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Blowup in diffusion equations: A survey

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Abstract

This paper deals with quasilinear reaction–diffusion equations for which a solution local in time exists. If the solution ceases to exist for some finite time, we say that it blows up. In contrast to linear equations blowup can occur even if the data are smooth and well-defined for all times. Depending on the equation either the solution or some of its derivatives become singular. We shall concentrate on those cases where the solution becomes unbounded in finite time. This can occur in quasilinear equations if the heat source is strong enough. There exist many theoretical studies on the question on the occurrence of blowup. In this paper we shall recount some of the most interesting criteria and most important methods for analyzing blowup. The asymptotic behavior of solutions near their singularities is only completely understood in the special case where the source is a power. A better knowledge would be useful also for their numerical treatment. Thus, not surprisingly, the numerical analysis of this type of problems is still at a rather early stage. The goal of this paper is to collect some of the known results and algorithms and to direct the attention to some open problems.

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1. Introduction

Parabolic equations of the type

$$u_t - \Delta u = f(x,t,u,\nabla u) \quad \text{in } D \times \mathbb{R}^+, \tag{1.1}$$

$$u(x,t) = 0 \quad \text{on } \partial D \times \mathbb{R}^+, \tag{1.2}$$

$$u(x,0) = u_0(x) \geqslant 0. \tag{1.3}$$

model a great number of physical problems. The most common interpretation is to think of u as the temperature of a substance in a recipient $D \in \mathbb{R}^N$ subject to a chemical reaction. A positive term

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f represents a heat source due to an exothermic reaction. Otherwise the reaction is endothermic. If f depends on the gradient, then convection effects are taken into consideration. In population dynamics similar equations are used to describe the concentrations of species. The Laplace operator is often replaced by the nonlinear operator Δu^m, first studied in the context of flows through porous media. It represents a slow or fast diffusion depending on whether $m>1$ or $m<1$. Several existence theories for local solutions are available. Classical solutions can be established in the setting of Hölder continuous functions. If D is smooth and bounded and if f is locally Lipschitz and defined for all u and ∇u, then for any $u_0 \in C^\alpha(\bar{D})$, $u_0 = 0$ on ∂D there exists a unique classical solutions for small t, say $t<T$. If $T=\infty$, the solution will be called *global*. By the Schauder estimates [30, 14] the solution ceases to exist if either $\lim_{t\to T-} \sup_{x\in D} |u(x,t)| = \infty$ or $\lim_{t\to T-} \sup_{x\in D} |\nabla u(x,t)| = \infty$. In this case, we shall say that it *blows up*. In this survey we shall concentrate on the first case. It is much easier to treat, especially if the function f is independent of the gradient. The more complicated case where f depends also on the gradient has received attention only recently [12, 42].

Let us first have a look at the pure reaction equation with a positive source term, $du/dt = f(u) \geqslant 0$, $u(0) = u_0$. This solution blows up in finite time if and only if

$$\mathcal{T}(u_0) = \int_{u_0}^{\infty} f^{-1}(s)ds < \infty. \tag{1.4}$$

The diffusion and the boundary condition have the tendency to decrease the solution. Since in Problem (1.1)–(1.3) all mechanisms such as reaction, diffusion and the cooling at the boundary act together simultaneously, it is natural to ask which one prevails. Blow-up phenomena for reaction–diffusion problems in bounded domains have been studied for the first time in a seminal paper by Kaplan [26]. He showed that for convex source terms $f = f(u)$ satisfying (1.4) diffusion cannot prevent blow up if the initial state is large enough. The next corner stone was the fundamental work of Fujita [17, 18]. He proved that the Cauchy problem ($D = \mathbb{R}^N$) with $f = u^p$ has no global positive nontrivial solutions if $1 < p < 1 + 2/N$. Every solution with arbitrarily small initial data blows up. The same is true for $p = 1 + 2/N$ as it was shown by Hayakawa [24]. When $p > 1 + 2/N$, solutions with small initial conditions tend to zero as time increases. Notice that for $p \in [0, 1]$ all solutions with bounded initial data are global.

Several papers contain numerous references on blow-up results (cf., e.g., [32, 7, 15]). An extensive literature on quasilinear problems of the type $u_t - \Delta\phi(u) = f(u)$ for various choices of ϕ and f is found in [44] (compare in particular the comments to Chapter 4).

As we have already indicated in the Abstract, the purpose of this survey is to highlight important results and tools in the analysis of blowup; thus, it is not meant to provide a chronological description of the development of the theory of blowup in nonlinear parabolic equations.

The physical interpretation of blowup is generally thought of as a dramatic increase in temperature which leads to ignition of a chemical reaction. In this context the following questions come up:

- *When does blowup occur*?
- *Where are the blow-up points*?
- *What is the asymptotic behavior of the solutions at blow-up time*?
- *Is it possible to extend the solution in some weak sense beyond the blow-up time*?
- *Is it possible to compute a blow-up solution numerically*?

The first question is fairly well-understood. During the last two decades great progress has been made in establishing criteria for blowup. Kaplan and Fujita's results have been generalized to various

cases. Much more difficult is the location of blow-up points and the discussion of the nature of the asymptotics. If u can be continued after the blow-up time true ignition does not take place. As will be seen below and in Section 6.3, many problems remain open in the numerical treatment of blow-up problems.

The *numerical analysis* and the *computational solution* of parabolic problems of the form (1.1)–(1.3) are still at a rather early stage, and most papers on these subjects have only dealt with the special case

$$u_t - \Delta u = f(u) \quad \text{in } Q_T := D \times (0,T), \tag{1.5}$$

$$u(x,t) = 0 \quad \text{in } \partial Q_T \times (0,T), \tag{1.6}$$

$$u(x,0) = u_0(x) \quad \text{in } D, \tag{1.7}$$

where $D \subset \mathbb{R}^N$ is bounded. In fact, the discussion has almost exclusively been restricted to the case $N = 1$ (but compare [56, 58, 50]). The discretization of (1.5)–(1.7) (or, more generally, of (1.1)–(1.3)) usually involves two separate steps: semidiscretization in space leads to an initial-value problem for a (large) system of nonlinear ordinary differential equations whose special structure (unbounded stiffness of its linear part; large, rapidly increasing nonlinear part) requires special care in the choice of time discretization methods. To be more specific, let D_h be a mesh (or grid) for the spatial domain D, and denote the interior mesh points of D_h by $\{P_i\colon\ i = 1,\dots,M\} =: \mathscr{P}_h$, where in general, M will be large. An approximation to the exact solution $u(x,t)$ at (P_i,t) will be denoted by $U_i(t)$, and we set $U = U_h := (U_1,\dots,U_M)^{\mathrm{T}}$. Spatial discretization of (1.5) with respect to D_h yields

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} + A_h U(t) = F_h(U(t)), \quad t \in (0,T),$$

$$U(0) = U_0 = (u_0(P_1),\dots,u_0(P_M))^{\mathrm{T}}. \tag{1.8}$$

The precise form of this system of ODEs will depend on the particular method used for the spatial semidiscretization of (1.5): this "method of lines" (MOL) approach may be based on classical finite-difference approximations to spatial derivatives ([68, 55, 56, 51, 50, 49]; see also [62, 63, 76]), on spatial collocation or (lumped mass) finite-element techniques ([69, 57, 58]), or on spectral and pseudospectral methods ([72]). Typically, the matrix $A_h \in \mathscr{L}(\mathbb{R}^M)$ is real, symmetric, sparse, and positive definite; its nonzero elements grow like $1/h^2$ where $h > 0$ is the mesh diameter in D_h. The last two properties imply that the linear part of (4.4) represents an unboundedly stiff system of ODEs.

Let now $J_\tau := \{0 = t_0 < t_1 < \cdots < t_n < \cdots\}$, with $\tau_n := t_{n+1} - t_n$, be a (nonuniform) temporal mesh, and assume that time discretization of (1.8) is by collocation in the space of piecewise linear C^0 functions $V = V(t) \in \mathbb{R}^M$. We use this approach partly due to ease of exposition and to exhibit the basic computational problems, but also because most standard time-stepping methods used in the literature (see also Section 6.1) may be viewed as special cases of these collocation methods. Collocation will be at the points $X_\tau := \{t_n + c_1\tau_n\colon\ n \geqslant 0\}$, where $c_1 \in [0,1]$ is given. Set

$$V(t_n + s\tau_n) = V_n + s\tau_n V'(t_n + c_1\tau_n), \quad s \in [0,1], \tag{1.9}$$

where $V_n := V(t_n) = (V(P_1,t_n),\dots,V(P_M,t_n))^{\mathrm{T}} \in \mathbb{R}^M$ approximates the solution $U(t_n)$ of (1.8). The collocation equation for (1.8),

$$V'(t_n + c_1\tau_n) + A_h V(t_n + c_1\tau_n) = F_h(V(t_n + c_1\tau_n)),$$

may be written as

$$V_{n+1} + \tau_n c_1 A_h V_{n+1} = V_n - \tau_n(1-c_1)A_h V_n + \tau_n F_h((1-c_1)V_n + c_1 V_n) \quad (n \geqslant 0),$$

with $V_0 = U_0$. Alternatively, we have for $n \geqslant 0$,

$$(I + \tau_n c_1 A_h)V_{n+1} = (I - \tau_n(1-c_1)A_h)V_n + \tau_n F_h((1-c_1)V_n + c_1 V_{n+1}), \tag{1.10}$$

where $I = I_M$ denotes the identity in $\mathscr{L}(\mathbb{R}^M)$. Eqs. (1.9) and (1.10) represent a one-parameter family of (continuous) one-stage Runge–Kutta methods for (1.8). The choices $c_1 = 0$ and $c_1 = 1$ yield, respectively, the (continuous) explicit and implicit Euler methods, while for $c_1 = \frac{1}{2}$ we obtain the (continuous) implicit midpoint method for (1.8). Almost all of the methods proposed in the literature employ one of the Euler methods; notable exceptions are [57, 58, 54] (see also Section 6.1 below for details). These brief remarks on the discretization of the simple blowup problem (1.5)–(1.7) give rise to the following questions:

- *Choice of the spatial and temporal meshes?*
  While early numerical approaches to (1.8) were based on uniform spatial meshes D_h (cf. [68, 69, 75, 55, 56]; see also [50, 49]) and suitably chosen nonuniform temporal mesh J_τ, the (often) known space–time structure of the analytical solution near (x_b, T_b) (in the case of single-point blowup) has recently been exploited to devise moving mesh methods (dynamic regridding); see [57, 58, 53, 54].
- *Choice of the time integrator?*
  Is the above class of "classical" time-stepping methods feasible for solving semidiscretized blow-up problems? Should one employ implicit or explicit time-stepping methods (stiffness versus contraction mapping principle: unboundedly stiff A_h and large values of $F_h(V)$ lead, respectively, to very small time steps $\{\tau_n\}$)?
- *Numerical simulation and detection of blow-up?*
  Compare [74, 80, 81], as well as [77] and its references. If for some given spatial discretization the solution of the resulting system of ODEs (1.8) does not exhibit blowup, is this true also for the solution of the original problem (1.5)–(1.7)?
- *Convergence analyses?*
  Open for most of the more sophisticated (moving mesh) methods ([51, 57, 58, 54]). Compare also Section 6.1 for existing convergence results.
- *Adaptive methods and a posteriori error control?*
  Can the recently developed adaptive techniques (based on a posteriori error estimates) of [59–61] be used in blow-up problems? More specifically, given $\varepsilon > 0$, design an adaptive method for (1.5)–(1.7) such that the computed blow-up time $\hat{T}_b$ satisfies $|T_b - \hat{T}_b| < \varepsilon$.

Related questions concern the design and analysis of methods for bounded spatial domains $D \subset \mathbb{R}^N$ with $N = 2, 3$, for parabolic problems with nonlocal boundary conditions [85] or memory terms ([79, 83, 84]; see also [82]), and for systems of reaction–diffusion equations with blow-up solutions. In Section 6 we will look at the history and the "state of the art" of the numerical analysis and computational solution of the problems described above.

2. Preliminaries: Local solutions

The goal of this section is to sum up some results on the existence of solutions for small values of t.

If the data are not smooth or, as in the case of slow and fast diffusion, some caution is required in defining what is meant by a solution. Weaker notions have to be used. Several definitions are common. We shall indicate the most standard ones.

Multiplying Problem (1.1)–(1.3) by any test function $\chi \in C_0^\infty(Q_T)$, where Q_T stands for the parabolic cylinder $D \times (0,T)$, and integrating formally by parts we obtain

$$\int_{Q_T} (u_t\chi + (\nabla u, \nabla\chi) - f\chi)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{2.1}$$

u is called a *weak solution* in the Sobolev space $W^{1,p}(Q_T)$ if $u, u_{x_i}, u_t \in L^p(Q_T)$ and (2.1) holds for all $\chi \in C_0^\infty(Q_T)$. Moreover, $u = 0$ on $\Gamma_T := \partial D \times (0,T)$ in the sense of the trace and $\lim_{t\to 0} \|u(x,t) - u_0(x)\|_{L^p(D)} = 0$.

Another definition which is used especially in the context of problems with slow and fast diffusion solutions requires the solution u to belong to $C([0,T]; L^1(D)) \cap L^\infty(Q_T))$ and satisfies

$$\int_D (u\chi)(x,t)\,\mathrm{d}x - \int_{Q_t} (u\chi_\tau + u^m\Delta\chi)\,\mathrm{d}x\,\mathrm{d}\tau \int_D u_0(x)\chi(x,0)\,\mathrm{d}x + \int_{Q_t} f\chi\,\mathrm{d}x\,\mathrm{d}\tau$$

for all test functions χ in $C^\infty(Q_T)$ with $\chi = 0$ on ∂D.

The third approach uses the Green's function $G(x,y,t)$. For fixed $y \in D$, it satisfies

$$\left(\frac{\partial}{\partial t} - \Delta_x\right) G(x,y,t) = \delta(y)\delta(t) \quad \text{in } Q_\infty, \qquad G(x,y,t) = 0 \quad \text{if } x \in \partial D.$$

Here $\delta(\cdot)$ denotes the Dirac function. The solution of (1.1)–(1.3) can then be written as

$$u = S(t)u_0 + \int_0^t S(t-s)f\,\mathrm{d}s \quad \text{where } S(t)\phi = \int_D G(x,y,t)\phi(y)\,\mathrm{d}y. \tag{2.2}$$

In the case $D = \mathbb{R}^N$ the Green's function is replaced by the heat kernel and hence

$$S(t)\phi = \frac{1}{(4\pi N t)^{N/2}} \int_{\mathbb{R}^N} \mathrm{e}^{-(|x-y|^2)/4t}\phi(y)\,\mathrm{d}y. \tag{2.3}$$

Eq. (2.2) can be interpreted as a dynamical system in a suitable Banach space. Different theories have been developed for establishing local solutions [48]. For smooth data all the concepts coincide. Moreover if f is locally Lipschitz, then for every $u_0 \in C^1(\overline{D})$ the solution is classical on $(0,T)$. Weissler's approach allows also to handle singular initial data $u_0 \in L^s(D)$. If s is sufficiently large he is able to prove the existence of a solution in $W^{1,s}(Q_T)$. He also observed that there are functions $u_0 \in L^s(D)$ with small s such that the problem does not have a local solution. It is interesting to note that even for simple problems such as $f = |u|^{p-1}u$ it is not yet clear whether there is always a local solution for "very" singular u_0 which does not belong to $L^1(D)$ [8].

3. Tools and blowup criteria

We shall describe some of the most widely used methods for establishing blowup. In short, we can distinguish between two different approaches: the first consists in constructing lower bounds for the solution which become infinite at finite time and the second derives differential inequalities for time-dependent integrals involving the solution.

3.1. Method of upper and lower solutions

In the sequel we shall put for short $L = \partial/\partial t - \Delta$.

Definition 1. $\underline{u}$ is called a lower solution of (1)–(3) if $L\underline{u} \leqslant f(x,t,\underline{u},\nabla\underline{u})$ in Q_T, $\underline{u} \leqslant 0$ on $\Gamma_T, \underline{u}(x,0) \leqslant u_0$. Similarly $\overline{u}$ is called an upper solution if the inequality signs are reversed.

As for the solutions we can define weak upper and lower solutions. In bounded domains Q_T it follows from *the maximum principle* for parabolic equations and its generalizations that, if f is continuous in the domain of definition and differentiable in u and u_{x_i}, then $\underline{u} \leqslant \overline{u}$ in Q_T.

In unbounded domains the same comparison holds true by a Phragmèn-Lindelöf-type argument provided that $(\overline{u} - \underline{u})(x,t) \geqslant -B\exp\{\beta|x|^2\}$ in Q_T for some positive B and β.

For classical solutions this comparison principle is found in [41]. For weak upper and lower solutions it is proved in [10] under the assumption that f does not depend on ∇u and $u \in W^{1,2}(Q_T)$. It can also be extended to problems with fast and slow diffusion [1]. We also direct the reader to the survey paper [25] on weak solutions to quasilinear degenerate parabolic equations.

If an upper and lower solution exists, there is a solution u such that $\underline{u} \leqslant u \leqslant \overline{u}$. This statement goes back to Sattinger [45] and has been extended to weak solutions, e.g. [1, 10].

Strategy I: *Construct a lower solution $\underline{u}$ which blows up at some finite time $\overline{\tau}$. Then $\overline{\tau}$ is an upper bound for the actual blow-up time T, e.i. $T \leqslant \overline{\tau}$. Likewise if $\overline{u}$ is an upper solution with blow-up time $\underline{\tau}$, then $\underline{\tau}$ is a lower bound for T, i.e. $\underline{\tau} \leqslant T$.*

A further important observation which has been used in many qualitative studies is the following:

If u_0 is a lower solution, i.e $\Delta u_0 + f(x,t,u_0,\nabla u_0) \geqslant 0$, then the function $t \to u(x,t)$ is increasing. Vice versa if u_0 is an upper solution, $t \to u(x,t)$ is decreasing.

It seems to have been stated for the first time in [45]. For its generalization to weak solutions compare [37].

We first describe some candidates for *lower solutions* $\underline{u}$ which blowup in finite time:

1. In the autonomous case $\underline{u} = z(v,w)$ where $z_v = f(z)$, $z(0,w) = w$ and $w(x,t)$ and $v(x,t)$ have to be chosen appropriately. A straightforward calculation yields,

$$\underline{u}_t - \Delta\underline{u} - f(\underline{u}) = f(\underline{u})(v_t - \Delta v) + \frac{f(\underline{u})}{f(w)}(w_t - \Delta w) + f(\underline{u})\left(\frac{f'(w)}{f(w)^2}|\nabla w|^2 - f'(\underline{u})\left|\nabla v + \frac{\nabla w}{f(w)}\right|^2\right) - f(\underline{u}).$$

Such lower solutions were explored in [33, 34, 39] for bounded and unbounded domains. Consider for instance the case of an increasing function f such that $f(0) > 0$. Take $w = 0$ and v to be the solution of $Lv = 1$ in Q_∞, $v = 0$ on Γ_∞, $v(x,0) = 0$. Then the function $\underline{u}$ constructed above is a lower solution for every nonnegative initial condition. It blows up in finite time, if for some

t_0, $\max_D \underline{u}(x,t_0) > \mathscr{T}(0)$, $\mathscr{T}(0)$ being defined in (1.4). We then have $T \leqslant t_0$. *For bounded initial data this condition is always be fulfilled in sufficiently large domains.*

Another interesting choice is $v = te^{-\lambda_1 t}\phi(x)/\phi_{\max}$ and $w = w_0 e^{-\lambda_1 t}\phi(x)/\phi_{\max}$ where λ_1 is the lowest eigenvalue and ϕ the corresponding eigenfunction of $\Delta\phi + \lambda\phi = 0$ in D, $\phi = 0$ on ∂D. If f is convex and $f(0) > 0$, then u blows up in finite time provided there exists t_0 such that $t_0 e^{-\lambda_1 t_0} > \mathscr{T}(w_0 e^{-\lambda_1 t_0})$. In this case $T < \mathscr{T}(w_0 e^{-\lambda_1 t_0})$. *This condition can always be achieved by taking large initial data.* More generally, Meier [34] noted

Blowup criterion 1. *Suppose that* $z(v,w) =: \underline{u}$ *is a lower solution and that* $\sup_D(v(x,t_0) - \mathscr{T}(w(x,t_0))) \geqslant 0$. *Then* $\underline{u}$, *resp.* u *blows up in finite time* $T \leqslant t_0$.

2. Self-similar functions of the form $h(t)k(x/t^{\alpha})$ have been studied for power-like nonlinearities [27]. They have also proved to be useful especially in problems with the porous media operator [42, 44] (see in particular the notes in Ch. 4).
3. If $\underline{u_1}(x,t)$ and $\underline{u_2}(x,t)$ are lower solutions then $\max\{\underline{u_1}, \underline{u_2}\}$ is also a lower solution. This shows that solutions in large domains are more likely to blowup. In fact by taking $\max\{u(x,t), 0\}$ as a lower solution we can prove:

Blowup criterion 2. *Let* $\tilde{u}$ *be the solution of* (1.1)–(1.3) *in* $\tilde{D} \supset D$ *with the initial condition* $\tilde{u}_0 = u_0$ *in* D *and* $\tilde{u}_0 = 0$ *in* $\tilde{D} \backslash D$. *If* u *blows up then so does* $\tilde{u}$.

Candidates for *upper solutions* are:

1. If u_0 is bounded and f is independent of x and t, the solution of

$$\mathrm{d}z/\mathrm{d}t = f(z,0), \qquad z(0) = \max u_0(x),$$

is an upper solution. It blows up under condition (1.4).

2. If f is independent of t and has a positive stationary solution $U(x)$, then by the comparison theorem, every solution $u(x,t)$ with $u(x,0) = u_0(x) \leqslant U(x)$ is global.
3. Separable solutions $T(t)W(x)$ are successful if the nonlinearity behaves like a power.
4. Similarly as for lower solutions we have: if $\overline{u_1}$ and $\overline{u_2}$ are two upper solutions, then $\min\{\overline{u_1}, \overline{u_2}\}$ is an upper solution.
5. Let $u(x,t)$ solve (1.1)–(1.3) in Q_T. Then by the comparison principle the solution of

$$v_t - \Delta v = \left(\max_{Q_{T'}} \frac{f(u)}{u}\right) v, \quad v = 0 \quad \text{on } \partial D \times R^+, \quad v(x,0) > u_0(x)$$

is an upper solution in $(0,T')$ with $T' < T$. Especially in unbounded domains it is useful to have an upper solution which blows up at finite time and depends not only on t but also on x. If $D = \mathbb{R}^N$, then $v = \|u_0\|_{\infty} e^{\alpha(T')t}(t+1)^{-N/2} e^{-|x|^2/4(t+1)}$ is a solution of the above problem [36]. Hence $u(x,t) \leqslant v(x,t)$.

3.2. Fourier coefficient

This method was devised by Kaplan [26] in the case of a bounded domain D. Let $\phi > 0$ be the first eigenfunction of the Laplacian and let λ_1 be the corresponding eigenvalue. Suppose that $\int_D \phi \, \mathrm{d}x = 1$ and that $f = f(u)$ is convex ($f'' > 0$). Multiplying (1.5)–(1.7) by ϕ, integrating over D and using Jensen's inequality we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_D u\phi \, \mathrm{d}x + \lambda_1 \int_D u\phi \, \mathrm{d}x = \int_D f(u)\phi \, \mathrm{d}x \geqslant f\left(\int_D u\phi \, \mathrm{d}x\right).$$

This is a differential inequality for $\chi(t)=\int_D u\phi\,\mathrm{d}x$. If the solution has the series expansion $u(x,t)=\sum_{n=1}^{\infty} a_n(t)\phi_n(x)$ where ϕ_n is a normalized eigenfunction of the Laplacian, $\|\phi_n\|_{L^2(D)}=1$, then $\chi(t)$ coincides up to a normalization factor to $a_1(t)$. The differential inequality leads immediately to the following.

Blowup criterion 3. *Assume f to be convex and* (1.4) *to be satisfied. Let t_0 be the largest zero of $f(s)-\lambda_1 s$. If $\chi(0)>t_0$, then $u(x,t)$ blows up in finite time.*

It becomes evident that the first Fourier coefficient is large if either the initial data u_0 or the first eigenvalue and therefore the domain is large. Then the process is governed by the reaction and the diffusion cannot prevent blowup.

If D is unbounded ϕ does not exist in general. Nevertheless, this method is still applicable. It suffices to take for ϕ a positive function such that $\Delta\phi+c\phi\geqslant 0$ in D and $\phi=0$ on ∂D. Suitable choices are:

1. $\phi=\exp\{-k|x|^2\}$, if $D=\mathbb{R}^N$ [17];
2. $\phi=\exp\{-k|x|^2\}|x|^m\psi(\theta)$ if D is a cone in $\mathbb{R}^N$, where m,k are suitably chosen positive constants and $\psi(\theta)$ is a function on, S^{N-1} [4];
3. any smooth, radially symmetric and nonincreasing function $0\leqslant\phi\leqslant 1$, $\phi=1$ for $|x|\leqslant\rho$ and $\phi=0$ in $|x|\geqslant 2\rho$. Recently, Qi [43] was able to prove Fujita-type results by means of such a general function.

3.3. Concavity methods

The concavity method is an elegant tool to derive norm estimates and gives criteria for blowup [31] (see also [19]). For simplicity, we shall describe it in the case where $f=f(x,u)$. An extension to degenerate operators is found in [13]. Let us first introduce some *notation*:

$$F(x,u):=\int_0^u f(x,s)\,\mathrm{d}s, \qquad (u(t),v(t)):=\int_D u(x,t)v(x,t)\,\mathrm{d}x,$$

$$\|u(t)\|^2=(u,u), \quad \text{and } J(t):=\frac{1}{2}\|\nabla u(t)\|^2-\int_D F(x,u)\,\mathrm{d}x.$$

By multiplying (1.1)–(1.3) by u, or u_t, and integrating over D we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2=-\|\nabla u(t)\|^2+\int_D f(x,u)u\,\mathrm{d}x, \tag{3.1}$$

$$\|u_t(t)\|^2=-\frac{\mathrm{d}}{\mathrm{d}t}J(t). \tag{3.2}$$

Hence,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2&=(2+\varepsilon)\left(\int_0^t\|u_t(s)\|^2\,\mathrm{d}s+\{J(t)-J(0)\}\right)\\&\quad-\|\nabla u(t)\|^2+\int_D f(x,u)u\,\mathrm{d}x=(2+\varepsilon)\int_0^t\|u_t(s)\|^2\,\mathrm{d}s\\&\quad+\frac{\varepsilon}{2}\|\nabla u(t)\|^2+\int_D\{f(x,u)u-(2+\varepsilon)F(x,u)\}\,\mathrm{d}x-(2+\varepsilon)J(0).\end{aligned}$$

If

$$f(x,u)u-(2+\varepsilon)F(x,u)\geqslant -c^2 \quad \text{for all positive } u,\ \varepsilon>0, \tag{3.3}$$

then by Poincaré's inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2\geqslant(2+\varepsilon)\int_0^t\|u_t(s)\|^2\,\mathrm{d}s+\frac{\varepsilon\lambda_1}{2}\|u(t)\|^2-c^2|D|-(2+\varepsilon)J(0). \tag{3.4}$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2\geqslant\varepsilon\lambda_1\|u(t)\|^2-k \quad \text{where } k=2c^2|D|+2(2+\varepsilon)J(0)$$

and, thus,

$$\|u(t)\|^2\geqslant\exp\{\varepsilon\lambda_1 t\}\left\{\|u_0\|^2-\frac{k}{\lambda_1\varepsilon}\right\}+\frac{k}{\lambda_1\varepsilon}.$$

Since ε is positive, it then follows that $\|u(t)\|^2$ becomes arbitrarily large if $k<0$. Denote $M(t):=\int_0^t\|u(s)\|^2\,\mathrm{d}s$. For t sufficiently large, the previous estimates together with Schwarz's inequality imply

$$M(t)M''(t)\geqslant(1+\varepsilon/2)(M'(t)-M'(0))^2\geqslant(1+\varepsilon/8)M'(t)^2,$$

which shows that $M^{-\varepsilon/8}(t)$ is concave. By an elementary argument, $M(t)$ must blow up in finite time.

This argument was also used for discussing under what conditions global solutions exist which are not unifomly bounded, a question raised by [13]. From these considerations it is important to remember the following

Blowup criterion 4. *Assume* (3.3). *If*

$$(2+\varepsilon)\left(\frac{1}{2}\|\nabla u_0\|^2-\int_D F(x,u_0)\,\mathrm{d}x\right)+c^2|D|<0$$

for some t_0, *then blowup occurs in finite time.*

4. Fujita's phenomenon

Assumption. Throughout this section we shall assume that f is nonnegative.

Definition 2. Problem (1.1)–(1.2) has the Fujita property, if every nontrivial solution blows up in finite time.

Heuristically, it is clear that for the Fujita property to hold, not only the growth of f at infinity plays a crucial role, but also its behavior near zero. It has to be sufficiently large, to prevail the dissipative effect of the Laplace operator on solutions with small initial data.

As already mentioned in the Introduction Fujita [17, 18] was the first to show that the Cauchy problem with $f(u)=u^p$, $p\in(1,1+2/N]$ has this property. Extensions in various directions have been derived since. The Fujita property has been established in the following cases:

1.
$$D=\mathbb{R}^N, \quad f=|x|^\sigma t^s u^p, \ s>0, \ \sigma>\begin{cases}-1 & \text{if } N=1,\\ -2 & \text{if } N\geqslant 2,\end{cases}$$
and $p\in(1,1+(2+\sigma+2s)/N]$ (cf. [33]).
2. The result in 1 remains true in exterior domains with compact boundary; and see [43, 46] where also the degenerate case is treated.
3. $D=\{x: x\neq 0, \ x/|x|\in\Omega\subset\{|x|=1\}\}$ is a cone with vertex at 0, $f(u)=u^p, 1<p\leqslant 1+2/(N+\gamma)$ where γ is the positive root of $\gamma(\gamma+N-2)=\omega$, ω being the first eigenvalue of the Laplace–Beltrami operator in Ω with Dirichlet boundary conditions (cf. [3]).
4. $D=D_1\times D_2$, $f(u)=u^p$ and p_i^* are the critical exponents such that the Fujita property holds in D_i for $1<p<p_i^*$ and fails for $p>p_i^*$. Then the critical exponent for D satisfies $\{p^*(D_1\times D_2)-1\}^{-1}=\{p_1^*-1\}^{-1}+\{p_1^*-1\}^{-1}$ [39].
5. D is bounded, $f(u)=\mu g(u)$ where g is convex, monotone, and $g(0)>0$. Let $\mu*>0$ be the critical value such that the stationary problem $\Delta v+\mu g(v)=0$, $v=0$ on ∂D has solutions only if $\mu<$ (or $\leqslant$) $\mu*$. Then for $\mu>\mu*$ the corresponding parabolic problem has the Fujita property [28].

5. Qualitative results

5.1. Influence of the geometry

As already observed blowup is more likely to occur in large domains and for large initial values. It turns out that the shape of the domain D is also crucial. In fact, the following statement is true.

Assume f to be convex, nondecreasing and $f(0)>0$, and let $u_0(x)=0$. Then among all domains D of given volume the blowup time is smallest for the ball.

This can be shown by means of symmetrization techniques [2]. The heuristic reason is that the boundary prevents the solution from blowing up. In the sphere the center where blowup occurs is "far away" from it.

Narrow domains however always possess global solutions even if their volume is big. The proof is based on constructing a global upper solution in a strip. Further results in this direction have been obtained in [40] as a consequence of a maximum principle.

5.2. Blow-up points

Definition 3. A point $x_0\in D$ is called a blow-up point if there exists sequences $x_n\in D$, $t_n\in(0,T)$ such that

$$x_n\to x_0, \quad t_n\to T, \quad u(x_n,t_n)\to\infty \quad \text{as } n\to\infty.$$

It is in general difficult to localize the blow-up points unless the solution preserves the shape of the initial condition. It has been shown in [16], that if D is a ball, $u_0=u_0(|x|)$ is radially decreasing and some additional assumptions on f that the center is the only blow-up point (see also [20]). If the

problem possesses some symmetry it is also reflected in the blowup set. The moving plane method of Gidas et al. [22] is applicable to parabolic equations. Extensions to unbounded domains are still open. If D is unbounded, blow-up at infinity cannot be excluded.

Giga and Kohn [23] proved that *for* $D=\mathbb{R}^N$, $f(u)=u^p$ *with* $p<(N+2)/(N-2)$ *if* $N>2$ *and* $u_0 \in H^1(\mathbb{R}^N)$, *then the set of blow-up points is compact.* Also, in cones it is not known if the blow-up points can tend to the vertex or if they have to stay away from the boundary. This question is still open for bounded domains and general nonlinearities. Some partial results are found in [16] and in [23]. In particular, it is shown that for power-like nonlinearities and for star-shaped domains blowup points cannot lie on the boundary.

5.3. Asymptotic shape

There are few results concerning the behavior of the solutions near the blow-up points. A seminal study is by Giga and Kohn [23]; see also the references cited there. One of their main results is that if u is a solution of (1.1)–(1.3), $f=|u|^{p-1}u$, $p<(N+2)/(N-2)$ and D is star-shaped then a is a blowup point if and only if

$$\lim_{t\to T}(T-t)^{1/(p-1)}u(a+y(T-t)^{1/2},t)=(p-1)^{-1/(p-1)} \text{ uniformly in } |y|\leqslant C.$$

The results of [28] for the Cauchy problem indicate that the asymptotic behavior near a blow-up point is governed in first order approximation by the reaction problem. The case of an exponential function has been treated in [7]. For general domains and general nonlinearities the exact asymptotic behavior is still open.

5.4. Continuation after blowup

In order to study possible continuation after blowup a natural concept of a generalized solution has to be introduced. A possibility is to consider partial blowup.

Definition 4. Let $u(x,t)$ blowup at time $t=T$, i.e. $\lim_{t\to T}\|u(t)\|_\infty=\infty$. The blowup will be called incomplete if there exists $x\in D$ such that (cf. (2.2))

$$u(x,t)=S(t)u_0+\int_0^t S(t-s)f(u)\,\mathrm{d}s<\infty \quad \text{for some } t>T.$$

If no such x exists then there is complete blowup.

In bounded domains complete blowup takes place for large classes of problems with bounded initial conditions. Baras and Cohen [6] proved complete blowup for reactions of the form $f(u)\sim u^p$ as $u\to\infty$ and $1<p<(N+2)/(N-2)$. Lacey and Tzanetis [29] investigated more general nonlinearities. For another, more general definition of complete blowup compare [21]. In this paper Galaktionov and Vázquez [21] studied the Cauchy problem and extended the results to slow diffusion problems. The question of continuation after blowup is very much related to the existence of local solutions with singular initial data. It should be mentioned that the problem $\Delta u+u^p=0$ possesses for $1<p<(N+2)/(N-2)$ infinitely many positive, radial solutions in a ball, which vanish

on the boundary. It turns out that the singularities are of the form $u \sim |x|^{-(N-2)}$ if $1 < p \leqslant N/(N-2)$, or $u \sim |x|^{-2/(p-1)}$ if $N/(N-2) < p < (N+2)/(N-2)$, cf. [5] for a classification of all radial solutions. The singular solutions solve the parabolic problem in a weak sense for all times. In particular, if $u_0 \in L^q(D)$, $q > N/2(p-1)$ and $q \geqslant p$, then (1.1)–(1.3) has a unique solution in $C([0,T]; L^q(D))$.

6. Numerical analysis of blowup

6.1. Time-stepping methods

For ease of reference we recall the semidiscretized problem (1.8),

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} + A_h U(t) = F_h(U(t)), \quad t > 0, \quad U(0) = U_0 \in \mathbb{R}^M, \tag{6.1}$$

where $A_h \in \mathscr{L}(\mathbb{R}^M)$ is positive definite, with largest eigenvalue tending to $+\infty$ as $M \to \infty$; $M \gg 1$ denotes the number of the mesh points D_h lying in D.

The paper by Stuart and Floater [73] gives an illuminating account of why numerical methods using a fixed stepsize are not appropriate when solving nonlinear ODEs whose solutions blow up in finite time. That the choice of the time steps $\{\tau_n\}$ in the discretization of the system of nonlinear ODEs (6.1) is governed by the growth of the analytical solution had already been recognized in Nakagawa's pioneering paper [68] of 1976 and subsequently in [69, 75] and [55] (see also the thesis [56]). Nakagawa solves (1.5)–(1.7), with $N=1$, $D=(0,1)$, $f(u)=u^2$, with a uniform spatial mesh ($h=1/(M+1)$) and using the *explicit* Euler method (i.e. (1.10) with $c_1=1$) for the time integration of the system (6.1):

$$V_{n+1} = (I - \tau_n A_h) V_n + \tau_n F_h(V_n) \quad (n \geqslant 0), \tag{6.2}$$

where $V_n = (V_{1,n}, \ldots, V_{M,n})^{\mathrm{T}}$ denotes the vector with the approximations to the exact values $u(x_1, t_n), \ldots, u(x_M, t_n)$). Let $\|V_n\|_q := (\sum_{j=1}^{M} h(V_{j,n})^q)^{1/q}$ $(q \geqslant 1)$, and suppose that the exact solution blows up as $t \to T_{\mathrm{b}}^- < \infty$.

Nakagawa's time-stepping criterion [68]. *Consider* (1.5)–(1.7) *with* $N=1$, $f(u)=u^2$, *and assume that its solution blows up in finite time* T_{b}. *Let* $\lambda := \tau/h^2 \in (0, \frac{1}{2}]$ *be fixed, and choose the time steps* $\{\tau_n\}$ *by*

$$\tau_n = \tau \cdot \min\left\{1, \frac{1}{\|V_n\|_2}\right\} \quad (n \geqslant 0). \tag{6.3}$$

If

$$\hat{T}_{\mathrm{b}} = \hat{T}_{\mathrm{b}}(\tau) := \lim_{n \to \infty} \sum_{j=0}^{n-1} \tau_n, \quad \textit{then} \quad \lim_{\tau \to 0^+} \hat{T}_{\mathrm{b}}(\tau) = T_{\mathrm{b}}.$$

The proof of this first convergence result for the numerical blow-up time is based on a discrete analogue of the Kaplan–Fujita–Tsutsumi criterion; a similar technique is employed in [69] and [55]. The results of Nakagawa's 1976 paper were extended in a number of ways (in all these cases, the underlying spatial mesh is fixed a priori and is uniform). In [69] the nonlinearity is assumed to

satisfy $f(u) \geqslant Cu^{1+\alpha}$ $(\alpha > 0)$, and spatial discretization is by a finite element method of lumped mass type; Euler's explicit method (6.2) is used for time stepping. While [75] closely follows Nakagawa [68], Chen's paper [55] introduces a number of new features: we still have $N=1$, but both Dirichlet and Neumann boundary conditions are studied, and the *linearly implicit* version of the implicit Euler method ((1.10) with $c_1=1$) is used for solving (6.1):

$$(I + \tau_n A_h)V_{n+1} = V_n + \tau_n F_h(V_n) \quad (n \geqslant 0). \tag{6.4}$$

Note that, by (1.8) and the remark following it, A_h is positive definite and hence $(I+\tau_n A_h)^{-1}$ exists for all $\tau_n > 0$. For $f(u)=u^{1+\alpha}$ $(\alpha > 1)$, we have (using the notation introduced above):

Chen's time-stepping criterion [55]. *Suppose the solution of* (1.5)–(1.6), *with* $N=1$ *and* $f(u)=u^{1+\alpha}$ $(\alpha > 1)$, *and with either (homogeneous) Dirichlet or Neumann boundary conditions, blows up in finite time* T_b, *and assume that* $\lambda := \tau/h^2$ *is fixed. If the time steps are chosen according to*

$$\tau_n = \tau \cdot \min\left\{1, \frac{1}{\|V_n\|_q^{\alpha}}\right\} \quad (n \geqslant 0), \quad \textit{then} \lim_{\tau \to 0^+} \hat{T}_b(\tau) = T_b. \tag{6.5}$$

Here, the parameter q *is taken as* $q=2$ *in the case of the Dirichlet condition, and* $q=1$ *for the Neumann condition.*

Chen goes much further: numerical blowup takes place in some sharply defined sense. Since exact (single-point) blowup is assumed to occur at the point $x_b = \frac{1}{2}$ of $D=(0,1)$, let the mesh point x_{m_b} coincide with x_b. It is shown in particular that the following results hold:

(a) *If* $\alpha > 1$, *then the computed solution* V_n *remains bounded except at the mesh point* x_{m_b}.

(b) *If* $0 < \alpha \leqslant 1$, *then the computed solution blows up at* x_{m_b} *and also at the adjacent mesh points* $x_{m_b \pm 1}$. Consider now the *fully implicit* collocation method (1.9), (1.10) with $\frac{1}{2} \leqslant c_1 \leqslant 1$ (which is dictated by the stiffness of the matrix $-A_h$). In order to compute the solution of this continuous implicit one-stage Runge–Kutta method for (6.1), we must solve a system of M nonlinear algebraic equations at each time level $t=t_n$. This will usually be done by direct fixed-point iteration (cf. [50]) or by a modified Newton iteration. As was pointed out by Stuart and Floater [73], time-stepping based on a fixed step $\tau > 0$ will, as $\|V_n\|$ becomes large, lead to a loss of uniqueness, or even existence, of the solution V_{n+1}; hence, the time step $\tau_n = t_{n+1} - t_n$ in (1.10) must be chosen such that the mapping from $\mathbb{R}^M$ to $\mathbb{R}^M$ that defines V_{n+1} is contractive. Let $\rho_n := \max_{1 \leqslant m \leqslant M} |V(P_m, t_n)|$, denote by $L=L(Q)$ the Lipschitz constant of f in $[0,Q]$, and consider the ball $B(\beta\rho_n) := \{w \in \mathbb{R}^M : |w| < \beta\rho_n\}$ for a given $\beta > 1$. It was shown in [50] that if the time steps $\{\tau_n\}$ in (1.10) satisfy the

Time-stepping criterion for one-point collocation:

$$\tau_n < \min\left\{\frac{(\beta-1)\rho_n}{c_1(4N\beta\rho_n/h^2 + f(\beta\rho_n))}, \frac{1}{c_1(4N/h^2 + L(\beta\rho_n))}\right\}, \tag{6.6}$$

$(n \geqslant 0)$, then the above mapping from $B(\beta\rho_n)$ to $B(\beta\rho_n)$ corresponding to fixed-point iteration based on (1.10) is a contraction. (The "optimal" choice of β is discussed in [50].) We note that a similar criterion was obtained in [62, 63] (compare also [44]) for an implicit (nonlinear) finite-difference method applied to the initial-boundary-value problem for

$$u_t - (\partial^2/\partial x^2)u^{\sigma+1} = u^p \quad (\sigma > 0,\ p > 1).$$

The analysis in [50] can be used constructively to obtain a robust numerical method for computing blow-up (in particular, convergent approximations to the blow-up time T_b) in $\mathbb{R}^N$ $(N \geqslant 1)$ not only

for (1.5) but also for more general equations (1.1) containing (linear) convection terms and constant delays [86].

The above results (6.3), (6.5), and (6.6) give rise to the question as to whether time-stepping for the semidiscretized problem (6.1) should be done by an *explicit method* (where the time steps τ_n are also constrained by the (unbounded) stiffness of the matrix $-A_h$; compare also [57, 58] which employs a three-stage explicit Runge–Kutta method of order two possessing an extended interval of stability), or by an *implicit (A-stable) method.* Note that the degree of stiffness in $-A_h$ does not change as t increases, while in an implicit method the contractivity constraint on τ_n (see (6.6)) becomes more and more severe as $U(t)$ becomes large. It is clear that a better understanding of this problem is needed (but see also [70] and Section 6.2 below). It may well turn out that an unconventional approach, along the lines of the adaptive time-stepping techniques of [59–61] based on error control via a posteriori error estimates for the *original* problem (1.1)–(1.3) (or (1.5)–(1.7)), will contain much of the key to this problem.

When does the solution $U(t)$ of the *semidiscretized problem* (6.1) blowup in finite time? This question is studied in [49] where, for the problem (1.5)–(1.7) with $N=1$, necessary and sufficient conditions for finite-time blow of the semidiscrete solution $U(t)$ are derived. However, the blow-up dynamics of the original problem (1.5)–(1.7) is in general quite different from that of its semidiscretized version corresponding to a given (fixed) spatial mesh D_h. As a simple illustration, let $N=1$, $D=(0,1)$, $u_0(x)=A\sin(\pi x)$ $(A>0)$, and choose the mesh D_h with $M=1$. For $f(u)=u^p$ $(p>1)$, (6.1) then reduces to the scalar initial-value problem

$$\dot{U}(t)+(2/h^2)U(t)=U^p(t), \quad t>0, \quad U(0)=U_0=A, \tag{6.7}$$

where $h=1/2$ and so $U(t)$ approximates $u(1/2,t)$.

Blowup for a semidiscrete problem. *Assume that $\lambda_h>0$, $\varepsilon>0$, and $U_0>0$. Then the solution of the (scalar) initial-value problem*

$$\dot{U}(t)+\lambda_h U(t)=\varepsilon U^p(t), \quad t>0, \quad U(0)=U_0,$$

blows up in finite time if, and only if, $U_0>(\lambda_h/\varepsilon)^{1/(p-1)}$. The blow-up time is then given by

$$T_b^h=\frac{1}{-\lambda_h(p-1)}\ln\left(1-\frac{\lambda_h}{\varepsilon U_0^{p-1}}\right).$$

Thus, the solution of the semidiscretized initial-value problem (6.7) blows up in finite time if, and only if, $U_0=A>(2/h^2)^{1/(p-1)}=8^{1/(p-1)}$. On the other hand, the solution of the original problem (1.5)–(1.7) with the above data and $p=2$ blows up when A reaches a value near $A=11.6$ (see [68, 77]), compared with $A=8.0$ for (6.7). For D_h with $M=20$, the blowup threshold for (6.1) is near $A=11.5$ [77].

This simple example is an indication that the *numerical detection* of blowup is a difficult and essentially unresolved problem, even for very simple model problems, and many challenging questions remain to be answered. (See also [76] for a different approach to blowup detection which is based on certain extrapolation and least-squares techniques.)

The major drawback of the discretization approaches discussed in this section are that these methods are based on uniform spatial meshes D_h (mostly with $N=1$) and that their design does not take into account the local structure of the exact blowup solution as it approaches its (local)

singularity (recall Section 5.3). Since the late 1980s several approaches have been proposed which attempt to rectify this situation (see [51, 57, 58, 54] and their references to earlier work on mesh adaptation). We shall briefly describe these methods of static and dynamic "regridding" in the following section.

6.2. *Adaptive mesh refinement*

Solutions to the parabolic equation

$$u_t - u_{xx} = u^p \ (p>1), \quad u(0,t)=u(1,t)=0, \quad u(x,0)=u_0(x)>0, \tag{6.8}$$

possess a *scale-invariance property*: if $u=u(x,t)$ is a solution, then so is $u_\gamma(x,t) := \gamma^{2/(p-1)}u(\gamma x, \gamma^2 t)$ for any $\gamma>0$. This observation forms the basis of the *rescaling method* of Berger and Kohn [51]. The method is similar to a mesh refinement technique (using multiple meshes) used for first-order hyperbolic systems in one spatial dimension (see [51] for precise references). Assuming that the initial function $u_0(x)>0$ is symmetric on $D=(-1,1)$, with $xu_0'(x)<0$, we have single-point blowup at $x_b=0$. Using the standard spatial semidiscretization on a uniform grid D_h, the solution of the resulting system of ODEs (6.1) is advanced (with uniform timestep) until $\|V_n\|_\infty$ reaches a given threshold $M_b>0$. For a scale factor λ chosen so that $\lambda^{-1}>1$ is a small integer, the solution is now rescaled, using $\gamma=\lambda$ in (6.8); since, this scaling stretches both the spatial and the temporal mesh, it is necessary to refine the mesh by introducing new mesh points, and to generate the necessary boundary conditions. The same time-stepping method (the explicit Euler method: (1.10) with $c_1=0$) is then used when the process is continued iteratively, with $\gamma \in \{\lambda^2, \lambda^3, \ldots\}$. The original approximate solution and the new one are advanced independently, each on its own grid.

This method of *static regridding* yields, for the above one-dimensional problem with single-point blowup, a very powerful algorithm, as shown by many numerical experiments. However, the underlying convergence and error analysis appears to be very complex and is yet to be established.

The *space–time structure* of the (approximate) *self-similar solutions* to higher-dimensional problems with radial symmetry and single-point blowup,

$$u_t - \frac{1}{r^{N-1}}(r^{N-1}u_r)_r = f(u), \qquad u(r,0)=u_0(r) \geqslant 0,$$

with $f(u)$ of the form $(1+u)\ln^p(1+u)$, $(\lambda+u)^p$, or $\exp(u)$ $(p>1, \lambda \geqslant 0)$ was exploited in [57, 58] to obtain a mesh refinement procedure that is consistent with this space–time structure (compare also the moving mesh method of [54] described below in which the scaling invariance is inherited from the original problem): the spatial steplength (in r) is chosen so that it corresponds to a uniform mesh with respect to the similarity variable i (see also Section 5.3 of the present paper). The resulting algorithm – finite-element discretization in space and time-stepping using an explicit three-stage Runge–Kutta method of order two with extended stability interval – shows an impressive performance in numerous numerical experiments [58]. However, as in the case of the rescaling method of Berger and Kohn, the theoretical foundation (convergence and error analysis) is still lacking.

Current work on the computation of blow-up solutions in parabolic problems (1.1)–(1.3) focuses on *moving mesh methods* (dynamic regridding) in which the spatial mesh is generated by appropriately chosen moving mesh PDEs (see, for example, [64–67]). So far, the use of these methods

has been limited to one-dimensional problems, and once again the underlying theory is not well yet understood. Consider again the model problem (6.8). It has been known for some time (see [54] and the references to the pioneering work of Dold and others) that the evolution of the (single) blow-up peak for the solution $u(x,t)$ of (6.8) near (x_b, T_b) can be characterized in terms of the so-called *ignition kernel*,

$$\mu = \mu(x,t) := (x - x_b)[(T_b - t)(\alpha - \ln(T_b - t))]^{-1/2},$$

where α denotes a constant depending on the given initial function u_0.

Theorem (Budd et al. [54]). *If $x = x(t)$ is chosen to keep the ignition kernel $\mu(x,t)$ constant, then the solution to* (6.8) *satisfies*

$$(T_b - t)^{\beta} u(x,t) \to \beta^{\beta} \left(1 + \frac{\mu^2}{4p\beta}\right)^{-\beta} \quad \text{as } t \to T_b^-,$$

with $\beta := 1/(p-1)$. Moreover, if $|x - x_b|$ is small but fixed and independent of t, then

$$u(x,t) \to u(x,T_b) := \left(4p\beta^2 \frac{|\alpha - 2\ln|x - x_b||}{|x - x_b|^2} \cdot (1 + \mathcal{O}(|x - x_b|^2))\right)^{\beta},$$

as $t \to T_b^-$.

In [54] numerical methods for (6.8) are designed so as to exploit the local symmetries of the solution (the method inherits the scaling invariance of the given problem), leading to the appropriate choice of the (local) mesh: for t close to T_b^- and x close to the blow-up point x_b, the moving mesh

$$x_i := x(i/M, t), \quad i = 0, 1, \ldots, M,$$

is placed at those points for which $\mu(x,t)$ is constant. This mesh is defined by a differentiable mesh transformation $x(\xi,t) := DD[0,1] \to [0,1]$, with x representing the physical coordinate and ξ being the computational one, subject to the constraint $\partial x/\partial \xi > 0$ (to avoid mesh crossings). In the MMPDE approach, a partial differential equation for $x(\xi,t)$ is solved simultaneously with the original problem (6.8). This MMPDE is based on an *equidistribution principle*: for a given monitor function $\Phi = \Phi(\xi,t)$ one requires that

$$\int_0^{x(\xi,t)} \Phi(z,t)\,\mathrm{d}z = \xi \int_0^1 \Phi(z,t)\,\mathrm{d}z, \quad x \in [0,1],$$

or, equivalently,

$$\frac{\partial}{\partial \xi}\left(\Phi(x(\xi,t),t)\frac{\partial}{\partial \xi}x(\xi,t)\right) = 0, \qquad x(0,t) = 0, \quad x(1,t) = 1 \ (t > 0).$$

The choice of the moving mesh PDE (and hence that of the monitor function Φ), as well as the way it is solved numerically, are clearly crucial for the success of this approach. Various choices of MMPDEs have been proposed and analyzed; see e.g., [64, 53]).

6.3. Extensions and future work

This brief expository presentation of the various approaches to the computational treatment of rather particular cases of blow-up problems for (1.1)–(1.3) reveals that the present state of the art encompasses only simple situations: most methods are for one-dimensional problems ($N=1$), and they assume single-point blowup, with known blow-up point $x_b \in D$. For all of the methods using variable spatial and temporal meshes (e.g. [51, 57, 58, 54]), a convergence analysis and computable error bounds (a posteriori error estimates) for the blow-up time T_b have yet to be found.

Most blowup problems arising in realistic modelling processes are not yet covered by the existing numerical methods. We conclude by listing a representative selection of such problems whose solution will lead to a better understanding of the numerical analysis of blowup.

- The general problem (1.1)–(1.3) (for example, PDEs with nonlinear gradient terms; compare [12, 4]);
- Blow-up problems on bounded $D \subset \mathbb{R}^N$ with $N=2,3$, including problems with multiple blowup points (cf. Section 5.2);
- Blow-up problems on unbounded domains $D \subset \mathbb{R}^N$ (for example, on cones, with $N=2,3$; see Section 5.2);
- Blow-up problems with nonlocal (nonlinear) boundary conditions (see [85]) or nonlocal reaction terms ([79–84, 91, 92]), as well as nonlinear Volterra integral equations with blowup solutions ([89, 90]).

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