation of Theorem 5.3, suppose $T \in \mathbb{T}_\lambda$, $\lambda \in lP^+$, such that $n_{i,k}(T) \in l\mathbb{N}$, $1 \le i \le n$, $1 \le k \le n - i + 1$. Then, T is a concatenation of l-columns, where an l-column is the lth exponent (for concatenation) of a single column.

Proof. By construction, the entries t of the ith row of a semistandard tableau verify $i \le t \le n + 1$. By hypothesis, $n_{i,k}(T) \in l\mathbb{N}$; hence each entry t_i, $i + 1 \le t_i \le n + 1$, in the ith row occurs an l-multiple of times. As $\lambda \in lP^+$, the number of boxes in the ith row is a multiple of l. So, each entry in the ith row occurs an l-multiple of times. By (R), the first l columns are the lth exponent of a single column. The result is obtained by reverse induction. ∎

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REFERENCES

1. J. Alev and F. Dumas, Sur le corps des fractions de certaines algèbres quantiques, *J. Algebra* **170** (1994), 229–265.
2. N. Bourbaki, "Groupes et Algèbres de Lie," Chap. VI, Masson, Paris, 1981.
3. N. Bourbaki, "Groupes et Algèbres de Lie," Chap. VIII, Masson, Paris, 1981.
4. N. Burroughs, Relating the approaches to quantized algebras and quantum groups, *Comm. Math. Phys.* **133** (1990), 91–117.
5. P. Caldero, Générateurs du centre de $U_q(sl(N+1))$, *Bull. Sci. Math.* **118** (1994), 177–208.
6. P. Caldero, "Algèbres enveloppantes quantifiées, action adjointe et représentations," Thesis, Université Paris VI, 1993.
7. P. Caldero, Sur le centre de $U_q(\mathfrak{n})$, *Beit. Algebra Geom.* **35** (1994), 13–23.
8. P. Caldero, Etude des q-commutations dans l'algèbre $U_q(\mathfrak{n})$, *J. Algebra* **178** (1995), 444–457.
9. P. Caldero, Invariants in the enveloping algebra of a semi-simple Lie algebra for the adjoint action of a nilpotent Lie subalgebra, *Comm. Math. Phys.*, to appear.

10. P. Caldero, On the Gelfand–Kirillov conjecture for quantum algebras, preprint.
11. V. Chari and A. Pressley, "A Guide to Quantum Groups," Cambridge Univ. Press, Cambridge, 1995.
12. C. de Concini and V. G. Kac, Representations of quantum groups at roots of 1, Colloque Dixmier, *Prog. Math.* **92** (1990), 471–506.
13. C. de Concini, V. G. Kac, and C. Procesi, Quantum coadjoint action, *J. Amer. Math. Soc.* **5** (1992), 151–189.
14. V. G. Drinfeld, On almost cocommutative Hopf algebras, *Leningrad Math. J.* **1**(2) (1990), 321–342.
15. A. Joseph, A generalization of the Gelfand–Kirillov conjecture, *Amer. J. Math.* **99** (1977), 1151–1165.
16. A. Joseph, "Quantum Groups and Their Primitive Ideals," Vol. 29, Springer-Verlag, Berlin/New York, 1995.
17. A. Joseph, Sur une conjecture de Feigin, *C. R. Acad. Sci. Paris Sér. I Math.* **320** (1995).
18. S. Z. Levendorskii and Y. S. Soibelman, Some applications of quantum Weyl group, *J. Geom. Phys.* **7** (1990), 241–254.
19. G. Lusztig, Quantum groups at roots of 1, *Geom. Dedicata* **35** (1990), 1–25.
20. C. M. Ringel, Hall algebras and quantum groups, *Invent. Math.* **101** (1990), 583–592.
21. T. Tanisaki, Killing forms, Harish–Chandra isomorphisms, and universal R-matrices for quantum algebras, *Internat. J. Modern Phys. A* **7** (Suppl. 1B) (1992), 941–961.