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ScienceDirect
Journal of Algebra 319 (2008) 4391–4403
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

On the cohomology of Brill–Noether loci over Quot schemes

Cristina Martínez [a,b,*]

[a] *Institut for Matematike Fag (CTQM), Aarhus Universitet, Ny Munkegade, 8000 Aarhus, Denmark*
[b] *Max Planck Institute for Mathematics, Vivatgasse 7, Bonn 53111, Germany*

Received 4 July 2007

Available online 4 March 2008

Communicated by Laurent Moret-Bailly

Abstract

Let C be a smooth projective irreducible curve over an algebraic closed field k of characteristic 0. We consider Brill–Noether loci over the moduli space of morphisms from C to a Grassmannian $G(m,n)$ of m-planes in k^n and the corresponding Quot schemes of quotients of a trivial vector bundle on C compactifying the spaces of morphisms. We study in detail the case in which $m=2$, $n=4$. We prove results on the irreducibility and dimension of these Brill–Noether loci and we address explicit formulas for their cohomology classes. We study the existence problem of these spaces which is closely related with the problem of classification of vector bundles over curves.

Keywords: Quot schemes; Brill–Noether theory; Vector bundles

1. Introduction

The theory of Brill–Noether over the space of stable vector bundles or semistable bundles has been very much studied (see for example [BGMN,BGN,Mer,Te]). Let C be a non-singular projective curve defined over an algebraically closed field k of characteristic 0, and let $M(r,d)$ denote the moduli space of stable vector bundles over C of rank r and degree d. When r and d are not coprime, $M(r,d)$ is not compact and can be compactified to a scheme $\widetilde{M}(r,d)$ by adding

* Address for correspondence: Institut for Matematike Fag (CTQM), Aarhus Universitet, Ny Munkegade, 8000 Aarhus, Denmark.

E-mail addresses: cmartine@imf.au.dk, cmartine@mpim-bonn.mpg.de.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.01.041

equivalence classes of semistable vector bundles of rank r and degree d. For E a rank r and degree d vector bundle, the slope of E is defined as $\mu(E)=\frac{d}{r}$. The notion of stability, semistability and S-equivalence was first introduced by Mumford, Seshadri and Narasimhan [Mum,NS,Ses]. A vector bundle is stable (respectively semistable) when for every proper subbundle F:

$$\mu(F)=\frac{\deg(F)}{\operatorname{rank}(F)}<\mu(E), \quad \text{respectively} \leqslant .$$

Any semistable vector bundle E has an S-filtration, that is, a filtration by subbundles:

$$0=E_0\subset E_1\subset\cdots\subset E_k=E,$$

whose factors (E_i/E_{i-1}) are all stable with $\mu(E_i/E_{i-1})=\mu(E)$. The isomorphism class of the direct sum $\operatorname{gr} E:=\bigoplus_{i=1}^k E_i/E_{i-1}$ is independent of the filtration, and two semistable bundles are called S-equivalent if

$$\operatorname{gr}(E)\cong\operatorname{gr}(F).$$

The Brill–Noether loci over the moduli space of stable bundles are defined as:

$$B_{r,d,k}=\bigl\{E\in M(r,d)\bigm| h^0(E)\geqslant k\bigr\}$$

for a fixed integer k, and over the moduli space of semistable vector bundles it is defined by:

$$\widetilde{B}_{r,d,k}=\bigl\{[E]\in M(r,d)\bigm| h^0\bigl(\operatorname{gr}(E)\bigr)\geqslant k\bigr\}.$$

By the Semicontinuity theorem, these Brill–Noether loci are closed subschemes of the appropriate moduli spaces, and in particular it is not difficult to describe them as determinantal loci which allows us to estimate their dimension.

The main object of Brill–Noether theory is the study of these subschemes, in particular questions related to their non-emptiness, connectedness, irreducibility, dimension, topological and geometric structure. In the case of line bundles in which the moduli spaces are all isomorphic to the Jacobian, these questions have been completely answered when the underlying curve is generic (see for instance [ACGH]). Montserrat Teixidor i Bigas proved in 1991 a theorem that gives a rather general solution to the problem, but it holds only for a generic curve (see [Te]). Brambila-Paz, Grzegorczyk and Newstead in 1995 gave a solution to the problem for every curve when $\frac{d}{r}\leqslant 1$ (cf. [BGN]).

We can define in an analogous way Brill–Noether loci over the space of morphisms $\operatorname{Mor}_d(C,X)$ of fixed degree d from a curve to a projective variety, and the corresponding Quot schemes compactifying these spaces of morphisms. In particular, we are going to consider a Brill–Noether stratification over the space of morphisms $R^0_{C,d}$ from a genus g, smooth projective irreducible curve C to the Grassmannian $G(m,n)$ of m-dimensional subspaces of k^n. Moreover, we study in detail the case in which $m=2$, $n=4$. In this case, this theory is connected with the geometry of ruled surfaces in $\mathbb{P}^3$. Since $G(2,4)$ is endowed with the universal quotient bundle $\mathcal{Q}$, a natural compactification for this space is provided by the Quot scheme $R_{C,d}$, parametrizing epimorphisms of vector bundles $\mathcal{O}_C^4\to E\to 0$. Quot schemes have been shown by Grothendieck, to be fine moduli spaces for the problem of parametrizing quotients of a fixed sheaf, and as such, to carry universal structures.

Given a morphism f in $R^0_{C,d}$, the pull-back $E := f^*\mathcal{Q}^\vee$ is a vector bundle of rank 2 over C. We can consider its Segre invariant $s(E)$ which is defined as the minimal degree of a twist $E^\vee \otimes L$ with a line bundle such that the resulting bundle has a non-zero section. The Brill–Noether stratum is then defined by the following locally closed condition:

$$R^0_{C,d,s} := \{ f \in R^0_{C,d} \mid \mathbf{s}(f^*\mathcal{Q}) = s \}.$$

We consider the closure of the stratum $R_{C,d,s}$ in the Quot scheme compactification. In the first three sections of the paper some basic results on the loci $R_{C,d,s}$ are presented. In particular $R_{C,d,s}$ is exhibited as the degeneration locus of a natural and appropriate morphism of vector bundles.

In section four the fundamental class of $R_{C,d,s}$ is computed in the cohomology ring of $R_{C,d}$ under the assumption that $R_{C,d,s}$ is either empty or of the expected codimension in $R_{C,d}$ (for large d depending on s), (Theorem 4.2). We give a partial solution to the existence problem in the rank two case, that is, when we are considering the Grassmannian $G(2,4)$. We see that the problem is closely related with the problem of classification of vector bundles over curves.

2. Brill–Noether loci

Consider the universal exact sequence over the Grassmannian $G(m,n)$:

$$0 \to \mathcal{N} \to \mathcal{O}^n_G \to \mathcal{Q} \to 0. \tag{1}$$

For every morphism $f \in M_d := \mathrm{Mor}_d(C, G(m,n))$, we take the pull-back of the sequence 1:

$$0 \to f^*\mathcal{N} \to f^*\mathcal{O}^n_{G(m,n)} \to f^*\mathcal{Q} \to 0. \tag{2}$$

The next lemma describes the bundles E which arise from this construction.

Lemma 2.1. *Given E, a degree d, rank r bundle over C, there exists a morphism $f \in \mathrm{Mor}_d(C, G(n-r,n))$ such that $f^*\mathcal{Q} = E$ if and only if E is generated by n global sections or equivalently is given by a quotient,*

$$\mathcal{O}^n_C \to E \to 0.$$

We consider the Grassmannian $G(m,n)$ where $m = n-r$. By the universal property of the Grassmannian, there exists a morphism $f \in M_d$ such that $f^*\mathcal{Q} \cong E$, where $\mathcal{Q}$ is the universal quotient bundle over the Grassmannian $G(m,n)$.

Conversely, for all $f \in M_d$, $f^*\mathcal{Q}$ is generated by global sections, since $\mathcal{Q}$ is given by a quotient:

$$\mathcal{O}^n_{G(m,n)} \to \mathcal{Q} \to 0.$$

These quotients are parametrized by Grothendieck's Quot schemes $Q_{d,r,n}(C)$ of degree d, rank r quotients of $\mathcal{O}^n_C$ compactifying the spaces of morphisms M_d.

In the genus 0 case, by a theorem of Grothendieck, every vector bundle E over $\mathbb{P}^1$ decomposes as a direct sum of line bundles and therefore for $E \cong \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i)$ to be generated by global sections means that $a_i \geqslant 0$. In the genus 1 case, the bundles generated by global sections are the indecomposable bundles of degree $d > r$ (since $h^0(E) = d > r$ for an indecomposable bundle of positive degree [At]), the trivial bundle $\mathcal{O}_C$ and the direct sums of such bundles. In genus greater than or equal to 2, certain restrictions on the bundle imply that it is generated by global sections. For example, we can tensorize with a line bundle of degree m such that $E(m) = E \otimes \mathcal{O}_C(m)$ is generated by global sections and $h^1(E(m)) = 0$ [Ses]. Moreover, if d is sufficiently large and E is semistable, then $h^1(E) = 0$ and E is generated by its global sections; in fact, it is sufficient to take $d > r(2g - 1)$.

We *define the Brill–Noether loci over the spaces of morphisms M_d* as:

$$M_{d,a} = \{ f \in M_d \mid h^0(C, f^*\mathcal{Q}) \geqslant a \} \tag{3}$$

for a fixed integer a. More generally, we can tensorize the bundle with a fixed line bundle L over C and consider the following Brill–Noether loci:

$$M_{d,a}(L) = \{ f \in M_d \mid h^0(C, f^*\mathcal{Q} \otimes L) \geqslant a \}. \tag{4}$$

3. A Brill–Noether stratification over the Quot scheme

In [Mar2], we consider the space of morphisms $R_d^0 := \mathrm{Mor}_d(\mathbb{P}^1, G(2,4))$ and two different compactifications of this space, the Quot scheme compactification and the compactification of stable maps given by Kontsevich. We consider the following Brill–Noether loci inside the space of morphisms R_d^0:

$$R_{d,a}^0 = \{ f \in R_d^0 \mid h^0(f^*\mathcal{Q}^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(a)) \geqslant 1,\ h^0(f^*\mathcal{Q}^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(a-1)) = 0 \}, \tag{5}$$

for a fixed integer a.

Note that we are considering here rank two bundles, but the definition can be generalized easily to bundles of arbitrary rank r.

It is easy to see that this set can be defined alternatively as the $f \in R_d^0$ with $f^*\mathcal{Q} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(d-a)$, for $a \leqslant \frac{d}{2}$, and the parameter a gives a stratification of the space R_d^0.

Geometric interpretation

The image of a curve C by f is a geometric curve in the corresponding Grassmannian or equivalently a rational ruled surface in $\mathbb{P}^3$ for the Grassmannian of lines. Fixing the parameter a we are fixing the degree of a minimal directrix in the ruled surface. The spaces $R_{d,a}^0$ are locally closed again by the Semicontinuity Theorem and they can be shown as the degeneration locus of a morphism of bundles by means of the universal exact sequence over the corresponding Quot scheme R_d and we find that the expected dimension of $R_{d,a}^0$ as a determinantal variety is $3d + 2a + 5$. These spaces are considered in [Mar1] as parameter spaces for rational ruled surfaces in order to solve the following enumerative problems:

(1) The problem of enumerating rational ruled surfaces through $4d + 1$ points, or equivalently computing the degree of R_d^0 inside the projective space of surfaces of fixed degree d,

$$R_d^0 \to \mathbb{P}^{\binom{d+3}{3}-1}.$$

(2) Enumerating rational ruled surfaces with fixed splitting type. This problem raises the question of defining Gromov–Witten invariants for bundles with a fixed splitting type.

From now, the underlying curve C will be a smooth, irreducible projective curve of genus greater than or equal to 1, and we will be studying only the case $n = 2, m = 4$. We call $R^0_{C,d}$ the spaces of morphisms $\mathrm{Mor}_d(C, G(2,4))$. For a vector bundle E of rank 2 over C, its Segre invariant is the integer s such that the minimal degree of a line quotient $E \to L \to 0$ is $\frac{d+s}{2}$, or the maximal degree of a twist $E^\vee \otimes L$, having a non-zero section. Note that $s(E) \equiv \deg(E) \bmod 2$ and that E is stable (respectively semistable) if and only if $s(E) \geqslant 1$ (respectively $s(E) \geqslant 0$) [LN,CS].

If T is any algebraic variety over k and $\mathcal{A}$ is a vector bundle of rank r on $C \times T$, then the function $s : T \to \mathbb{Z}$ defined by $s(t) = s(\mathcal{A}|_{C\times t})$ is lower semicontinuous.

Given $f \in R^0_{C,d}$ we consider the Segre invariant s of the bundle $f^*\mathcal{Q}$ which is the maximal degree of a twist $f^*\mathcal{Q}^\vee \otimes L$ with a generic line bundle L of degree $\frac{d+s}{2}$ such that the resulting bundle has a non-zero section.

We *define the corresponding Brill–Noether loci over* $R^0_{C,d}$ as the subsets:

$$R^0_{C,d,s} = \left\{ f \in R^0_{C,d} \;\middle|\; h^0\big(f^*\mathcal{Q}^\vee \otimes L\big) \geqslant 1,\ L \text{ of minimal degree } \frac{d+s}{2} \right\}. \tag{6}$$

The subvarieties $R^0_{C,d,s}$ are locally closed and are a natural generalization for arbitrary genus of the subvarieties $R^0_{d,a}$, defined previously, taking $a = \frac{d+s}{2}$ which is the degree of the line bundle L.

The Zariski closure of $R^0_{C,d,s}$ is given by the set [Mar2]:

$$\overline{R}^0_{C,d,s} = \left\{ f \in R^0_{C,d} \;\middle|\; h^0\big(f^*\mathcal{Q}^\vee \otimes L\big) \geqslant 1,\ \deg(L) = \frac{d+s}{2} \right\}.$$

Observe that L is allowed to vary on the variety of line bundles of fixed degree, which is isomorphic to the Jacobian, which is compact. The closure describes exactly the set of morphisms $f \in R^0_{C,d}$ for which $f^*\mathcal{Q}$ is a rank two vector bundle with Segre invariant less or equal than s, that is,

$$\overline{R}^0_{C,d,s} = R^0_{C,d,s} \cup R^0_{C,d,s-1} \cup \cdots.$$

Evenmore, $R^0_{C,d,s-1} \subset R^0_{C,d,s}$, and therefore $\overline{R}^0_{C,d,s-1} \subset \overline{R}^0_{C,d,s}$.

Let us consider the universal exact sequence on $R_{C,d} \times C$,

$$0 \to \mathcal{K} \to \mathcal{O}^n_{R_{C,d}\times C} \to \mathcal{E} \to 0, \tag{7}$$

which satisfies that for all k-scheme S, the set of morphisms $f : S \to R_{C,d}$ is in correspondence one to one with the set of isomorphism classes of short exact sequences, $\mathcal{O}^4_{S\times C} \to \mathcal{E}_{S\times C} \to 0$, where $\mathcal{E}_{S\times C}$ is flat over S.

Since the universal quotient sheaf $\mathcal{E}$ is flat over $R_{C,d}$, for each $q \in R_{C,d}$, $E_q := \mathcal{E}\,|_{\{q\}\times C}$ is a coherent sheaf over C. By flatness, $h^0(E_q) - h^1(E_q)$ is constant on any connected component of $R_{C,d}$. The Riemann–Roch theorem allows us to compute its value:

$$h^0(E_q) - h^1(E_q) = d + 2(1-g),$$

for every $q \in R_{C,d}$.

We consider the Zariski closure $R_{C,d,s}$ of the sets $R^0_{C,d,s}$ inside the Quot scheme compactification of the space of morphisms:

$$\begin{aligned} R_{C,d,s} &= \left\{ q \in R_{C,d} \;\middle|\; h^0\big(C, E_q^\vee \otimes L\big) \geqslant 1,\ \deg L = \frac{d+s}{2} \right\} \\ &= \left\{ q \in R_{C,d} \;\middle|\; h^1\big(C, E_q \otimes K_C \otimes L^{-1}\big) \geqslant 1,\ \deg L = \frac{d+s}{2} \right\} \\ &= \left\{ q \in R_{C,d} \;\middle|\; h^0\big(C, E_q \otimes K_C \otimes L^{-1}\big) \geqslant d+3-2g,\ \deg L = \frac{d+s}{2} \right\}. \end{aligned}$$

The first equality is due to the duality theorem, and the second one is due to the Riemann–Roch theorem.

The existence problem

The existence problem here means that, given s, there exist a vector bundle E of rank 2 with Segre invariant s and a morphism $f \in R^0_{C,d}$ such that $f^*\mathcal{Q} := E$. Therefore the existence problem for $R^0_{C,d,s}$ is closely related with the existence problem of vector bundles of rank 2 with Segre invariant s and consequently with the problem of classification of vector bundles over curves. The contents of the next proposition are well known in the context of vector bundles (see for example [La] for the rank 2 case and [RT] for the general case).

Proposition 3.1.

(1) *If $s > g$ then $R^0_{C,d,s}$ is empty.*
(2) *If C is a smooth elliptic curve, $d \equiv s \bmod 2$ and $d \geqslant 3$ then $R^0_{C,d,1}$ and $R^0_{C,d,0}$ are non-empty.*
(3) *If C is a smooth curve of genus $g \geqslant s \geqslant 0$, $d \equiv s \bmod 2$ and $d > 2(2g-1)$, then $R^0_{C,d,s}$ is non-empty.*

Proof. Given $f \in R^0_{C,d}$, the pull-back $f^*\mathcal{Q}^\vee$ of the dual of the universal quotient bundle over $G(2,n)$ is a rank 2 bundle over C of degree $-d$. We can tensorize it with a line bundle of degree $\frac{d+s}{2}$ such that $E' := f^*\mathcal{Q}^\vee \otimes L$ is a normalized bundle of degree s, that is, $h^0(E') \neq 0$ but for all invertible sheaves $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$, we have $H^0(E' \otimes L) = 0$. Since E' is a normalized bundle there are two possibilities for E':

- If E' is decomposable as a direct sum of two invertible sheaves, then $E' \cong \mathcal{O}_C \oplus F$ for some F with $\deg F \leqslant 0$, therefore $s(E') \leqslant 0$ and all values of $s(E) \leqslant 0$ are possible.
- If E' is indecomposable, then $2-2g \leqslant s \leqslant g$ (see [Har]).

In particular, this implies that there does not exist a vector bundle of rank 2 over C with Segre invariant $s > g$, and consequently $R^0_{C,d,s}$ is empty as we stated in (1).

If C is an elliptic curve, for each value $0 \leqslant s \leqslant 1$ there is a bundle E over C of rank 2 with Segre invariant s. Therefore by Lemma 2.1, the sets $R^0_{C,d,1}$ and $R^0_{C,d,0}$ are non-empty. This proves (2).

If C is a smooth curve of genus $g \geqslant 2$, $0 \leqslant s \leqslant g$ and $d > 2(2g-1)$, there is a semistable vector bundle of rank 2 over C with Segre invariant s and again by Lemma 2.1, $R^0_{C,d,s}$ is non-empty. $\square$

The next theorem will exhibit the $R_{C,d,s}$ as determinantal varieties which allows us to estimate their dimensions.

Let K_C be the canonical bundle over C and π_1, π_2 be the projection maps of $R_{C,d} \times C$ over the first and second factors respectively. Tensorizing the sequence (7) with the linear sheaf $\pi_2^*(K_C \otimes L^{-1})$ gives the exact sequence:

$$0 \to \mathcal{K} \otimes \pi_2^*\big(K_C \otimes L^{-1}\big) \to \mathcal{O}^n_{R_{C,d}\times C} \otimes \pi_2^*\big(K_C \otimes L^{-1}\big) \to \mathcal{E} \otimes \pi_2^*\big(K_C \otimes L^{-1}\big) \to 0. \quad (8)$$

Here L is a generic line bundle on C of fixed degree $\frac{d+s}{2}$. The π_{1*} direct image of the above sequence yields the following long exact sequence on $R_{C,d}$:

$$\begin{aligned} 0 &\to \pi_{1*}\big(\mathcal{K} \otimes \pi_2^*\big(K_C \otimes L^{-1}\big)\big) \to \pi_{1*}\big(\mathcal{O}^n_{R_{C,d}} \otimes \pi_2^*\big(K_C \otimes L^{-1}\big)\big) \\ &\to \pi_{1*}\big(\mathcal{E} \otimes \pi_2^*\big(K_C \otimes L^{-1}\big)\big) \to R^1\pi_{1*}\big(\mathcal{K} \otimes \pi_2^*\big(K_C \otimes L^{-1}\big)\big) \\ &\to R^1\pi_{1*}\big(\mathcal{O}^n_{R_{C,d}} \otimes \pi_2^*\big(K_C \otimes L^{-1}\big)\big) \to R^1\pi_{1*}\big(\mathcal{E} \otimes \pi_2^*\big(K_C \otimes L^{-1}\big)\big) \to 0. \end{aligned}$$

Theorem 3.2. *For d sufficiently large depending on s, $R_{C,d,s}(L)$ is the locus where the map*

$$R^1\pi_{1*}\big(\mathcal{K} \otimes \pi_2^*\big(L^{-1} \otimes K_C\big)\big) \to R^1\pi_{1*}\big(\mathcal{O}^n_{R_{C,d}\times C} \otimes \pi_2^*\big(L^{-1} \otimes K_C\big)\big) \quad (9)$$

is not surjective. It is irreducible and has expected codimension $2g - s - 1$ as a determinantal variety.

Proof. The map (9) is not surjective in the support of the sheaf

$$R^1\pi_{1*}\big(\mathcal{E} \otimes \pi_2^*\big(K_C \otimes L^{-1}\big)\big),$$

that is, in the points $q \in R_{C,d}$ such that $h^1(\mathcal{E} \otimes \pi_2^*(K_C \otimes L^{-1})|_{\{q\}\times C}) \geqslant 1$, or equivalently in $R_{C,d,s}(L)$ by Serre duality. In other words, by semicontinuity there is an open set $P_s \subset \mathbb{P}(R^1\pi_{1*}(\mathcal{E} \otimes \pi_2^*(K_C \otimes L^{-1})))$ parametrizing classes of quotients $\mathcal{O}^4_C \to E_s \to 0$ such that E_s is a rank 2 bundle with Segre invariant s, modulo the canonical operation of k^*.

Let $P_s \xrightarrow{f} R_{C,d,s}$ be the surjective morphism such that the image by f of each quotient is its class of isomorphism in $R_{C,d,s}(L)$. The fibers are isomorphic to $\mathbb{P}(\mathrm{Aut}(E_s))$.This proves that the subschemes $R_{C,d,s}(L)$, being the image of an irreducible variety by a morphism, are irreducible.

By Serre duality it follows that

$$\big(R^1\pi_{1*}\big(\mathcal{K} \otimes \pi_2^*\big(L^{-1} \otimes K_C\big)\big)\big)^\vee \cong \pi_{1*}\big(\mathcal{K}^\vee \otimes \pi_2^*L\big)$$

and by the Base Change Theorem, their fibers are isomorphic to

$$H^0\big(C, \mathcal{K}^\vee \otimes \pi_2^*L|_{C\times\{p\}}\big), \quad p \in R_{C,d}$$

and have dimension $2d+s+m(1-g)$, where $m = \mathrm{rank}(\mathcal{K}^\vee)$. It is enough to take $d+s > 2m(g-1)$ to ensure the vanishing of $h^1(C, \mathcal{K}^\vee \otimes \pi_2^*L|_{C\times\{p\}})$. As a consequence, $\pi_{1*}(\mathcal{K}^\vee \otimes \pi_2^*L)$ is a bundle of rank $2d + s + 2(1-g)$ (note that we are specializing m and n to be 2, 4 respectively).

Again by Serre duality we see that $R^1\pi_{1*}(\mathcal{O}^n_{R_{C,d}\times C}\otimes\pi_2^*(L^{-1}\otimes K_C))\cong\pi_{1*}(\mathcal{O}^n_{R_{C,d}}\otimes\pi_2^*L)$ and it is a bundle with fiber isomorphic to

$$H^0\big(C,\mathcal{O}^n_{R_{C,d}}\otimes\pi_2^*L|_{C\times\{p\}}\big)$$

of dimension $2d+2s-4g+4$. Therefore we have the following morphism of bundles:

$$\pi_{1*}\big(\mathcal{K}^\vee\otimes\pi_2^*L\big)\xrightarrow{\phi}\mathcal{O}^{2d+2s-4g+4}_{R_{C,d}}.$$

The expected codimension of $R_{C,d,s}$ as determinantal variety is

$$\big((2d-2g+s+2)-(2d+2s-4g+3)\big)\cdot\big((2d+2s-4g+4)-(2d+2s-4g+3)\big)$$
$$=2g-s-1. \qquad \square$$

Remark 3.3. Note that as s increases, the bundle E becomes more general, so the codimension decreases. In particular when $s=g-1$, the codimension is g for fixed L or 0 when we allow L to vary.

4. Cohomology of the varieties $R_{C,d,s}$

In this section we determine the cohomology classes of the Brill–Noether loci in terms of some natural elements in the cohomology ring of $R_{C,d}$ defined by the universal bundle.

Let $\{1,\delta_k,1\leqslant k\leqslant 2g,\eta\}$ be a basis for the cohomology of C, where η represents the class of a point. We will also denote by $\{1,\delta_k,1\leqslant k\leqslant 2g,\eta\}$ the pull-backs to $R_{C,d}$ by the projection morphism.

Let us consider the classes t_i,u_{i-1},s_i^j in $H^*(R_{C,d};\mathbb{Q})$ defined by the Künneth decomposition of the Chern classes of $\mathcal{K}^\vee$:

$$c_i\big(\mathcal{K}^\vee\big)=t_i+\sum_{j=1}^{2g}s_i^j\delta_j+u_{i-1}\eta,\quad t_i\in H^{2i}(R_{C,d};\mathbb{Q}),\ s_i^j\in H^{2i-1}(R_{C,d};\mathbb{Q}),$$
$$u_{i-1}\in H^{2i-2}(R_{C,d};\mathbb{Q}).$$

Every class z in the cohomology ring $H^*(R_{C,d};\mathbb{Q})$ can be written in the form

$$z=c+\sum_{j=1}^{2g}b^j\delta_j+f\,\eta$$

where $c=\pi_*(\eta\,z)$ and $f=\pi_*(z)\in H^*(R_{C,d};\mathbb{Q})$. In particular, $t_i=\pi_*(\eta\,c_i(\mathcal{K}^\vee))$ and $u_{i-1}=\pi_*(c_i(\mathcal{K}^\vee))$, $u_0=\pi_*(c_1(\mathcal{K}^\vee))=d$.

Conjecture 4.1. *The elements t_1, t_2, u_1, s_i^j $(1\leqslant j\leqslant 2g,\ i=1,2)$ generate $H^*(R_{C,d};\mathbb{Q})$.*

The evidence for the conjecture is that in the genus 0 case it is true [Str] and s_i^j ($1 \leqslant j \leqslant 2g$, $i = 1, 2$) are the generators that appear when we consider curves in genus higher than 0. In addition, for the ordinary moduli space $M(2, d)$ with d odd, the s_i^j are all needed, together with t_2 and one of t_1, u_1 as well (see [Za]). For d even, the conjecture is less plausible, and it is unlikely to be true for integer cohomology.

For L a line bundle over C of degree $a = \frac{d+s}{2}$, its first Chern class is given by

$$c_1(\pi_2^* L) = a\eta.$$

Theorem 4.2. *If C is any smooth curve of genus g, and $R_{C,d,s}$ is either empty or generically reduced and of the expected codimension $2g - s - 1$, $R_{C,d,s}$ has fundamental class*:

$$[R_{C,d,s}] = -c_{2g-s-1}\big(\pi_{1*}\big(\mathcal{K}^\vee \otimes \pi_2^* L\big)\big).$$

Proof. By assumption, $R_{C,d,s}$ has the expected codimension as a determinantal variety and is irreducible as we have seen in Theorem 3.2; therefore, it does not have components contained at infinity and the Porteous formula gives the fundamental class of the varieties $R_{C,d,s}$ in terms of the Chern classes of the bundles given by Theorem 3.2. We get that

$$[R_{C,d,s}] = \Delta_{2g-s-1,1}\big(c_t\big(-\pi_{1*}\big(\mathcal{K}^\vee \otimes \pi_2^* L\big)\big)\big),$$

where

$$\Delta_{p,q}(a) = \det \begin{pmatrix} a_p & \dots & a_{p+q-1} \\ \vdots & & \vdots \\ a_{p-q+1} & \dots & a_p \end{pmatrix},$$

for any formal series $a(t) = \sum_{k=-\infty}^{k=+\infty} a_k t^k$. $\square$

Remark 4.3. Note that the assumption that $R_{C,d}$ is generically reduced and of expected codimension $2g - s - 1$ is satisfied for large d (relative to s and g). In that case, $R_{C,d}$ is a smooth projective bundle over the Jacobian J^d of degree d line bundles, and the intersection numbers on $\mathrm{Quot}_{d,r,n}(C)$ correspond to certain counts of maps from C to $G(r, n)$. The intersection of the t classes has been studied extensively in [Ber], where it is shown that the evaluation of a top-degree monomial in the t classes on the fundamental cycle has enumerative meaning. It is the number of degree d maps from C to the Grassmannian $G(r, n)$ which sends fixed points on C to special Schubert varieties of $G(r, n)$.

4.0.1. Computations of Chern classes

Applying Grothendieck–Riemann–Roch theorem to the projection morphism π_1, it follows that

$$\mathrm{ch}\big(\pi_{1*}\big(\mathcal{K}^\vee \otimes \pi_2^* L\big)\big) = \pi_{1*}\big(\mathrm{Td}(R_{C,d} \times C)/R_{C,d}\big) \cdot \mathrm{ch}\big(\mathcal{K}^\vee \otimes \pi_2^* L\big). \tag{10}$$

First we compute the Chern classes of $\mathcal{K}^\vee \otimes \pi_2^* L$:

$$c_1\big(\mathcal{K}^\vee \otimes \pi_2^* L\big) = t_1 + \left(\sum_{j=1}^{2g} s_1^j \delta_j\right) + (d+2a)\eta,$$

$$c_2\big(\mathcal{K}^\vee \otimes \pi_2^* L\big) = t_2 + \left(\sum_{j=1}^{2g} s_2^j \delta_j\right) + u_1\eta + a\eta t_1 + a\left(\sum_{j=1}^{2g} s_1^j \delta_j\right)\eta.$$

Let α_1 and α_2 be the classes $\sum_{j=1}^{2g} s_1^j \delta_j$ and $\sum_{j=1}^{2g} s_2^j \delta_j$ respectively. The intersection numbers for the δ_i imply the following relations:

$$\alpha_1^2 = -2A\eta, \quad A = \sum_{j=1}^{g} s_1^j s_1^{j+g} \in H^2(R_{C,d}; \mathbb{Q}), \qquad \alpha_1^3 = 0,$$

$$\alpha_2^2 = -2\gamma\eta, \quad \gamma = \sum_{i=1}^{g} s_2^j s_2^{j+g} \in H^6(R_{C,d}; \mathbb{Q}), \qquad \alpha_2^3 = 0,$$

$$\alpha_1\alpha_2 = B\eta, \quad B = \left(\sum_{i=1}^{g} -s_1^i s_2^{i+g} + s_1^{i+g} s_2^i\right) \in H^4(R_{C,d}; \mathbb{Q}).$$

Let E be a bundle of rank n and denote by $x_1, \dots, x_n$ its Chern roots. The Chern character $\mathrm{ch}(E)$ is defined by the formula:

$$\mathrm{ch}(E) = \sum_{i=1}^{n} e^{x_i},$$

where $e^x = \sum_{n \geqslant 0} \frac{x^n}{n!}$, and $x_1, \dots, x_n$ are the Chern roots of E. The first few terms are:

$$\mathrm{ch}(E) = r + c_1 + \frac{1}{2}\big(c_1^2 - 2c_2\big) + \frac{1}{3}\big(c_1^3 - 3c_1c_2 + 3c_3\big) + \cdots,$$

where $c_i = c_i(E)$. The n^{th} term is $\frac{p_n}{n!}$, where p_n is determined inductively by Newton's formula (see [Mac]):

$$p_n - c_1 p_{n-1} + c_2 p_{n-2} - \cdots + (-1)^{n-1} c_{n-1} p_1 + (-1)^n n c_n = 0.$$

Finally, the Todd class $\mathrm{td}(E)$ of a bundle with Chern polynomial $c_t(E) = \prod_i (1 + x_i t)$ is defined by

$$\mathrm{td}(E) = \prod_i \frac{x_i}{1 - e^{-x_i}}.$$

Let us denote by ch_i the i-homogeneous part of the Chern character of a bundle. Then, we have

$$\begin{aligned}
\operatorname{ch}_0\big(\mathcal{K}^\vee \otimes \pi_2^* L\big) &= m = 2, \\
\operatorname{ch}_1\big(\mathcal{K}^\vee \otimes \pi_2^* L\big) &= t_1 + \alpha_1 + \eta(d+2a), \\
\operatorname{ch}_2\big(\mathcal{K}^\vee \otimes \pi_2^* L\big) &= \frac{1}{2}\big[t_1^2 + \alpha_1^2 + 2\,t_1\,\alpha_1 + 2t_1\eta(d+2a) - 2t_2 - 2\alpha_2 - 2u_1\eta - 2a\eta t_1\big], \\
\operatorname{ch}_3\big(\mathcal{K}^\vee \otimes \pi_2^* L\big) &= \frac{1}{6}\big[t_1^3 + \alpha_1 t_1^2 + 3(d+2a)\eta t_1^2 + 2\alpha_1 t_1^2 + 3\alpha_1^2 t_1 + \alpha_1^3 \\
&\quad - 3t_1t_2 - 3t_1\alpha_2 - 3\eta u_1 t_1 - 3\alpha_1\alpha_2 - 3\eta(2a+d)t_2 + 3a\eta\, t_1^2 - 3\alpha_1 t_2\big].
\end{aligned}$$

We observe that $\operatorname{td}(R_{C,d}) = 1$ and $\operatorname{td}(R_{C,d} \times C) = 1 + (1-g)\eta$, putting this together with the computations of the Chern classes of $\mathcal{K}^\vee \otimes \pi_2^* L$ and the Grothendieck–Riemann–Roch formula, yields:

$$\operatorname{ch}\big(\pi_{1*}\big(\mathcal{K}^\vee \otimes \pi_2^* L\big)\big) = \pi_{1*}\big(1 + (1-g)\eta\big)\big(\operatorname{ch}\big(\mathcal{K}^\vee \otimes \pi_2^* L\big)\big).$$

The i-homogeneous term of $\operatorname{ch}(\pi_{1*}(\mathcal{K}^\vee \otimes \pi_2^* L))$ is given by the formula:

$$\operatorname{ch}_i\big(\pi_{1*}\big(\mathcal{K}^\vee \otimes \pi_2^* L\big)\big) = (1-g)\operatorname{ch}_i\big(\mathcal{K}^\vee \otimes \pi_2^* L|_{\{q\}\times R_{C,d}}\big) + \operatorname{coeff}_\eta\big(\operatorname{ch}_{i+1}\big(\mathcal{K}^\vee \otimes \pi_2^* L\big)\big).$$

The first few terms are:

$$\begin{aligned}
\operatorname{ch}_0 &= \operatorname{rank}\big(\pi_{1*}\big(\mathcal{K}^\vee \otimes \pi_2^* L\big)\big) = d + 2a + 2(1-g), \\
\operatorname{ch}_1 &= t_1(d+2a) + \alpha_1(d+2a) - at_1 - a\alpha_1 - u_1 + (1-g)\alpha_1, \\
\operatorname{ch}_2 &= (1-g)\left[\frac{1}{2}t_1^2 + \frac{1}{2}\alpha_1^2 + t_1\alpha_1 - t_2 - \alpha_2 + 3(d+2a)t_1^2 + 3(d+2a)\alpha_1^2\right] \\
&\quad + 6(d+2a)\alpha_1 t_1 - 3u_1t_1 - 3at_1^2 - 3\alpha_1^2 - 6a\alpha_1 t_1 - 3u_1\alpha_1 - 3(d+2a)t_2 \\
&\quad - 3(d+2a)\alpha_2 + 3u_2 + 3at_2 + 3\alpha_2, \\
\operatorname{ch}_3 &= (1-g)\big[t_1^3 + \alpha_1^3 + 3t_1\alpha_1^2 + 3t_1^2\alpha_1 - 3t_1t_2 - 3\alpha_1\alpha_2 - 3\alpha_1 t_2 + 3t_3 + 3\alpha_3\big], \\
&\ \ \vdots
\end{aligned}$$

Finally, we get that the Chern classes of $\pi_{1*}(\mathcal{K}^\vee \otimes \pi_2^* L)$ are given by the recursive formula:

$$c_n\big(\pi_{1*}\big(\mathcal{K}^\vee \otimes \pi_2^* L\big)\big) = -\sum_{r=1}^{n} \frac{(-1)^{r-1}}{n} r!\operatorname{ch}_r\big(\pi_{1*}\big(\mathcal{K}^\vee \otimes \pi_2^* L\big)\big)c_{n-r}\big(\pi_{1*}\big(\mathcal{K}^\vee \otimes \pi_2^* L\big)\big).$$

Let σ_i be the i-symmetric function, that is,

$$\sum_{r=0}^{n} \sigma_r t^r = \prod_{i=1}^{n}(1 + x_i t),$$

for each $r \geqslant 1$ the rth power sum is:

$$p_r = \sum x_i^r = m_{(r)}.$$

The generating function for the p_r is:

$$p(t) = \sum_{r \geqslant 1} p_r t^{r-1} = \sum_{i \geqslant 1} \sum_{r \geqslant 1} x_i^r t^{r-1} = \sum_{i \geqslant 1} \frac{d}{dt} \log \frac{1}{1 - x_i t}, \tag{11}$$

$$P(t) = \frac{d}{dt} \prod_{i \geqslant 1} (1 - x_i t)^{-1} = \frac{d}{dt} \log H(t) = \frac{H'(t)}{H(t)},$$

$$P(-t) = \frac{d}{dt} \log E(t) = \frac{E'(t)}{E(t)}. \tag{12}$$

From (11) and (12), we get that

$$n\sigma_n = \sum_{r=1}^{n} (-1)^{r-1} p_r \sigma_{n-r}.$$

This is a standard formula for symmetric functions (see [Mac]). The first Chern classes are:

$$c_1\big(\pi_{1*}\big(\mathcal{K}^\vee \otimes \pi_2^* L\big)\big) = (a + d + 1 - g)t_1 + \alpha_1 - u_1 + \eta(1 - g)(d + 2a),$$

$$\begin{aligned} c_2\big(\pi_{1*}\big(\mathcal{K}^\vee \otimes \pi_2^* L\big)\big) &= \frac{1}{2}c_1^2 - \frac{1}{2}(1 - g + d - a)t_1^2 - \frac{1}{2}(1 - g)\alpha_1^2 \\ &\quad - (1 - g)\big(\alpha_1 + t_1 + (a + d)\eta t_1 - t_2 - \alpha_2 - u_1\eta\big)u_1 t_1 - \frac{1}{2}(u_2 + t_2), \end{aligned}$$

$$\vdots$$

Corollary 4.4. *If C is a curve of genus* 1*, then*

$$[R_{C,d,0}] = -c_1\big(\pi_{1*}\big(\mathcal{K}^\vee \otimes \pi_2^* L\big)\big) = -(d + a)t_1 - \alpha_1 + u_1,$$

$$\begin{aligned} [R_{C,d,-1}] &= -c_2\big(\pi_{1*}\big(\mathcal{K}^\vee \otimes \pi_2^* L\big)\big) \\ &= -\frac{1}{2}\big((d + a)t_1 + \alpha_1 - u_1\big)^2 - \frac{1}{2}(d - a)t_1^2 + u_1 t_1 - \frac{1}{2}(u_2 + t_2). \end{aligned}$$

Acknowledgments

I would like to thank Prof. Peter Newstead for comments and suggestions on both content and presentation of this paper. I also want to thank Rajagopalan Parthasarathy for pointing out the reference [Mac]. Finally I would like to thank Prof. J.E. Andersen and Alex Bene for reading a last version of the paper and some corrections. This work has been partially supported by The European Contract Human Potential Programme, Research Training HPRN-CT-2000-00101 and MPIM grant.

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