Homogenization of the fluid flow and heat transfer in transpiration cooled multi-layer plates

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Abstract

To predict the aerothermal behavior of transpiration cooled plates, a multi-scale approach based on the homogenization method of periodic material structures is developed. This method allows one to calculate effective equivalent thermophysical properties either for each layer or for the multi-layer of superalloy, bondcoat and TBC. A general formulation is developed here for the fluid flow through a porous media which is able to deduce as equivalent macroscopic behavior either a Darcy law with a constant permeability or a more general Fochheimer law with a permeability function of the mean velocity. Effective Darcy permeabilities are calculated by solving special Stokes flow problems on a unit cell. Finally, effective conductivities and permeabilities are determined for different configurations of cylindrical and shaped transpiration cooling channels.

Keywords: Homogenization; Porous media; Permeabilities

1. Introduction

Intensive cooling of blades and combustion chambers in stationary modern gas turbines is required to guarantee the performance and an acceptable life time of the components in contact with the hot gas. Classical film cooling [8] can be improved by application of transpiration cooling technology to gas turbine components. In a first step, finest laser-drilled holes are applied to a coated multi-layer plate to obtain a homogeneous cooling film on the surfaces. In order to achieve the most homogeneous cooling gas protection film at the outlet face, several design configurations of the cooling holes have been investigated by application of a three-dimensional conjugate fluid and heat flow solver [2,10]. In order to predict correctly the interaction between hot and cooling gases, numerical models

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of $0.7–1.2$ million finite volumes for a $\frac{1}{2}$ row of seven shifted holes are made. The corresponding aerothermal analyses need several weeks on a supercomputer to reach the steady state. Thus, it is actually unrealistic to evaluate with the same mesh fineness the three-dimensional fluid flow and heat transfer in complex gas turbine blades cooled by more than 1000 holes.

In order to circumvent this difficulty, a multi-scale approach based on the homogenization method applied on periodic structures [4,5,11] has been adopted. This homogenization technique allows one to calculate effective equivalent thermophysical properties either for each layer separately or for the multi-layer. For this purpose a reference unit cell with periodic boundary conditions is extracted from the macro-model (see Fig. 1). Then the homogenization technique based on the asymptotic expansion of the temperature, velocity and pressure field with respect to $\varepsilon$, the ratio between the macro- and microscopic length scale, is developed on this unit cell.

Since the pioneering work of Sanchez-Palencia [11] in the early 1980s, several papers about homogenization technique applied on the heat and mass transport through porous media can be found in the literature [4,5]. But, to our knowledge, they do not numerically investigate the influence of the design of transpiration cooling channels on the effective permeability and thermal properties of the porous media. As the description of the homogenization method applied to advective heat transfer problems has been published in Refs. [9,10], this paper focuses on the implementation of the homogenization technique applied to the compressible fluid flow through a porous media. The aim of this formulation is to be so general that as macroscopic behavior, either a Darcy law with a constant permeability or a more general Fochheimer law with a permeability function of the mean velocity, can be derived. Effective permeabilities are then evaluated by solving special Stokes problems on the unit cell. To solve these problems, the well-known analogy between Stokes flow and incompressible elasticity is used [7]. Finally, effective thermal conductivities and permeabilities are evaluated and compared for different configurations of transpiration cooling channels in a multi-layer plate.

2. Homogenization of fluid flow through a porous material

2.1. Asymptotic expansion for flow through a porous material

The cooling channel (see Fig. 1) is filled with a compressible gas whose density is small but varies mainly with the temperature due to the low pressure drop which was obtained in the
three-dimensional conjugate heat and fluid flow analyses of the considered multi-layer plate. Thus, by adopting the Boussinesq approximation [4], following state equation is adopted for the cooling gas:

$$\rho^h = \rho_0(1 - \beta T^h)$$  \hspace{1cm} (1)

with $\beta$ being the thermal expansion coefficient of the fluid and $\rho_0$ the density at the reference temperature.

In a general multi-scale approach, the density and the viscosity depend on the aspect ratio $\varepsilon$ between the micro and macro scale as follows:

$$\mu^h = \varepsilon^m \mu_0 \quad \text{and} \quad \rho^h = \varepsilon \rho_0(1 - \beta T^h),$$  \hspace{1cm} (2)

where $m, r$ are real numbers to be determined.

Indeed, as mentioned by Ene and Sanchez-Palencia [5], the asymptotic process and the deduced macroscopic limit equations may be different if several small parameters are involved in the fluid flow problem. Liquid flow (large viscosity) in porous media is often very slow and inertia effects may be neglected. Then, the Navier–Stokes equations reduce to the Stokes one at the micro-scale. But, if the viscosity is small like for a cooling gas, the microscopic velocity may be large and causes nonnegligible nonlinear effects leading to a nonlinear form of Darcy’s law, like Forchheimer’s one, as shown recently by Giorgi [6] and Chen et al. [3].

Moreover, following the standard setup of the homogenization technique developed in Ref. [4], formal asymptotic expansions of the form

$$v^h(x) = v(x, y) = \varepsilon^n v^0(x, y) + \varepsilon^{n+1} v^1(x, y) + O(\varepsilon^{n+2}),$$  \hspace{1cm} (3)

$$p^h(x) = p(x, y) = p^0(x, y) + \varepsilon p^1(x, y) + O(\varepsilon^2),$$  \hspace{1cm} (4)

$$T^h(x) = T(x, y) = T^0(x, y) + \varepsilon T^1(x, y) + O(\varepsilon^2),$$  \hspace{1cm} (5)

where $x$ is the macroscopic variable, which varies slowly from unit cell to unit cell, and $y = x/\varepsilon$ the periodic microscopic variable, which varies quickly within each unit cell, and $n$ the real number to be determined, are specified for each physical field ($T^h$, $v^h$ and $p^h$) involved in the considered cooling problem through a porous material.

At first, neglecting the fluid flow dissipation in the energy conservation equation, the temperature field is the same as for the homogenization of the thermal conductivity [9,10] and is given by

$$T^h(x) = T^0(x) - \chi^k(y) \nabla_{\!x} T^0(x),$$  \hspace{1cm} (6)

where $\chi^k (k = 1, 2, 3)$ is a microscopic periodic displacement field. Introducing (6) in relation (2), the fluid state equation becomes

$$\rho^h = \varepsilon' \rho^*(x) - \varepsilon'^{+1} \beta \chi^k(y) \nabla_{\!x} T^0(x) + O(\varepsilon'^{+2}),$$  \hspace{1cm} (7)

where $\rho^*(x) = \rho_0[1 - \beta T^0(x)]$. 

The serial developments (3)–(4) and the differential operator: \( D/Dx_i = \nabla_{x_i} + \epsilon^{-1}\nabla_{y_i} \) are then introduced in the steady-state Navier–Stokes equations and in the no-slip boundary condition leading to a system of differential equations as follows:

(a) momentum:

\[
\rho^* \epsilon^{2n+r-1} (v_k^0 \cdot \nabla_{y_j} v_i^0) + \epsilon^{2n+r} \{ \rho^* [v_k^0 \cdot (\nabla_{x_i} v_j^0 + \nabla_{y_k} v_i^0)] + \rho_0 \beta_s^s \nabla_{x_k} T^0 (v_k^0 \cdot \nabla_{y_j} v_i^0) \} + \rho_0 \epsilon^{2n+r+1} (\cdots) \\
- \epsilon^{-1} \nabla_{y_i} p^0 - \epsilon^0 (\nabla_{x_i} p^0 + \nabla_{y_i} p^1) - \epsilon (\cdots) \\
+ \mu_0 \epsilon^{n+m-2} \Delta_{y_i} v_i^0 + \epsilon^{n+m-1} (2 \nabla_{x_k} \cdot \nabla_{y_k} v_i^0 + A_{y_i} v_i^1) \\
+ \mu_0 \epsilon^{n+m} (A_{y_i} v_i^0 + 2 \nabla_{x_k} \cdot \nabla_{y_k} v_i^1) + \cdots + \rho^* \epsilon^e g_i + \rho_0 \epsilon^{e+1} \beta g_i \beta_s^s \nabla_{x_k} T^0, \tag{8}
\]

(b) continuity:

\[
(\nabla_{x_i} + \epsilon^{-1} \nabla_{y_i}) (\rho^* \epsilon^e + \rho_0 \epsilon^{e+1} \beta_s^s \nabla_{x_k} T^0) (\epsilon^0 v_i^0 + \epsilon^{e+1} v_i^1 + \cdots) = 0, \tag{9}
\]

(c) boundary condition on \( \Gamma_{\xi} \):

\[
\epsilon^e v_i^0 (x, y) + \epsilon^{e+1} v_i^1 (x, y) + \cdots = 0. \tag{10}
\]

Before collecting the terms with the same power of \( \epsilon \) in equations (8) and (9), the values of parameters \( m, n \) and \( r \) must be determined. The macroscopic pressure gradient \( \nabla_{x_i} p^0 \) is of order zero, while the first term of the microscopic viscous forces, \( \mu_0 A_{y_i} v_i^0 \), is of order \( m + n + 2 \). To balance this pressure gradient, we assume: \( m + n + 2 = 0 \). As mentioned above, different scaling of \( \epsilon \) exists

- in the classical formulation of Sanchez-Palencia [11] only the viscous forces balance this pressure gradient; leading thus to \( m = 0, n = 2 \) and \( r = 0 \). The contribution of the inertial forces in the zero-order expression was thus fully ignored. Physically, this implies that the microscopic fluid flow is still in the Darcy regime and is governed by the Stokes equations;
- in order to model nonlinear effects, it is important to include also contributions of the microscopic inertial forces to balance the macroscopic pressure gradient. The first term in the inertial forces, \( \rho^* (v_k^0 \cdot \nabla_{y_k} v_i^0) \), is of order \( 2n + r - 1 \). As inertial forces and viscous forces should play a similar role, we have to set also \( 2n + r - 1 = 0 \). Giorgi [6] and Chen et al. [3] adopt in their formulation: \( m = \frac{3}{2} \) and \( n = \frac{1}{2} \), which implies that \( r = 0 \). This choice is adopted here, nevertheless, neglecting the effect of small gas density \( (r > 0) \), as suggested by Ene and Sanchez-Palencia [5].

With this parameter choice, the \( \epsilon^{-1} \) term of the momentum equation (8) becomes

\[
- \nabla_{y_i} p^0 (x, y) = 0 \rightarrow p^0 = p^0 (x). \tag{11}
\]

Thus \( p^0 \) is independent of the periodic variable \( y \) and all terms involving \( \nabla_{y_i} p^0 (x, y) \) vanish. Moreover, taking the \( \epsilon^0 \) term of momentum equation (8), the \( \epsilon^{-1/2} \) term of continuity equation (9) and the \( \epsilon^{1/2} \) term of the boundary condition (10) lead now to specify the following microscopic problem
on the unit cell:
\[
\begin{align*}
\rho^*(v^0_k \cdot \nabla_y v^0_i) &= -\nabla_x P^0 - \nabla_y P^1 + \mu_0 A_{xy} v^0_i + \rho^* g_i \quad \text{in } Y_f, \\
\rho^* \nabla_y v^0_i &= 0 \quad \text{in } Y_f, \\
v^0_i &= 0 \quad \text{at } \Gamma_{f},
\end{align*}
\]  
(12a)

2.2. Variational formulation of the microscopic flow problem

In order to solve the microscopic boundary value problem (12) on the unit cell \(Y\), we split the heterogeneous velocity \(v^h\) in two components, as suggested by Giorgi [6]

\[
v(x, y) = \tilde{v}(x, y) + \langle v \rangle(x),
\]

(13)

where \(\langle v \rangle(x) = 1/|Y_f| \int_{Y_f} v(x, y) \, dy\) the average macroscopic velocity. Next, the inertial nonlinear term of (12a) is linearized as follows:

\[
\rho^*(x) v^0_k(x, y) \cdot \nabla_y v^0_i(x, y) \approx \rho^*(x) \langle v^0_k \rangle(x) \cdot \nabla_y v^0_i(x, y).
\]

(14)

In the process of deriving an equation for \(\langle v^0 \rangle(x)\), we keep track of the Darcy component of the flow. To do so, we split the first term of the velocity expansion as

\[
v^0(x, y) = a(x, y) + b(x, y, \langle v^0 \rangle)
\]

(15)

and the pressure term as

\[
\nabla_y p^1(x, y) = \nabla_y p^1_a(x, y) + \nabla_y p^1_b(x, y).
\]

(16)

Introducing (15) and (16) in the local momentum equation (12a), we obtain for the field \(a(x, y)\):

\[
- \mu_0 A_{xy} a_i(x, y) = -[\nabla_x, p^0(x) + \nabla_y p^1_a(x, y)] + \rho^* g_i
\]

(17)

and for \(b(x, y, \langle v^0 \rangle)\):

\[
- \mu_0 A_{xy} b_i + \rho^* \langle v^0_k \rangle(x) \nabla_y b_i = -\nabla_y p^1_b(x, y) - \rho^* \langle v^0_k \rangle(x) \nabla_y a_i(x, y).
\]

(18)

Eq. (17) is well studied, since Sanchez-Palencia [11] has shown that its variational formulation provides the macroscopic Darcy law. Indeed, it has a solution \(a(x, y) \in V_f\) given by

\[
a_i(x, y) = \frac{\omega_j(x)}{\mu_0} [ - \nabla_x, p^0(x) + \rho^* g_i],
\]

(19)

where \(V_f\) is the space of periodic functions, \(V_f = \{ \nabla_y q_i = 0, q_i = 0 \text{ on } \Gamma_{t}, q_i \text{ is } Y \text{ – periodic} \}\), the additional velocity fields \(\omega_j \in V_f, j = 1, 2, 3\) satisfy

\[
\int_{Y_f} \nabla_y \omega_j \cdot \nabla_y q \, dy = \int_{Y_f} q_j(y) \, dy \quad \text{for any } q \in V_f.
\]

(20)

The average of solution (19) over the unit cell gives exactly Darcy’s law

\[
\mu_0 \langle a_i \rangle(x) = P_{ij} [ - \nabla_x, p^0(x) + \rho^* g_j],
\]

(21)
where \( P_{ij} \) is the permeability tensor:

\[
P_{ij} = \langle \omega_j^i \rangle = \frac{1}{Y} \int_Y \omega_j^i(y) \, dy.
\]  
(22)

Note. The constant permeability tensor \( P_{ij} \) defined as the mean value of the component \( i \) of the velocity \( \omega_j^i \) is a symmetric, positive definite matrix. Its components depend only on the geometry of the unit cell \( Y \) and not on thermophysical data like viscosity or density.

After introducing solution (19) in Eq. (18) a variational formulation is also used to solve this equation. Giorgi [6] shows that new additional fields \( H_j^i(y; \langle v^0 \rangle) \) must be determined on \( Y_f \):

\[
\begin{align*}
\mu_0 \int_{Y_f} \nabla_y H_j^i \cdot \nabla_y q \, dy + \rho^* \langle v^0_k \rangle(x) \int_{Y_f} \nabla_y H_j^i \cdot q \, dy \\
= - \rho^* \langle v^0_k \rangle \int_{Y_f} \nabla_y \omega_j^i \cdot q \, dy \quad \forall q \in V_F.
\end{align*}
\]  
(23)

These fields allow to evaluate \( b(x; y; \langle v^0 \rangle) \) in a similar way as for \( a(x, y; H_j(y, \langle v^0 \rangle)) \) replaces \( \omega_j^i \) in expression (19). Finally, collecting the different velocity contributions and averaging over \( Y \) provide a generalized Forchheimer law for \( \langle v^0 \rangle(x) \):

\[
\mu_0 \langle v^0_i \rangle(x) = [ - \nabla_y p^0(x) + \rho^* g_j ] [ \omega_j^i + H_j^i(\langle v^0 \rangle) ].
\]  
(24)

This expression generalizes Darcy’s law by adding a term which is nonlinear in the macroscopic velocity, thus leading to a mean velocity-dependent permeability definition

\[
P_{ij} = \langle \omega_j^i \rangle + \langle H_j^i \rangle(\langle v^0 \rangle).
\]  
(25)

### 2.3. Numerical evaluation of the permeability tensor

In this paper Darcy’s constant permeability tensor, defined by expression (22), will be evaluated numerically on the unit cell \( Y \) as a first approximation of its more general expression (25). This definition needs to determine the unknown velocity fields \( \omega_j^i \) with \( j = 1, 2, 3 \), verifying the weak formulation (20). Each variational expression (20) corresponds to a special Stokes flow problem on the unit cell \( Y \):

\[
A_y \omega_j^i = e^j \text{ with } e^j \text{ a unit vector, } \nabla_y \omega_j^i = 0 \text{ on } Y_f \text{ and } \omega_j^i = 0 \text{ on } \Gamma_k.
\]  
(26)

It corresponds to the flow of a unit viscosity fluid where either negative unit macro-pressure gradient or positive unit body forces in direction \( j \) are applied on the unit cell \( Y \). Periodicity boundary conditions must be imposed on \( \omega_j^i \) components and on the pressure field.

As a stationary Stokes flow problem has to be solved on the unit cell, a well-known analogy with incompressible elasticity can be used [7]. Thus the velocity vector of this problem is related to the displacement vector in the analogous problem. Continuity equation (incompressibility) is then satisfied by using Poisson’s ratio very close to 0.5. As the viscosity is one, the corresponding elastic properties are: \( G = 1 \) and \( E = 3 \). This analogous elasticity problem is solved with ABAQUS [1] by using hybrid linear volume elements, suitable for modeling near-incompressible materials. Regarding the input of ABAQUS, it is more convenient to specify on the unit cell unitary body forces in different directions \( j \) than a pressure gradient.
The ABAQUS displacement results are finally transferred to the homogenization program HOMAT [9] in order to evaluate by averaging the effective permeability tensor (22) of the porous media.

3. Design of the plate and conjugate heat and fluid flow analysis

In this study a multi-layer plate is investigated. Its substrate layer is made of superalloy CMSX-4 with a thickness of 2.0 mm (Fig. 2). The bondcoat consists of a MCrAlY layer (0.15 mm thick). The thermal barrier coating (TBC) is an yttrium stabilized ZrO$_2$ layer with 0.25 mm thickness.

The plate is perforated by laser treatment with seven rows of cooling holes with a diameter of 0.2 mm. Altogether, four different cooling configurations with cylindrical or shaped holes are investigated. As shown in Table 1, the cylindrical hole configurations vary with the distance between two rows of holes and their inclination angle. The configuration with shaped holes is based on configuration C30. The shaping is applied only in the region of the TBC, the design parameters can be also found in Table 1.

The boundary conditions for the fluid flow are derived from modern gas turbine combustion chambers and first turbine stage flow conditions. For the main flow on the plate surface, the total inlet temperature is set to 1302.8°C and the total inlet pressure is set to 2.0125 MPa. The hot gas flow channel has a height of 5 mm. At this distance above the plate surface, an adiabatic wall boundary condition is applied [2]. In this investigation, the plenum of the cooling fluid is not modeled so that for each hole a unique inlet boundary condition is adopted. The static inlet temperature for the cooling fluid is 450°C and its velocity $v_c$ is given in Table 1. The velocity of the hot gas flow yields approximately 75 m/s which leads to a Ma-number of 0.1 in the main flow.

As detailed in Refs. [2,10], the results of the conjugate analysis of the five cooling configurations show that the decrease of the inclination angle of the cooling channels leads to a reduction of the intensity of the kidney vortices which are induced by cooling jets. The shaping of the holes in the outlet region causes an additional reduction of the vortex intensity. Thus a homogeneous cooling film can be observed. In comparison to known cooling techniques [8], the analysis of the different cooling designs proves a significant decrease in the amount of cooling fluid to obtain an endurable maximum temperature of the superalloy.

Fig. 2. Geometry of the investigated multi-layer plate.
Table 1
Design parameters of the different investigated configurations

<table>
<thead>
<tr>
<th>Configuration</th>
<th>L</th>
<th>α (°)</th>
<th>β₁ (°)</th>
<th>β₂ (°)</th>
<th>B</th>
<th>vₑ (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>C45-10</td>
<td>10</td>
<td>D 45</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>5</td>
</tr>
<tr>
<td>C45</td>
<td>6</td>
<td>D 45</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>5</td>
</tr>
<tr>
<td>C30</td>
<td>6</td>
<td>D 30</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>5</td>
</tr>
<tr>
<td>S30-F</td>
<td>6</td>
<td>D 30</td>
<td>8</td>
<td>0</td>
<td>0.6088 D</td>
<td>2.5</td>
</tr>
<tr>
<td>S30-FL</td>
<td>6</td>
<td>D 30</td>
<td>8</td>
<td>10</td>
<td>0.6088 D</td>
<td>3.5</td>
</tr>
</tbody>
</table>

Thereby, the inclination and the shaping of the cooling holes are the principal factors influencing the reduction of the temperature level in the plate. Both lead to a similar, significant decrease of the maximum temperature.

4. Homogenization results for different designs of the cooled plate

4.1. Effective thermal conductivities

Effective thermal conductivities are evaluated for the designs of the transpiration cooled plate illustrated in Fig. 2 by using the homogenization method detailed in Refs. [9,10]. In this investigation, the parallelepiped unit cell with four \( \frac{1}{4} \) channels and one central hole is used. For each configuration, effective values are calculated either for the multi-layer or for each mono-layer of the perforated plate.

Concerning the mono-layer results, the obtained effective thermal conductivities, their mean mono-layer temperatures and the corresponding isotropic conductivity are reported in Table 2. In the last row the effect of the cooling holes on the thermal conductivity is expressed. In the case of cylindrical cooling holes each layer shows a quasi-isotropic behavior. But, the shape designs present a significant orthotropic, equivalent behavior, which cannot be neglected. Due to better cooling and thus lower mean temperature, configuration C30 leads, for example, to a reduction of 10.95% of the effective thermal conductivity of CMSX-4 compared to design C45. The additional shaping of the holes in the TBC layer increases this reduction.

Contrary to the mono-layer homogenization where implicit fluxes are evaluated only at the solid/gas interfaces, additional fluxes in the \( z \) direction are generated in the multi-layer at the solid interlaminar faces. These high fluxes lead to a larger \( \chi^z \) displacement field inducing an extension of the unit cell mainly near the middle hole (see Fig. 3). This field induces an important reduction of the mean isotropic conductivity in the \( z \) direction: −23.4% for C45 (see Table 3). This way, the multi-layer has a pronounced orthotropic equivalent behavior. But, in the layer plane \( x–y \) the effective conductivities of the multi-layer are quasi-isotropic. Indeed, as shown in Fig. 3, displacements \( \chi^x \) and \( \chi^y \) are, respectively, along their axis in a compressive state. As displacement \( \chi^x \) is smaller than \( \chi^y \), a smaller correction of the isotropic conductivity is induced in the \( x \) direction.
Table 2
Effective thermal conductances [W/(mK)] of each layer for different cooling configurations

<table>
<thead>
<tr>
<th>Design</th>
<th>Layer</th>
<th>$k_{x}^{hom}$</th>
<th>$k_{y}^{hom}$</th>
<th>$k_{z}^{hom}$</th>
<th>$T$ (°C)</th>
<th>$\tilde{k}$</th>
<th>$\Delta k$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>C45-10</td>
<td>TBC</td>
<td>1.305</td>
<td>1.292</td>
<td>1.320</td>
<td>858.8</td>
<td>1.367</td>
<td>−4.37</td>
</tr>
<tr>
<td></td>
<td>Bond</td>
<td>31.85</td>
<td>31.50</td>
<td>32.30</td>
<td>833.0</td>
<td>33.51</td>
<td>−3.61</td>
</tr>
<tr>
<td></td>
<td>CMSX-4</td>
<td>18.73</td>
<td>18.51</td>
<td>18.96</td>
<td>821.2</td>
<td>19.69</td>
<td>−3.72</td>
</tr>
<tr>
<td>C45</td>
<td>TBC</td>
<td>1.267</td>
<td>1.243</td>
<td>1.292</td>
<td>875.7</td>
<td>1.369</td>
<td>−7.21</td>
</tr>
<tr>
<td></td>
<td>Bond</td>
<td>30.95</td>
<td>30.33</td>
<td>31.56</td>
<td>853.4</td>
<td>33.69</td>
<td>−6.08</td>
</tr>
<tr>
<td></td>
<td>CMSX-4</td>
<td>18.38</td>
<td>18.00</td>
<td>18.74</td>
<td>841.9</td>
<td>19.96</td>
<td>−6.14</td>
</tr>
<tr>
<td>C30</td>
<td>TBC</td>
<td>1.234</td>
<td>1.190</td>
<td>1.271</td>
<td>763.0</td>
<td>1.359</td>
<td>−9.66</td>
</tr>
<tr>
<td></td>
<td>Bond</td>
<td>29.79</td>
<td>28.43</td>
<td>30.37</td>
<td>739.0</td>
<td>33.11</td>
<td>−8.60</td>
</tr>
<tr>
<td></td>
<td>CMSX-4</td>
<td>16.61</td>
<td>15.87</td>
<td>16.61</td>
<td>723.8</td>
<td>18.40</td>
<td>−8.62</td>
</tr>
<tr>
<td>C30-F</td>
<td>TBC</td>
<td>1.173</td>
<td>1.092</td>
<td>1.270</td>
<td>664.4</td>
<td>1.359</td>
<td>−13.44</td>
</tr>
<tr>
<td></td>
<td>Bond</td>
<td>29.48</td>
<td>28.49</td>
<td>29.85</td>
<td>665.2</td>
<td>32.72</td>
<td>−8.45</td>
</tr>
<tr>
<td></td>
<td>CMSX-4</td>
<td>15.80</td>
<td>15.21</td>
<td>16.80</td>
<td>655.5</td>
<td>18.40</td>
<td>−8.45</td>
</tr>
<tr>
<td>C30-FL</td>
<td>TBC</td>
<td>1.153</td>
<td>1.072</td>
<td>1.315</td>
<td>635.8</td>
<td>1.359</td>
<td>−13.84</td>
</tr>
<tr>
<td></td>
<td>Bond</td>
<td>29.19</td>
<td>28.29</td>
<td>29.73</td>
<td>636.4</td>
<td>32.47</td>
<td>−10.08</td>
</tr>
<tr>
<td></td>
<td>CMSX-4</td>
<td>15.43</td>
<td>14.91</td>
<td>15.40</td>
<td>627.7</td>
<td>17.14</td>
<td>−8.49</td>
</tr>
</tbody>
</table>

The change of hole inclination from 45° to 30° induces a significant reduction of the equivalent conductivity of the multi-layer, mainly in the $x$–$y$ plane. Compared to hole distance in flow direction, the inclination is the most sensitive design parameter.

4.2. Effective permeabilities

To determine effective Darcy permeabilities, the rhombical unit cell with only one central hole (see Fig. 1) is used instead of the parallelepiped one due to its easier definition of the boundary conditions in ABAQUS (only $v = 0$ at $\Gamma_{e}$ is there specified). For each design of the cooled plate
Table 3
Effective conductances (W/(mK)) of the multi-layer for different cooling configurations

<table>
<thead>
<tr>
<th>Design</th>
<th>$k_{xx}^{\text{hom}}$</th>
<th>$k_{yy}^{\text{hom}}$</th>
<th>$k_{zz}^{\text{hom}}$</th>
<th>$\Delta k_{zz}^\text{Y} (%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>C45-10</td>
<td>17.77</td>
<td>17.57</td>
<td>13.34</td>
<td>-24.3</td>
</tr>
<tr>
<td>C45</td>
<td>17.37</td>
<td>17.01</td>
<td>13.34</td>
<td>-24.3</td>
</tr>
<tr>
<td>C30</td>
<td>15.82</td>
<td>15.12</td>
<td>12.72</td>
<td>-17.6</td>
</tr>
<tr>
<td>S30-F</td>
<td>15.12</td>
<td>14.56</td>
<td>12.17</td>
<td>-18.0</td>
</tr>
<tr>
<td>S30-FL</td>
<td>14.79</td>
<td>14.28</td>
<td>11.71</td>
<td>-19.5</td>
</tr>
</tbody>
</table>

Table 4
Permeabilities $P_{ij}$ ($\mu$m$^2$) of each layer and of the multi-layer for different cooling designs and FE discretizations

<table>
<thead>
<tr>
<th>Design</th>
<th>Layer</th>
<th>$P_{xx}$</th>
<th>$P_{yy}$</th>
<th>$P_{zz}$</th>
<th>$P_{xz}$</th>
<th>$P_{zx}$</th>
<th>$Y_f/Y$</th>
<th>$\alpha_{xz}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>C45-10</td>
<td>TBC</td>
<td>23.97</td>
<td>2.597</td>
<td>23.57</td>
<td>20.49</td>
<td>22.88</td>
<td>3.7</td>
<td>44.7</td>
</tr>
<tr>
<td></td>
<td>Bond</td>
<td>16.77</td>
<td>0.007</td>
<td>17.26</td>
<td>17.24</td>
<td>16.76</td>
<td>3.7</td>
<td>44.6</td>
</tr>
<tr>
<td></td>
<td>CMSX-4</td>
<td>20.57</td>
<td>0.368</td>
<td>20.89</td>
<td>20.48</td>
<td>20.38</td>
<td>3.7</td>
<td>44.7</td>
</tr>
<tr>
<td>C45</td>
<td>TBC</td>
<td>39.95</td>
<td>4.33</td>
<td>39.29</td>
<td>34.15</td>
<td>38.15</td>
<td>6.2</td>
<td>44.8</td>
</tr>
<tr>
<td></td>
<td>Bond</td>
<td>27.96</td>
<td>0.012</td>
<td>28.77</td>
<td>28.73</td>
<td>27.93</td>
<td>6.2</td>
<td>44.6</td>
</tr>
<tr>
<td></td>
<td>CMSX-4</td>
<td>34.23</td>
<td>0.605</td>
<td>34.83</td>
<td>34.14</td>
<td>33.97</td>
<td>6.2</td>
<td>44.7</td>
</tr>
<tr>
<td>C30</td>
<td>TBC</td>
<td>82.41</td>
<td>5.92</td>
<td>28.30</td>
<td>41.28</td>
<td>47.10</td>
<td>8.7</td>
<td>29.3</td>
</tr>
<tr>
<td></td>
<td>Bond</td>
<td>61.75</td>
<td>0.009</td>
<td>21.35</td>
<td>36.89</td>
<td>35.63</td>
<td>8.7</td>
<td>28.6</td>
</tr>
<tr>
<td></td>
<td>CMSX-4</td>
<td>74.46</td>
<td>0.846</td>
<td>25.56</td>
<td>46.19</td>
<td>42.95</td>
<td>8.7</td>
<td>30.6</td>
</tr>
<tr>
<td>S30-F</td>
<td>TBC</td>
<td>140.08</td>
<td>25.22</td>
<td>29.61</td>
<td>45.94</td>
<td>45.49</td>
<td>13.4</td>
<td>22.5</td>
</tr>
<tr>
<td></td>
<td>Bond</td>
<td>69.91</td>
<td>0.003</td>
<td>24.66</td>
<td>41.47</td>
<td>40.64</td>
<td>8.7</td>
<td>30.4</td>
</tr>
<tr>
<td></td>
<td>CMSX-4</td>
<td>75.52</td>
<td>0.802</td>
<td>25.93</td>
<td>43.79</td>
<td>43.55</td>
<td>8.7</td>
<td>30.2</td>
</tr>
<tr>
<td>S30-FL</td>
<td>TBC</td>
<td>157.11</td>
<td>44.32</td>
<td>26.55</td>
<td>44.65</td>
<td>42.57</td>
<td>15.8</td>
<td>21.2</td>
</tr>
<tr>
<td></td>
<td>Bond</td>
<td>69.91</td>
<td>0.003</td>
<td>24.66</td>
<td>41.47</td>
<td>40.64</td>
<td>8.7</td>
<td>30.4</td>
</tr>
<tr>
<td></td>
<td>CMSX-4</td>
<td>75.52</td>
<td>0.802</td>
<td>25.93</td>
<td>43.79</td>
<td>43.55</td>
<td>8.7</td>
<td>30.2</td>
</tr>
</tbody>
</table>

defined in Table 1, effective permeabilities are calculated for each layer and for the multi-layer, respectively.

In Table 4, the diagonal and principal off-diagonal terms as well as the volume ratio between cooling channel $Y_f$ and the unit cell $Y$ are reported. In the last column, the angle $\alpha_{xz}$ of the principal permeability direction is also given. These results show that, independent of the design configuration, the Darcy permeability tensor is anisotropic. This behavior can be explained by the analysis of the analogous displacements of the incompressible elastic material on the unit cell. For example, in Fig. 4, the displacements in $x$, $y$, and $z$ directions are given on the deformed shape for the cooling channels C30 and S30-F at their outflow (TBC) face. Both displacement fields are induced by a unit body force in the $x$ direction. The $x$ component for design C30 is a little larger than the $z$ one, due to the channel inclination. But, both components are significantly larger than the values
in \( y \) direction, which is normal to the channel inclination. Due to the shaping of design S30-F, larger body forces are applied in \( x \) direction in the TBC layer leading to greater \( x \) displacements than for C30. These larger displacements combined with the material incompressibility lead to a different upper face deformation, mainly in the \( z \) direction. Moreover, for design S30-FL, larger \( y \) body forces are induced due to channel opening in this direction. Thus, for this design, \( P_{yy} \) of the TBC layer becomes larger than \( P_{zz} \).

A quasi-isotropic behavior is predicted in the plane \( x-z \) for the designs C45-10 and C45 with a \( 45^\circ \) inclination angle. Compared to C45, design C30 presents a pronounced orthotropic behavior in the plane \( x-z \). Indeed, \( P_{xx} \) increases significantly but \( P_{zz} \) reduces a little. For example, for the TBC layer we have for design C30 a ratio: \( P_{xx}/P_{zz} = 2.9 \). The additional shaping of designs S30-F and S30-FL increases this orthotropic equivalent behavior: \( P_{xx}/P_{zz} = 4.75 \) for S30-F and \( 5.92 \) for S30-FL.

Note that the principal off-diagonal terms \( (P_{xz} \text{ and } P_{zx}) \) show that, due to numerical discretization errors, the permeability tensor is not exactly symmetric as theoretically expected. Diagonalizing the permeability tensor leads to specifying the angle \( \alpha_{xz} \) of the principal permeability direction in the \( x-z \) plane. For the cylindrical designs, this angle corresponds quasi to the channel inclination. Further shaping of the TBC layer induces a significant reduction of this angle indicating that the direction of greatest permeability tends to the hot gas direction. Moreover, the permeability of the bondcoat layer is for each design smaller than for the other two layers. This is due to the fact that no specific periodicity condition is specified in the \( z \) direction on the unit cell (see Fig. 1). The deformation of the TBC and CMSX-4 channel parts can be so large due to free deformation of their upper or lower faces, respectively.
5. Conclusions

A general multi-scale approach based on the homogenization method applied to the flow of a compressible gas through a periodic porous media has been presented. This formulation allows one to deduce as macroscopic behavior either a Darcy law with constant permeability tensor or a Forchheimer law with a permeability function of the mean velocity. Significant nonlinear inertial effects at the micro-scale induce these more general equivalent law. Darcy permeabilities are calculated by solving special Stokes flow problems and by using the well-known analogy with the incompressible elasticity. The developed strategy has been applied successfully to the prediction of permeabilities and effective thermal conductivities of a transpiration cooled plate. The thermal homogenization results show that each equivalent mono-layer has a quasi-isotropic behavior, but that the equivalent multi-layer has pronounced orthotropic conductivities. The permeability tensor of each layer is anisotropic for the analyzed cooling configurations. Next, the adopted strategy will be extended to calculate the more complex Forchheimer permeability tensor and to formulate a new ‘loss model’ for the fluid boundary layer for the three-dimensional conjugate fluid flow analysis.

References