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Lower Bounds for Some Ramsey Numbers

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Let $n, r, u_1, u_2, \ldots, u_k$ be positive integers satisfying $u_i \geqslant r$ for $i = 1, 2, \ldots, k$. The symbol $n \to (u_1, u_2, \ldots, u_k)^r$ means that, for any partition of the r-subsets of an n-set S into k classes $C_1, C_2, \ldots, C_k$, there is a u_i-subset of S all of whose r-subsets belong to C_i for some i, $1 \leqslant i \leqslant k$. A theorem of F. P. Ramsey asserts that, if $r, u_1, u_2, \ldots, u_k$ are given, then $n \to (u_1, u_2, \ldots, u_k)^r$ for all sufficiently large n. $n \not\to (u_1, u_2, \ldots, u_k)^r$ denotes the negation of $n \to (u_1, u_2, \ldots, u_k)^r$. In this paper a number of results of the form $n \not\to (u_1, u_2, \ldots, u_k)^3$ are obtained.

1. INTRODUCTION

If S is a set, S denotes the cardinality of S, and, if r is a positive integer, [S]' is the family of all r-subsets of S. $[S]^r = (C_1, C_2, \ldots, C_k)$ denotes a partition of $[S]^r$ into k families $C_1, C_2, \ldots, C_k$. Let $n, r, u_1, u_2, \ldots, u_k$ be positive integers satisfying $u_i \geqslant r$ for $i = 1, 2, \ldots, k$. The symbol

$$n \to (u_1, u_2, \ldots, u_k)^r \tag{1}$$

means that, if S is an n-set, then for every partion $[S]^r = (C_1, C_2, \ldots, C_k)$ there is a q-subset U_i of S such that $[U_i]^r \subseteq C_i$ for some i. It is a well-known theorem of F. P. Ramsey [8] that, for given $r, u_1, u_2, \ldots, u_k$, (1) holds for all sufficiently large n. The least such n is called a Ramsey Number. The symbol

$$n \not\to (u_1, u_2, \ldots, u_k)^r \tag{2}$$

will be used to denote the negation of (1). Whenever $u_i = u$ for all i, (1) and (2) will be written as $n \to (u)_k^r$ and $n \not\to (u)_k^r$, respectively. By a $(u_1, u_2, \ldots, u_k)^r$ partition we shall mean a partition which establishes (2).

The case r = 2 has been studied extensively by many authors. In this paper we shall be concerned with the case $r = 3$, although most of our results extend with only minor modifications to the case $r > 3$. First we mention briefl some of the known results. P. Erdös [3] has proved by probabilistic methods that, if $\binom{n}{u} \leqslant 2^{\binom{u}{2}-1}$, then $n \mapsto (u)_2^2$. It has been pointed out by J. G. Kalbfleisch [6] (see also [2] and [7]) that the argument used by Erdös can be generalized so as to give the following result:

$$\text{if} \quad \sum_{i=1}^{k} \binom{n}{u_i} k^{-\binom{u_i}{r}} \leqslant 1, \qquad \text{then} \quad n \mapsto (u_1, u_2, \ldots, u_k)^r.$$

In particular, if $r = 3$ and $k = 2$ one gets, via a simple computation (c is an absolute constant[1]),

$$n \mapsto ([c(\log n)^{1/2}])_2^3, \tag{3}$$

and for arbitrary k one gets

$$n \mapsto \left(\left[c\left(\frac{\log n}{\log k}\right)^{1/2}\right]\right)_k^3. \tag{4}$$

We remark that, by using the methods of [I], (4) can be improved to

$$n \mapsto \left(\left[c\left(\frac{\log n}{k}\right)^{1/2}\right]\right)_k^3. \tag{5}$$

In [6], Kalbfleisch proved the following theorem:

Theorem 1 (Kalbfleisch). *If* $n \mapsto (u, v-1)^3$ and $m \mapsto (u-1, v)^3$, *then* $n + m \mapsto (u, v)^3$.

From Theorem 1 and the result $12 \mapsto (4)_2^3$ of J. R. Isbell [5], Kalbfleisch deduced that

$$6\binom{u+v-8}{u-4} + \binom{u+v-4}{u-2} \mapsto (u, v)^3. \tag{6}$$

One can check that, for small values of u and v, (6) is superior to the result of Erdös and, moreover, has the advantage of being constructive. However, simple calculations show that Erdös's result is stronger when u and v are large. For example, (6) gives only $n \mapsto ([c \log n])_2^3$.

In Section 2 of this paper we prove some theorems which, when combined with Theorem 1 of Kalbfleisch and the result of Isbell, yield results which are stronger than (6) for $u, v \geqslant 4$ (except in the case $v = 4$, $u = 4, 5, 6$). In Section 3 we consider the case $r = 3$, k large, and obtain a result which is substantially better than (5).

[1] The letter c will be used throughout to denote absolute constants. The numerical value of c will not necessarily be the same at each occurrence.

2. The Case $r = 3, k = 2$

Theorem 2. *If $n \mapsto (u)_2^3$ and $m \mapsto (v)_2^3$, then $mn \mapsto (u + v \quad 2)_2^3$*

Proof. Let S be an n-set. We may suppose without loss of generality that $S = \{1, 2, ..., n\}$. Let $[S]^3 = (C_1, C_2)$ be a $(u)_2^3$ partition of $[S]^3$. Let $T = \bigcup_{j=1}^{m} S_j$ be an mn-set consisting of n disjoint m-sets $S_j = \{x_{j1}, x_{j2}, ..., x_{jm}\}$. Let $S \cap T = \varnothing$ and let $[S_j]^3 = (C_1^{(j)}, C_2^{(j)})$ be a $(v)_2^3$ partition of $[S_j]^3$. Let $[T]^3 = (D_1, D_2)$ where D_1 and D_2 are determined as follows: Consider $A = \{x_{\alpha\beta}, x_{\gamma\delta}, x_{\lambda\mu}\} \in [T]^3$. If $\alpha = \gamma = \lambda$ (in which case β, δ, and μ are distinct and $A \in [S_\alpha]^3$) put A in D_1 or D_2 according as A is in $C_1^{(\alpha)}$ or $C_2^{(\alpha)}$. If α, γ, and λ are distinct (in which case $\{\alpha, \gamma, \lambda\} \in [S]^3$) put A in D_1 or D_2 according as $\{\alpha, \gamma, \lambda\}$ is in C_1 or C_2. If exactly two of α, γ, and λ are equal, say $\alpha = \gamma \neq \lambda$, put A in D_1 or D_2 according as $\alpha > \lambda$ or $A < \lambda$.

Suppose there were a $(u + v - 2)$-subset R of T such that $[R]^3 \subseteq D_1$ or $[R]^3 \subseteq D_2$. It follows from the above construction that $R \cap S_j \leqslant v - 1$ for $j = 1, 2, ..., n$. Now consider, for any α and β, $\alpha > \beta$, $x_{\alpha\gamma}, x_{\alpha\delta} \in S_\alpha$ and $x_{\beta\lambda}, x_{\beta\mu} \in S_\beta$. Then by our construction $\{x_{\alpha\gamma}, x_{\alpha\delta}, x_{\beta\lambda}\} \in D_1$ and $\{x_{\alpha\gamma}, x_{\beta\lambda}, x_{\beta\mu}\} \in D_2$; so we may therefore suppose that $R \cap S_j \leqslant 1$ for all except possibly one value of j. This means that there exist at least u distinct numbers $\alpha_1, \alpha_2, ..., \alpha_u$ such that $R \cap S_{\alpha_1} \leqslant v - 1$ and $R \cap S_{\alpha_j} = 1$ for $j = 2, 3, ..., u$. However, this would imply that there is a u-subset U of S (namely, $U = \{\alpha_1, \alpha_2, ..., \alpha_u\}$) such that $[U]^3 \subseteq C_1$ or $[U]^3 \subseteq C_2$, and this is a contradiction. This completes the proof of Theorem 2.

Theorem 3. *If $m \mapsto (v)_2^3$, then $3m \mapsto (v + 1)_2^3$.*

Proof. By letting $n = u = 3$ in Theorem 2 (and suppressing the hypothesis $n \mapsto (u)_2^3$), the same proof works if the partition (D_1, D_2) is modified thus: when exactly two of α, γ, λ are equal, say $\alpha = \gamma \neq \lambda$, put A into D_1 if (α, λ) equals $(1, 2)$, $(2, 3)$, or $(3, 1)$ and into D_2 otherwise.

We remark that the arguments used to prove Theorems 2 and 3 can be used to prove the following two results. We do not present the details.

Theorem 4. *If $n \mapsto (u_1, u_2)^3$ and $m \mapsto (v_1, v_2)^3$, then*

$$mn \mapsto (u_1 + v_1 - 2, u_2 + v_2 - 2)^3.$$

Theorem 5. *If $n \mapsto (u, v)^3$, then $3n \mapsto (u + 1, v + 1)^3$.*

Our next theorem will enable us to establish results of the form $n \mapsto (u, v)^3$ when v is small compared to u.

THEOREM 6. *If $n \mapsto (u, v)^3$ and $m \mapsto (w, v)^3$, then*

$$mn \mapsto ((u-1)(w-1)+1, v)^3.$$

Proof. Let S and T be as in the proof of Theorem 2 with (C_1, C_2) a $(u, v)^3$ partition and each $(C_1^{(j)}, C_2^{(j)})$ a $(w, v)^3$ partition. Define $[T]^3 = (D_1, D_2)$ as before for triples A with α, γ, λ all alike or all distinct. If two of α, γ, λ are equal put A in D_1. It is now a straightforward matter to verify that (D_1, D_2) is a $((u-1)(w-1)+1, v)^3$ partition of $[T]^3$. We leave the details to the reader.

We list in Table I some results of the form $n \mapsto (u, v)^3$ that can

TABLE 1

u	v	n	best previous n
4	4		12
5	4	—	16
5	5	36	32
6	4	—	21
6	5	57	53
6	6	144	107
7	4	36	27
7	5	93	80
7	6	237	186
7	7	456	372
8	4	43	34
8	5	136	114
8	6	373	300
8	7	829	672
8	8	1728	1344

be obtained from our theorems and the results of Kalbfleisch and Isbell. For the sake of comparison we list also the best previous result. The symbol "—" indicates that we have been able to get no improvement. We consider only $4 \leqslant v \leqslant u \leqslant 8$.

3. THE CASE $r = 3$, k LARGE

In this section we shall show that (5) can be substantially improved.

THEOREM 7. *If $u \geqslant 5$ and $n \mapsto (u)_k^3$, then $n^2 \mapsto (u)_{k+1}^3$.*

Proof. Let $S = \{1, 2, \ldots, n\}$ and let $[S]^3 = (C_1, C_2, \ldots, C_k)$ be a $(u)_k^3$ partition of $[S]^3$. Let $T = \bigcup_{j=1}^{n} S_j$ be an n^2-set consisting of n disjoint n-sets $S_j = \{x_{j1}, \ldots, x_{jn}\}$. Let $S \cap T = \varnothing$ and let $[S_j]^3 = (C_1^{(j)}, C_2^{(j)}, \ldots, C_k^{(j)})$ be a $(u)_k^3$ partition of $[S_j]^3$. Let $[T]^3 = (D_1, D_2, \ldots, D_{k+1})$ de defined as follows: Consider $A = \{x_{\alpha\beta}, x_{\gamma\delta}, x_{\lambda\mu}\} \in [T]^3$. If $\alpha = \gamma = \lambda$ (so that $A \in [S_\alpha]^3$) put A in D_i if A is in $C_i^{(\alpha)}$. If a, γ and λ are distinct (so that $\{\alpha, \gamma, \lambda\} \in [S]^3$) put A in D_i if $\{\alpha, \gamma, \lambda\}$ is in C_i. If exactly two of α, γ, and λ are equal, put A in D_{k+1}. It is now a routine matter to verify that $(D_1, D_2, \ldots, D_{k+1})$ is a $(u)_{k+1}^3$ partition of $[T]^3$.

In the case $u = 4$, the construction given in the proof of Theorem 7 does not work. However, the following result holds:

THEOREM 8. *If* $n \mapsto (4)_k^3$, *then* $n^2 \mapsto (4)_{k+2}^3$.

Proof. The construction is the same as in Theorem 7 except that if two of α, γ, and λ are equal, say $\alpha = \gamma \neq \lambda$, put A into D_{k+1} if $\alpha > \lambda$ and into D_{k+2} if $\alpha < \lambda$.

We now show how Theorem 7 can be used to improve (5). By choosing m sufficiently large we have, by (3),

$$m \mapsto ([c(\log m)^{1/2}])_2^3. \tag{7}$$

Henceforth m is fixed. It now follows from (7) and Theorem 7, by induction on k, that

$$m^{2^{k-2}} \mapsto ([c(\log m)^{1/2}])_k^3. \tag{8}$$

Let n be given and let k be defined by $m^{2^{k-3}} < n \leqslant m^{2^{k-2}}$. It then follows from (8) that

$$n \mapsto ([c(\log m)^{1/2}])_k^3.$$

Finally, since

$$\frac{\log n}{2^{k-2}} \leqslant \log m < \frac{\log n}{2^{k-3}},$$

we get

$$n \mapsto \left(\left[c\left(\frac{\log n}{2^k}\right)^{1/2}\right]\right)_k^3. \tag{9}$$

It is clear that (9) is substantially stronger than (5).

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