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Tannaka duals in semisimple tensor categories

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Abstract

Tannaka duals of finite-dimensional Hopf algebras inside semisimple tensor categories are used to construct orbifold tensor categories, which are shown to include the Tannaka dual of the dual Hopf algebras. The second orbifolds are then monoidally equivalent to the initial tensor categories in a canonical fashion. © 2002 Elsevier Science (USA). All rights reserved.

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Introduction

One of the major interests in recent studies of Hopf algebras is based on its use as quantum symmetry, which can be described more or less in terms of the notion of tensor category [3,12,14]. In this respect, finite group symmetry in tensor category is particularly interesting and provides the right place to take out quotients, known as the orbifold construction.

There have been many interesting researches on orbifolds of quantum symmetries, particularly in connection with conformal field theory (see [5,6,11] for example). There are also recent works such as [18,19,29], which deals with the subject related to tensor categories.

In our previous paper [42], we proposed a pure algebraic formulation of orbifolds of tensor categories with respect to finite group symmetry motivated

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by these works of physical interest, which recovers the combinatorial data of orbifolds in concrete examples such as ADE-models (see [12] for more information on the ADE classification).

More precisely, starting with a tensor category bearing a finite group symmetry inside, the associated orbifold is formulated as a tensor category of bimodules with actions of the preassigned symmetry group. When the relevant group is abelian, the dual group appears naturally inside our orbifold tensor category and hence it enables us to take the second orbifold which turns out to be monoidally equivalent to the initial tensor category, a duality for orbifolds in [42].

In the present paper, we shall extend this kind of duality to the symmetry governed by Hopf algebras.

Given a finite-dimensional semisimple Hopf algebra A with its Tannaka dual $\mathcal{A}$ realized inside a semisimple tensor category $\mathcal{T}$, we introduce the notion of $\mathcal{A}$-$\mathcal{A}$ modules in $\mathcal{T}$, which is formulated in terms of the existence of trivializing isomorphisms. In the group (algebra) case, this reflects the absorbing property of regular representations.

The totality of our $\mathcal{A}$-$\mathcal{A}$ modules then turns out to constitute a tensor category $\mathcal{T} \rtimes \mathcal{A}$ with the unit object given by an analogue of the regular representation of A. The notation indicates the fact that it is a categorical analogue of crossed products in operator algebras (see [39] for details). More explicitly, if a Hopf algebra (symmetry) A comes into through a coaction on an operator algebra M, then the crossed (or smash) product algebra $M \rtimes A^*$ and the fixed point algebra M^A are associated so that they act on M in a bimodule fashion. Moreover the $M \rtimes A^*$-M^A bimodule M obtained this way is imprimitive in the sense that $M \rtimes A^*$ and M^A are commutants of each other. The existence of such an imprimitivity bimodule enables us to change the acting algebras for operator-algebraic bimodules from $M \rtimes A^*$ into M^A or from M^A into $M \rtimes A^*$ without modifying the structure of tensor categories (cf. [2] for an algebraic formulation of these facts).

The crossed products vs. fixed point algebras reciprocity of this kind then (when it being suitably translated in terms of pure algebras) allows us to interpret $\mathcal{T} \rtimes \mathcal{A}$ as presenting the orbifold of $\mathcal{T}$ by the dual Hopf algebra A^* (cf. [38]).

The orbifold tensor category $\mathcal{T} \rtimes \mathcal{A}$ in turn admits a canonical realization of the Tannaka dual $\mathcal{B}$ of the dual Hopf algebra A^*, which allows us to take the second orbifold $(\mathcal{T} \rtimes \mathcal{A}) \rtimes \mathcal{B}$ and one of our main results shows the duality $(\mathcal{T} \rtimes \mathcal{A}) \rtimes \mathcal{B} \cong \mathcal{T}$.

In our previous paper [42], we proved this for finite abelian groups by counting the number of simple objects in the second dual $(\mathcal{T} \rtimes \mathcal{A}) \rtimes \mathcal{B}$. Here we shall give a more conceptual proof of duality. The idea has long been known in harmonic analysis of induced representations as imprimitivity bimodules [10,30].

By forgetting the bimodule action of $\mathcal{A}$ on the unit object to one-sided (say, right) $\mathcal{A}$-action, we can make it into a right $\mathcal{B}$-module M with the property of imprimitivity, $M \otimes_{\mathcal{B}} M^* \cong I$ and ${}_{\mathcal{B}}M^* \otimes M_{\mathcal{B}} \cong {}_{\mathcal{B}}I_{\mathcal{B}}$.

If we place M at an off-diagonal corner of a suitable bicategory so that it connects $\mathcal{T}$ and $(\mathcal{T} \rtimes \mathcal{A}) \rtimes \mathcal{B}$, then the duality is obtained quite easily, though it still bears rich information on orbifold constructions.

We notice here that another interesting categorical formulation of imprimitivity bimodules is worked out by D. Tambara [34], where a different notion of categorical module is used to get an imprimitivity bimodule which relates $\mathcal{A}$ and $\mathcal{B}$.

For future applications, we also investigate how the rigidity is inherited under the process of taking orbifolds: if the original tensor category $\mathcal{T}$ is rigid and semisimple, then so is for the orbifold tensor category $\mathcal{T} \rtimes \mathcal{A}$.

Basic assumptions

We shall work with the complex number field $\mathbb{C}$ as a ground field, though any algebraically closed field of characteristic zero can be used equally well.

By a tensor category, we shall mean a linear category with a compatible monoidal structure, which is assumed to be strict without losing generality by the coherence theorem (see [26] for example).

A tensor category is said to be semisimple if $\mathrm{End}(X) = \mathrm{Hom}(X, X)$ is a finite-dimensional semisimple algebra for any object X. Tensor categories in this paper are also assumed to be closed under taking subobjects and direct sums (which is not a real restriction for combinatorial structures): To an idempotent e of $\mathrm{End}(X)$, an object eX (the associated subobject) is assigned so that $\mathrm{Hom}(eX, fY) = f\,\mathrm{Hom}(X, Y)e$ and a finite family $\{X_j\}_{1 \leqslant i \leqslant m}$ of objects gives rise to an object $X_1 \oplus \cdots \oplus X_m$ so that

$$\mathrm{Hom}(X_1 \oplus \cdots \oplus X_m, Y_1 \oplus \cdots \oplus Y_n) = \bigoplus_{i,j} \mathrm{Hom}(X_i, Y_j).$$

The unit object I in a semisimple tensor category is assumed to be simple, i.e., $\mathrm{End}(I) = \mathbb{C}1_I$, without further qualifications.

Let A be a finite-dimensional semisimple Hopf algebra with the associated tensor category $\mathcal{A}$ of finite-dimensional A-modules (the Tannaka dual of A), see [27,36,37,40] for more information on Tannaka duals of Hopf algebras. Since the ground field is assumed to be of characteristic zero, the antipode of A is involutory [22,23] and then $\mathcal{A}$ admit dual objects in an involutory fashion: the accompanied rigidity pairings and copairings are given by the ordinary ones (i.e., those in vector spaces), which we shall denote by $\epsilon_V : V \otimes V^* \to \mathbb{C}$ and $\delta_V : C \to V^* \otimes V$, respectively. Note that, if we denote the transposed morphism of $f : V \to W$ by ${}^t f : W^* \to V^*$, then $(V^*)^* = V$ and ${}^t({}^t f)$. The quantum dimension $d(V)$ of an object V then coincides with the ordinary (vector space) dimension $\dim(V)$. (For the notion of rigidity and related subjects, we refer to [1,3,4,14,28,31] and references therein.)

Recall here that the dual object V^* of V is based on the dual vector space of the underlying vector space of V. Let U, V, and W be objects in $\mathcal{A}$. Our assumption then allows us to identify various triangular vector spaces

$$\mathrm{Hom}(U \otimes V, W), \qquad \mathrm{Hom}(V, U^* \otimes W), \qquad \mathrm{Hom}(V \otimes W^*, U^*),$$

and so on. The connecting isomorphisms are referred to as Frobenius transforms, which are obtained by switching input or output objects by pairings or copairings. By the involutivity of antipodes, we have the coherence for repeated applications of Frobenius transforms (see [41]).

Although our main concerns are centered around tensor categories, the notion of bicategories also comes into as a relevant language to describe categorical bimodules. Recall that a bicategory consists of a class of labels, $\mathcal{A}$, $\mathcal{B}$ and so on (which is considered to be the counterpart of objects in ordinary categories) and a family of categories $\{\mathrm{Hom}(\mathcal{A}, \mathcal{B})\}$ indexed by a pair of labels (which is an analogue of hom-sets in ordinary categories and referred to as hom-categories), which satisfies some reasonable axioms analogous to those for ordinary morphisms (see, for example, [26] for details on bicategories).

In the present paper, we shall adopt somewhat less formal notation (and convention) which makes it easier to trace the resemblance with tensor categories: Instead of $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$, we simply write ${}_{\mathcal{B}}\mathcal{H}_{\mathcal{A}}$. Then, given an object X in ${}_{\mathcal{B}}\mathcal{H}_{\mathcal{A}}$ and another object Y in ${}_{\mathcal{C}}\mathcal{H}_{\mathcal{B}}$, as an analogy to the composition of morphisms, we can associate the third object in ${}_{\mathcal{C}}\mathcal{H}_{\mathcal{A}}$, which is denoted by the notation of tensor product $Y \otimes X$. The operation is also supposed to be applied to morphisms in categories ${}_{\mathcal{B}}\mathcal{H}_{\mathcal{A}}$ so that, given $f : X \to X'$ in ${}_{\mathcal{B}}\mathcal{H}_{\mathcal{A}}$ and $g : Y \to Y'$ in ${}_{\mathcal{C}}\mathcal{H}_{\mathcal{B}}$, we have $g \otimes f : Y \otimes X \to Y' \otimes X'$.

The associativity of the "composition" in bicategory is then described by a completely same way as that of tensor categories: we are privileged to identify double "compositions" $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$ so that it satisfies the pentagonal identity (the coherence condition for triple "compositions").

To recover the original interpretation of hom-categories, given label objects $\mathcal{A}$ and $\mathcal{B}$, express the multiplicative nature of hom-categories in the matrix form

$$\begin{pmatrix} {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}} & {}_{\mathcal{A}}\mathcal{H}_{\mathcal{B}} \\ {}_{\mathcal{B}}\mathcal{H}_{\mathcal{A}} & {}_{\mathcal{B}}\mathcal{H}_{\mathcal{B}} \end{pmatrix}.$$

It is now clear that each ${}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}}$ is an ordinary tensor category and a bicategory of single object (label) is synonymous to a tensor category.

1. Bimodules in tensor categories

Let $\mathcal{T}$ be a semisimple tensor category (closed under taking subobjects and direct sums). By imbedding $\mathcal{T}$ into $\mathcal{T} \otimes \mathcal{V} = \mathcal{V} \otimes \mathcal{T}$ with $\mathcal{V}$ denoting the tensor category of finite-dimensional vector spaces, we can perform the tensor product $X \otimes V = V \otimes X$ of an object X in $\mathcal{T}$ and an object V in $\mathcal{V}$ so that

$$\mathrm{Hom}(X \otimes V, Y \otimes W) = \mathrm{Hom}(X, Y) \otimes \mathrm{Hom}(V, W).$$

Note here that the imbedding $\mathcal{T} \to \mathcal{T} \otimes \mathcal{V}$ gives an equivalence of tensor categories by the semisimplicity assumption on $\mathcal{T}$. We also remark that, given a representative set S of simple objects in $\mathcal{T}$, we have

$$X \cong \bigoplus_{s \in S} s \otimes \mathrm{Hom}(s, X)$$

in $\mathcal{T} \otimes \mathcal{V}$.

Let A be a finite-dimensional semisimple Hopf algebra with the associated tensor category $\mathcal{A}$ of finite-dimensional A-modules and consider a monoidal imbedding $F : \mathcal{A} \to \mathcal{T}$ (F being a fully faithful monoidal functor). Since the tensor category $\mathcal{A}$ admits the canonical Frobenius duality, the same holds for its image under F: we shall denote the accompanied rigidity pairings and copairings by $\epsilon_{F(V)} : F(V) \otimes F(V^*) \to I$ and $\delta_{F(V)} : I \to F(V^*) \otimes F(V)$, respectively.

By a *left $\mathcal{A}$-module* in $\mathcal{T}$ (relative to the imbedding F), we shall mean an object X in $\mathcal{T}$ together with a natural family of isomorphisms $\{\varphi_V : F(V) \otimes X \to X \otimes V\}$ (we forget the A-module structure of V, W and regard them just vector spaces when taking the tensor product with X) satisfying the associativity

$$\begin{tikzcd}
F(V) \otimes F(W) \otimes X \arrow[r, "1\otimes\varphi_W"] \arrow[d, "m^F_{V,W}\otimes 1"'] & F(V) \otimes X \otimes W \arrow[d, "\varphi_V \otimes 1"] \\
F(V \otimes W) \otimes X \arrow[r, "\varphi_{V\otimes W}"'] & X \otimes V \otimes W
\end{tikzcd}$$

and the condition that

$$\varphi_{\mathbb{C}} : F(\mathbb{C}) \otimes X = I \otimes X \to X = X \otimes \mathbb{C}$$

is reduced to the left unit constraint l_X in $\mathcal{T}$.

Let B be another finite-dimensional semisimple Hopf algebra with $\mathcal{B}$ the tensor category of B-modules and $G : \mathcal{B} \to \mathcal{T}$ be a monoidal imbedding. A *right $\mathcal{B}$-module* in $\mathcal{T}$ (through G) is, by definition, an object Y in $\mathcal{T}$ together with a natural family of isomorphisms $\{\psi_W : Y \otimes G(W) \to W \otimes Y\}$ such that $\psi_{\mathbb{C}} = r_Y$ ($=$ the right unit constraint for Y) and

$$\begin{tikzcd}
Y \otimes G(V) \otimes G(W) \arrow[r, "\psi_V\otimes 1"] \arrow[d, "m^G_{V,W}\otimes 1"'] & V \otimes Y \otimes G(W) \arrow[d, "1\otimes\psi_W"] \\
Y \otimes G(V \otimes W) \arrow[r, "\psi_{V\otimes W}"'] & V \otimes W \otimes Y.
\end{tikzcd}$$

An *$\mathcal{A}$-$\mathcal{B}$ bimodule* in $\mathcal{T}$ (relative to the imbeddings F, G) is an object X in $\mathcal{T}$ together with structures of a left $\mathcal{A}$-module and a right $\mathcal{B}$-module,

$$\varphi_V : F(V) \otimes X \to V \otimes X, \qquad \psi_W : X \otimes G(W) \to W \otimes X$$

such that the following diagram commutes:

$$\begin{tikzcd}
F(V)\otimes X\otimes G(W) \arrow[r] \arrow[d] & F(V)\otimes W\otimes X \arrow[r, equal] & W\otimes F(V)\otimes X \arrow[d] \\
X\otimes V\otimes G(W) \arrow[r, equal] & X\otimes G(W)\otimes V \arrow[r] & W\otimes X\otimes V.
\end{tikzcd}$$

We shall often write ${}_{\mathcal{A}}X_{\mathcal{B}}$ to indicate an $\mathcal{A}$-$\mathcal{B}$ bimodule based on an object X in $\mathcal{T}$ when no confusion arises for the choice of families $\{\varphi_V\}$, $\{\psi_W\}$. We also use the notation $\xi_{V,W} : F(V)\otimes X\otimes G(W) \to W\otimes X\otimes V$ to express the isomorphism in the above diagram, which is referred to as a *trivializing isomorphism* in the following.

Example 1.1. If A is the function algebra of a finite group H, then H is realized as a subset of the spectrum $\mathrm{Spec}(\mathcal{T})$ (the set of equivalence classes of simple objects) of $\mathcal{T}$ through the imbedding F and the functor F itself is identified with a lift of $H \subset \mathrm{Spec}(\mathcal{T})$. Similarly, if B is the function algebra of another finite group K, then the monoidal imbedding $G : \mathcal{B} \to \mathcal{T}$ is identified with a lift of $K \subset \mathrm{Spec}(\mathcal{T})$.

With this observation in mind, $\mathcal{A}$-$\mathcal{B}$ bimodules are naturally recognized as H-K bimodules in $\mathcal{T}$ in the sense of [41]: this case, the underlying vector spaces for simple A-modules are identified with the 1-dimensional vector space $\mathbb{C}$. Of course, when the A-module structure is concerned, we should distinguish them according to points in the spectrum set H of A and we shall write $\mathbb{C}_h$ to denote the simple A-module corresponding to an element $h \in H$, which forms a representative set of simple objects in the category $\mathcal{A}$ and there exists a natural way of identifications $\mathbb{C}_g \otimes \mathbb{C}_h = \mathbb{C}_{gh}$ for $g, h \in H$.

So, given a monoidal imbedding $F : \mathcal{A} \to \mathcal{T}$, we obtain a family of invertible objects $X_g = F(\mathbb{C}_g)$ parameterized by $g \in H$ with an associative family of multiplication morphisms $m_{g,h} : X_g \otimes X_h \to X_{gh}$. Now a left $\mathcal{A}$-module X, for example, is captured as an object in our target category $\mathcal{T}$ with an "H-module" structure governed by a family of isomorphisms $m_{g,X} : X_g \otimes X \to X$ satisfying the associativity

$$\begin{tikzcd}
X_g\otimes X_h\otimes X \arrow[r] \arrow[d] & X_g\otimes X \arrow[d] \\
X_{gh}\otimes X \arrow[r] & X.
\end{tikzcd}$$

Example 1.2. Let A be the group algebra of a finite group G with $\mathcal{A}$ the Tannaka dual of G. For notational economy, we write ${}_GV$ to express a (left) G-module with the underlying vector space V. Thus ${}_GV \otimes {}_GW$, for example, denotes the tensor product G-module of ${}_GV$ and ${}_GW$ whereas ${}_GV \otimes W$ means the G-module amplified by the vector space W, with the same underlying vector space $V \otimes W$.

Let ${}_G\mathbb{C}[G]$ be the left regular representation of G. Given an element $a \in G$ and a G-module ${}_GV$, define isomorphisms

$$\varphi_V^a : {}_GV \otimes {}_G\mathbb{C}[G] \to {}_G\mathbb{C}[G] \otimes V, \qquad \psi_V^a : {}_G\mathbb{C}[G] \otimes {}_GV \to V \otimes {}_G\mathbb{C}[G]$$

by

$$\varphi_V^a(v \otimes g) = g \otimes ag^{-1}v, \qquad \psi_V^a(g \otimes v) = ag^{-1}v \otimes g.$$

Then, for any given pair (a, b) of elements in G, the family $\{\varphi_V^a\}$ and $\{\psi_V^b\}$ makes ${}_G\mathbb{C}[G]$ into an $\mathcal{A}$-$\mathcal{A}$ bimodule in $\mathcal{A}$ (relative to the trivial imbedding), which is denoted by ${}_{\mathcal{A}}R^{a,b}{}_{\mathcal{A}}$. When the left (respectively right) action is forgotten in ${}_{\mathcal{A}}R^{a,b}{}_{\mathcal{A}}$, the resulting left (respectively right) $\mathcal{A}$-module is denoted by ${}_{\mathcal{A}}R^a$ (respectively $R^b{}_{\mathcal{A}}$).

Definition 1.3. Given Tannaka duals $\mathcal{A}$, $\mathcal{B}$ (of finite-dimensional semisimple Hopf algebras) in a semisimple tensor category $\mathcal{T}$ and $\mathcal{A}$-$\mathcal{B}$ bimodules ${}_{\mathcal{A}}X_{\mathcal{B}}$, ${}_{\mathcal{A}}Y_{\mathcal{B}}$ in $\mathcal{T}$, we call a morphism $f : X \to Y$ in $\mathcal{T}$ an $\mathcal{A}$-$\mathcal{B}$ *intertwiner* if the following diagram commutes:

$$\begin{tikzcd}
F(V) \otimes X \otimes G(W) \arrow[r, "1\otimes f\otimes 1"] \arrow[d] & F(V) \otimes Y \otimes G(W) \arrow[d] \\
W \otimes X \otimes V \arrow[r, "1\otimes f\otimes 1"'] & W \otimes Y \otimes V.
\end{tikzcd}$$

The category ${}_{\mathcal{A}}\mathcal{M}(\mathcal{T})_{\mathcal{B}}$ of $\mathcal{A}$-$\mathcal{B}$ bimodules in $\mathcal{T}$ is then defined by taking $\mathcal{A}$-$\mathcal{B}$ intertwiners as morphisms in ${}_{\mathcal{A}}\mathcal{T}_{\mathcal{B}}$. We use the notation $\mathrm{Hom}({}_{\mathcal{A}}X_{\mathcal{B}}, {}_{\mathcal{A}}Y_{\mathcal{B}})$ to stand for the hom-sets in the category ${}_{\mathcal{A}}\mathcal{M}(\mathcal{T})_{\mathcal{B}}$ while $\mathrm{Hom}(X, Y)$ is reserved to denote the hom-set in $\mathcal{T}$ related to the underlying objects X and Y in $\mathcal{T}$.

Example 1.4. Let G be a finite group and $\mathcal{A}$ be its Tannaka dual. For $h \in G$, denote by $\rho(h)$ the right regular representation of h: $\rho(h) : g \mapsto gh^{-1}$ for $g \in G \subset \mathbb{C}[G]$.

(i) For $a, b \in G$, we have

$$\mathrm{Hom}\big({}_{\mathcal{A}}R^a, {}_{\mathcal{A}}R^b\big) = \mathbb{C}\rho\big(b^{-1}a\big) = \mathrm{Hom}\big(R^a{}_{\mathcal{A}}, R^b{}_{\mathcal{A}}\big).$$

(ii) For $a', b' \in G$, we have

$$\mathrm{Hom}\big({}_{\mathcal{A}}R^{a',b'}{}_{\mathcal{A}}, {}_{\mathcal{A}}R^{a,b}{}_{\mathcal{A}}\big) = \begin{cases} \mathbb{C}\rho(a^{-1}a') & \text{if } a^{-1}a' = b^{-1}b', \\ 0 & \text{otherwise.} \end{cases}$$

Recall that the underlying vector space of $R^{a,b}$ is $\mathbb{C}[G]$.

2. Tensor products

We shall make the totality of ${}_{\mathcal{A}}\mathcal{M}(\mathcal{T})_{\mathcal{B}}$ for various Tannaka duals $\mathcal{A}$, $\mathcal{B}$ into a bicategory. To this end, we first introduce the notion of $\mathcal{A}$-tensor products. Let $X_{\mathcal{A}}$ be a right $\mathcal{A}$-module and ${}_{\mathcal{A}}Y$ be a left $\mathcal{A}$-module in $\mathcal{T}$. Given a simple A-module V and a basis $\{v_i\}$ of V, let $\{v_i^*\}$ be its dual basis. Then the linear operator $v_{i,j} = v_i \otimes v_j^*$ in V is identified with an element of A. These for various V form matrix units in the algebra A. We define $\widehat{v}_{ij} \in A^*$ by

$$\langle \widehat{v}_{ij}, w_{kl} \rangle = \begin{cases} \delta_{il}\delta_{jk} \dim V & \text{if } V \cong W, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\{\widehat{v}_{ij}\}_{V,i,j}$ forms a linear basis of A^*.

We now introduce an element $\pi(\widehat{v}_{ij}) \in \mathrm{End}(X \otimes Y)$ by the composition

$$X \otimes Y \xrightarrow{1\otimes\delta_{F(V)}\otimes 1} X \otimes F(V^*) \otimes F(V) \otimes Y \longrightarrow V^* \otimes X \otimes Y \otimes V \longrightarrow X \otimes Y,$$

where the last morphism in the diagram is given by the pairing with $\widehat{v}_{ij}$: if the composite of the first two morphisms is expressed as

$$\sum_{i,j} v_i^* \otimes t_{ij} \otimes v_j$$

with $t_{ij} \in \mathrm{End}(X \otimes Y)$, then we set $\pi(\widehat{v}_{ij}) = \dim(V) t_{ij}$ or, equivalently, the composite $X \otimes Y \to X \otimes F(V^*) \otimes F(V) \otimes Y \to V^* \otimes X \otimes Y \otimes V$ has the expression

$$\sum_{i,j} (\dim V)^{-1} v_i^* \otimes \pi(\widehat{v}_{ij}) \otimes v_j,$$

which is an element in

$$\mathrm{Hom}(X \otimes Y, V^* \otimes X \otimes Y \otimes V) = V^* \otimes \mathrm{End}(X \otimes Y) \otimes V.$$

It is immediate to check that the map π is basis-free and extended to the linear map of A^* into $\mathrm{End}(X \otimes Y)$, which is again denoted by π.

Lemma 2.1. *Let V, W be simple A-modules and $\{v_i\}$, $\{w_k\}$ be their bases. Then we have*

$$\pi(\widehat{v}_{ij})\pi(\widehat{w}_{kl}) = \pi(\widehat{v}_{ij}\widehat{w}_{kl}).$$

Here the multiplication in the right-hand side is the one obtained by dualizing the coproduct of A.

Proof. Let $U \xrightarrow{T} V \otimes W \xrightarrow{T^*} U$ give an irreducible decomposition of $V \otimes W$, i.e., $\{T, T^*\}$ is a family of morphisms such that $T^*T = 1_U$ and $\sum_T TT^* = 1_{V\otimes W}$. Then, for the rigidity copairing $\delta_{V\otimes W} : \mathbb{C} \to W^* \otimes V^* \otimes V \otimes W$, we have

$$\delta_{V\otimes W} = \sum_{T : U \to V\otimes W} (\overline{T} \otimes T)\delta_U,$$

where $\overline{T}$ is the transposed map of $T^*: V \otimes W \to U$. By the associativity and the naturality of $\mathcal{A}$-actions, we see that the composite morphism

$$\begin{aligned} X \otimes Y &\to X \otimes F(W^*) \otimes F(V^*) \otimes F(V) \otimes F(W) \otimes Y \\ &\to W^* \otimes V^* \otimes X \otimes Y \otimes V \otimes W \end{aligned}$$

is equal to

$$\sum_T \big(X \otimes Y \to U^* \otimes X \otimes Y \otimes U \xrightarrow{\overline{T} \otimes 1 \otimes T} W^* \otimes V^* \otimes X \otimes Y \otimes V \otimes W\big),$$

where $X \otimes Y \to U^* \otimes X \otimes Y \otimes U$ is given by the composition

$$X \otimes Y \to X \otimes F(U^*) \otimes F(U) \otimes Y \to U^* \otimes X \otimes Y \otimes U.$$

If we replace this with

$$\sum_{a,b} (\dim U)^{-1} u_a^* \otimes \pi(\widehat{u}_{ab}) \otimes u_b$$

and then compute $\pi(\widehat{v}_{ij})\pi(\widehat{w}_{kl})$, we obtain the formula

$$\begin{aligned} \pi(\widehat{v}_{ij})\pi(\widehat{w}_{kl}) &= \sum_T \sum_{a,b} (\dim U)^{-1} \big\langle \overline{T} u_a^* \otimes \pi(\widehat{u}_{ab}) \otimes T u_b, \widehat{v}_{ij} \otimes \widehat{w}_{kl} \big\rangle \\ &= \sum_T \sum_{a,b} \frac{d(V)d(W)}{d(U)} \big\langle \overline{T} u_a^*, v_i \otimes w_k \big\rangle \big\langle T u_b, v_j^* \otimes w_l^* \big\rangle \pi(\widehat{u}_{ab}). \end{aligned}$$

On the other hand, the definition of multiplication in A^* gives

$$\langle \widehat{v}_{ij} \widehat{w}_{kl}, x \rangle = \big\langle \widehat{v}_{ij} \otimes \widehat{w}_{kl}, \Delta(x) \big\rangle = \sum_T d(V)d(W) \big\langle v_j^* \otimes w_l^*, T x T^*(v_i \otimes w_k) \big\rangle$$

for $x \in A \cong \bigoplus_V \mathcal{L}(V)$. By using the obvious identity

$$T^*(v_i \otimes w_k) = \sum_a \big\langle u_a^*, T^*(v_i \otimes w_k) \big\rangle u_a,$$

the above expression takes the form

$$d(V)d(W) \sum_T \sum_a \big\langle v_j^* \otimes w_l^*, T x u_a \big\rangle \big\langle u_a^*, T^*(v_i \otimes w_k) \big\rangle,$$

or equivalently, we have another formula

$$\widehat{v}_{ij} \widehat{w}_{kl} = \sum_T \sum_{a,b} \frac{d(V)d(W)}{d(U)} \big\langle v_j^* \otimes w_l^*, T u_b \big\rangle \big\langle u_a^*, T^*(v_i \otimes w_k) \big\rangle \widehat{u}_{ab},$$

proving the assertion. □

Since the trivial representation of A is given by the counit ϵ, we see that $\pi(\epsilon)$ is equal to the identity morphism as the composition

$$X \otimes Y \to X \otimes I \otimes I \otimes Y \to \mathbb{C} \otimes X \otimes Y \otimes \mathbb{C} = X \otimes Y.$$

This, together with the previous lemma, shows that $\pi : A^* \to \mathrm{End}(X \otimes Y)$ is an algebra-homomorphism. Since A^* is semisimple by Larson and Radford [23], the component of the trivial representation of A^* gives rise to an idempotent $e_{\mathcal{A}}$ in $\mathrm{End}(X \otimes Y)$. The associated subobject of $X \otimes Y$ is then denoted by $X \otimes_{\mathcal{A}} Y$ and is referred to as the $\mathcal{A}$-*module tensor product* of X and Y.

Remark. (i) The idempotent $e_{\mathcal{A}}$ is realized as $\pi(e)$, where the idempotent e in A^* is given by the normalized invariant integral $e \in A^*$ of A:

$$\langle e, x\rangle = \sum_{[V]} \frac{\dim(V)}{\dim(A)} \mathrm{tr}(x_V), \quad x \in A.$$

(ii) Since the counit for A^* is given by the evaluation map at the unit 1_A of A, the idempotent $e_{\mathcal{A}}$ is non-zero if and only if there exists a simple object Z of $\mathcal{T}$ such that

$$\left\{f \in \mathrm{Hom}(Z, X \otimes Y);\ \pi(a^*) \circ f = a^*(1_A) f \text{ for any } a^* \in A^*\right\} \neq \{0\}.$$

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be Tannaka duals in the tensor category $\mathcal{T}$ and consider ${}_{\mathcal{A}}X_{\mathcal{B}}$, ${}_{\mathcal{B}}Y_{\mathcal{C}}$. The tensor product $X \otimes Y$ is then an $\mathcal{A}$-$\mathcal{C}$ module in an obvious manner and the associativity of biactions for X, Y gives the following lemma.

Lemma 2.2. *We have* $\pi(B^*) \subset \mathrm{End}({}_{\mathcal{A}}X \otimes Y_{\mathcal{C}})$.

In particular, the biaction of $\mathcal{A}$ and $\mathcal{C}$ on $X \otimes Y$ is reduced to the subobject $X \otimes_{\mathcal{B}} Y$, which is denoted by ${}_{\mathcal{A}}X \otimes_{\mathcal{B}} Y_{\mathcal{C}}$ and is referred to as the *relative tensor product* of bimodules. For morphisms $f : {}_{\mathcal{A}}X_{\mathcal{B}} \to {}_{\mathcal{A}}X'_{\mathcal{B}}$ and $g : {}_{\mathcal{B}}Y_{\mathcal{C}} \to {}_{\mathcal{B}}Y'_{\mathcal{C}}$, $f \otimes g \in \mathrm{Hom}({}_{\mathcal{A}}X \otimes Y_{\mathcal{C}}, {}_{\mathcal{A}}X' \otimes Y'_{\mathcal{C}})$ obviously commutes with $\pi(B^*)$ and hence induces the morphism

$$f \otimes_{\mathcal{B}} g : {}_{\mathcal{A}}X \otimes_{\mathcal{B}} Y_{\mathcal{C}} \to {}_{\mathcal{A}}X' \otimes_{\mathcal{B}} Y'_{\mathcal{C}},$$

which is the relative tensor product of morphisms.

The operation of taking relative tensor products is clearly associative. Thus the categories of bimodules in $\mathcal{T}$ constitute a bicategory $\mathcal{M}(\mathcal{T})$ if we can show the existence of unit objects. Here label objects of the bicategory $\mathcal{M}(\mathcal{T})$ are indexed by Tannaka duals (of finite-dimensional semisimple Hopf algebras) realized inside the tensor category $\mathcal{T}$.

3. Unit objects

Let $F:\mathcal{A}\to\mathcal{T}$ be a fully faithful imbedding of the Tannaka dual $\mathcal{A}$ of a Hopf algebra A. Given A-modules U, V and W, we use the notation

$$\begin{bmatrix} U \\ V\ W \end{bmatrix} = \mathrm{Hom}(U, V\otimes W).$$

Choose a representative set $\{V\}$ of irreducible A-modules and set

$$\mathbb{A} = \bigoplus_V F(V)\otimes V^*,$$

which is an object in $\mathcal{T}$ (more precisely in $\mathcal{T}\otimes\mathcal{V}$). Given an A-module U, define an isomorphism $F(U)\otimes\mathbb{A}\to\mathbb{A}\otimes U$ by the composition

$$\begin{aligned}
F(U)\otimes\mathbb{A} &= \bigoplus_V F(U)\otimes F(V)\otimes V^* \\
&\cong \bigoplus_V F(U\otimes V)\otimes V^* \\
&\quad \text{(by the multiplicativity of monoidal functor)} \\
&\cong \bigoplus_{V,X} F(X)\otimes \begin{bmatrix} X \\ U\ V \end{bmatrix}\otimes V^* \\
&\quad \text{(by the irreducible decomposition of } U\otimes V\text{)} \\
&\cong \bigoplus_{V,X} F(X)\otimes \begin{bmatrix} V^* \\ X^*\ U \end{bmatrix}\otimes V^* \quad \text{(by Frobenius transform)} \\
&= \bigoplus_X F(X)\otimes X^*\otimes U \\
&\quad \text{(by the irreducible decomposition of } X^*\otimes U\text{)} \\
&= \mathbb{A}\otimes U.
\end{aligned}$$

Similarly, we define an isomorphism $\mathbb{A}\otimes F(U)\to U\otimes\mathbb{A}$ by

$$\begin{aligned}
\mathbb{A}\otimes F(U) &= \bigoplus_V F(V)\otimes F(U)\otimes V^* \\
&\cong \bigoplus_{V,X} F(X)\otimes \begin{bmatrix} X \\ V\ U \end{bmatrix}\otimes V^* \\
&\cong \bigoplus_{V,X} F(X)\otimes \begin{bmatrix} V^* \\ U\ X^* \end{bmatrix}\otimes V^* \\
&= \bigoplus_X F(X)\otimes U\otimes X^* \\
&= U\otimes\mathbb{A}.
\end{aligned}$$

Here in the last line, we applied the commutativity $F(X) \otimes U = U \otimes F(X)$ and similarly in the top line.

Lemma 3.1. *The isomorphisms defined so far make $\mathbb{A}$ into an $\mathcal{A}$-$\mathcal{A}$ bimodule.*

Proof. We just check the compatibility of left and right isomorphisms: Given A-modules U and W, we shall prove the commutativity of the diagram

$$\begin{tikzcd}
F(U) \otimes \mathbb{A} \otimes F(W) \arrow[r] \arrow[d] & F(U) \otimes W \otimes \mathbb{A} \arrow[r, equal] & W \otimes F(U) \otimes \mathbb{A} \arrow[d] \\
\mathbb{A} \otimes U \otimes F(W) \arrow[r, equal] & \mathbb{A} \otimes F(W) \otimes U \arrow[r] & W \otimes \mathbb{A} \otimes U.
\end{tikzcd}$$

By the associativity of the monoidal functor F

$$\begin{tikzcd}
F(U) \otimes F(V) \otimes F(W) \arrow[r] \arrow[d] & F(U) \otimes F(V \otimes W) \arrow[d] \\
F(U \otimes V) \otimes F(W) \arrow[r] & F(U \otimes V \otimes W),
\end{tikzcd}$$

the problem is reduced to the equality of compositions

$$\begin{aligned}
\bigoplus_{V,X} F(X) \otimes \begin{bmatrix} & X & \\ U & V & W \end{bmatrix} \otimes V^* &\to \bigoplus_{V,X,Y} F(X) \otimes \begin{bmatrix} & X \\ U & Y \end{bmatrix} \otimes \begin{bmatrix} & Y \\ V & W \end{bmatrix} \otimes V^* \\
&\to \bigoplus_{V,X} F(X) \otimes \begin{bmatrix} & V^* & \\ W & X^* & U \end{bmatrix} \otimes V^*, \\
\bigoplus_{X,V} F(X) \otimes \begin{bmatrix} & X & \\ U & V & W \end{bmatrix} \otimes V^* &\to \bigoplus_{V,X,Y} F(X) \otimes \begin{bmatrix} & X \\ Y & W \end{bmatrix} \otimes \begin{bmatrix} & Y \\ U & V \end{bmatrix} \otimes V^* \\
&\to \bigoplus_{V,X} F(X) \otimes \begin{bmatrix} & V^* & \\ W & X^* & U \end{bmatrix} \otimes V^*.
\end{aligned}$$

By an easy manipulation of transposed morphisms (use the equality of left and right transposed morphisms), we see that these are the ones associated to the following composite Frobenius transforms:

$$\begin{bmatrix} & X & \\ U & V & W \end{bmatrix} \to \begin{bmatrix} & W^* & \\ X^* & U & V \end{bmatrix} \to \begin{bmatrix} & V^* & \\ W & X^* & U \end{bmatrix},$$
$$\begin{bmatrix} & X & \\ U & V & W \end{bmatrix} \to \begin{bmatrix} & U^* & \\ V & W & X^* \end{bmatrix} \to \begin{bmatrix} & V^* & \\ W & X^* & U \end{bmatrix}.$$

In fact, given a vector

$$f \otimes g \in \begin{bmatrix} & X \\ U & Y \end{bmatrix} \otimes \begin{bmatrix} & Y \\ V & W \end{bmatrix}$$

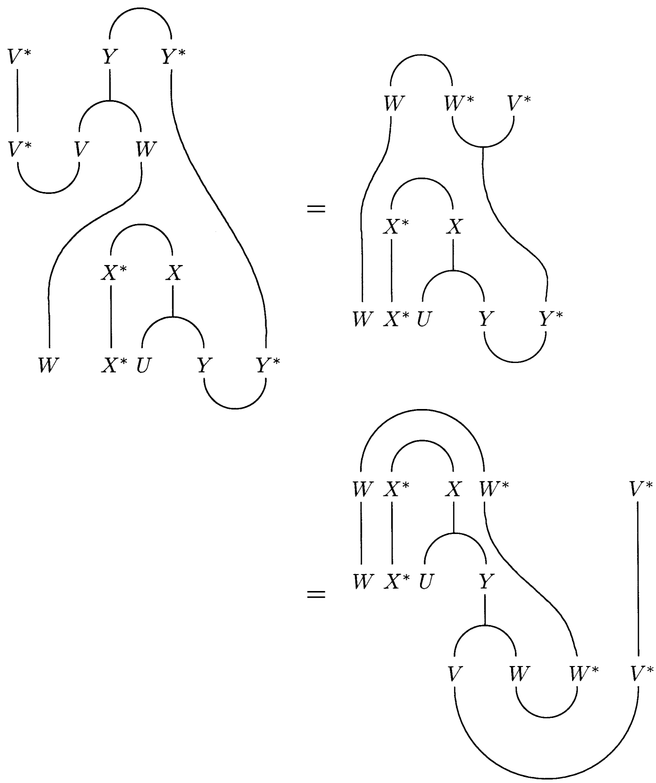


Fig. 1.

in the middle vector space, we need to identify the map

$$\begin{bmatrix} X \\ U\,V\,W \end{bmatrix} \ni (1 \otimes g) f \mapsto \bigl(1 \otimes \tilde{f}\bigr)\tilde{g} \in \begin{bmatrix} V^* \\ W\,X^*\,U \end{bmatrix},$$

where

$$\tilde{f} \in \begin{bmatrix} Y^* \\ X^*\,U \end{bmatrix}, \qquad \tilde{g} \in \begin{bmatrix} V^* \\ W\,Y^* \end{bmatrix}$$

are Frobenius transforms of f and g, respectively. Now Fig. 1 shows that the morphism $(1 \otimes \tilde{f})\tilde{g}$ is obtained by applying Frobenius transforms to $(1 \otimes g) f$ repeatedly.

Now the coincidence of these is further reduced to the equality of left and right transposed morphisms, which is a consequence of the involutiveness of antipodes for finite-dimensional semisimple Hopf algebras [22]. □

For later use, we record here the following formula for the inverse trivialization.

Lemma 3.2. *The inverse of the trivialization isomorphism* $\mathbb{A} \otimes W \to F(W) \otimes \mathbb{A}$ *is given by the following*:

$$\begin{aligned}\mathbb{A} \otimes W \;&=\; \bigoplus_V F(V) \otimes V^* \otimes W \to \bigoplus_{U,V} F(V) \otimes \begin{bmatrix} U^* \\ V^*\, W \end{bmatrix} \otimes U^* \\ &\to \bigoplus_{U,V} F(V) \otimes \begin{bmatrix} V \\ W\, U \end{bmatrix} \otimes U^* \to \bigoplus_U F(W \otimes U) \otimes U^* \\ &\to \bigoplus_U F(W) \otimes F(U) \otimes U^*.\end{aligned}$$

Here the isomorphisms are given by irreducible decompositions and Frobenius transforms as in the definition of trivialization isomorphisms.

Proof. This is immediate if we compute the composition with the trivialization isomorphism, which turns out to be the identity morphism. □

Remark. We have the following gauge ambiguity for the choice of trivializing isomorphisms: Given an invertible element $\theta \in \mathrm{End}(\mathbb{A})$, we can perturb the trivialization isomorphisms by the commutativity of the diagram

$$\begin{array}{ccc} F(U) \otimes \mathbb{A} \otimes F(W) & \xrightarrow{\alpha_{U,W}} & W \otimes \mathbb{A} \otimes U \\ {\scriptstyle 1\otimes\theta\otimes 1}\Big\uparrow & & \Big\uparrow{\scriptstyle 1\otimes\theta\otimes 1} \\ F(U) \otimes \mathbb{A} \otimes F(W) & \xrightarrow[\alpha^{\theta}_{U,W}]{} & W \otimes \mathbb{A} \otimes U. \end{array}$$

Note that, $\mathbb{A}$ being isomorphic to $\bigoplus_V F(V) \otimes V^*$ as an object in $\mathcal{T}$, we have the identification $\mathrm{Aut}(\mathbb{A}) = \prod_V \mathrm{GL}(V^*)$.

When $\mathcal{T}$ is a C*-tensor category (see [25] for example) and A is a C*-Hopf algebra, with the choice of θ defined by the family $\{\sqrt{d(V)}1_{V^*}\}_V$, the isomorphism $\alpha^{\theta}_{U,W}$ becomes a unitary. In fact, the unperturbed isomorphism are locally given by

$$\begin{bmatrix} X \\ V\, U \end{bmatrix} \otimes V^* \ni T \otimes v^* \mapsto \widetilde{T} v^* \in X^* \otimes U$$

with their norms (the inner products being associated to operator norms) by

$$\|T \otimes v^*\|^2 = \frac{1}{d(X)} \langle T^*T \rangle (v^*|v^*), \qquad \|\widetilde{T} v^*\|^2 = \frac{1}{d(V)} \langle T^*T \rangle (v^*|v^*),$$

where $\widetilde{T}$ denotes the Frobenius transform of T and $\langle T^*T \rangle$ the quantum trance of $T^*T \in \mathrm{End}(X)$.

4. Unit constraints

Given a left $\mathcal{A}$-module X in $\mathcal{T}$, we now introduce a morphism $\lambda : \mathbb{A} \otimes X \to X$ by the composition

$$\bigoplus_V F(V) \otimes X \otimes V^* \to \bigoplus_V X \otimes V \otimes V^* \to X,$$

where the last morphism is the one associated to the pairing map

$$\bigoplus_V V \otimes V^* \ni v \otimes v^* \mapsto \langle v, v^* \rangle \in \mathbb{C}.$$

Lemma 4.1. *We have*

$$\lambda \circ \pi(a^*) = a^*(1)\lambda : \mathbb{A} \otimes X \to X \quad \textit{for } a^* \in A^*.$$

Moreover, λ is $\mathcal{A}$-linear: the following diagram commutes:

$$\begin{tikzcd}
F(U) \otimes \mathbb{A} \otimes X \arrow[r, "1\otimes\lambda"] \arrow[d] & F(U) \otimes X \arrow[r] & X \otimes U \arrow[d, equal] \\
\mathbb{A} \otimes U \otimes X \arrow[r, equal] & \mathbb{A} \otimes X \otimes U \arrow[r, "\lambda\otimes 1"'] & X \otimes U.
\end{tikzcd}$$

Proof. Let $a^* = \widetilde{w}_{kl} \in A^*$ be an element associated to a simple A-module W. Then the composition $\lambda \circ \pi(\widetilde{w}_{kl})$ is given by

$$\begin{aligned}
\bigoplus_V F(V) \otimes V^* \otimes X &\to \bigoplus_V F(V \otimes W^*) \otimes F(W) \otimes X \otimes V^* \\
&\to \bigoplus_{U,V} F(U) \otimes \begin{bmatrix} & U \\ V & W^* \end{bmatrix} \otimes V^* \otimes X \otimes W \\
&\to \bigoplus_U F(U) \otimes W^* \otimes U^* \otimes W \otimes X \\
&\overset{\widehat{w}_{kl}}{\to} \bigoplus_U F(U) \otimes U^* \otimes X \\
&\overset{\lambda}{\to} X,
\end{aligned}$$

which is, by the naturality of $F(\cdot) \otimes X \to X \otimes (\cdot)$, equal to the composition

$$\begin{aligned}
\bigoplus_V F(V) \otimes V^* \otimes X &\longrightarrow \bigoplus_V X \otimes V \otimes V^* \\
&\xrightarrow{1\otimes\delta_W\otimes 1} \bigoplus_V X \otimes V \otimes W^* \otimes W \otimes V^*
\end{aligned}$$

$$
\begin{aligned}
&\longrightarrow \bigoplus_{U,V} X \otimes U \otimes W \otimes V^* \otimes \begin{bmatrix} U \\ V\,W^* \end{bmatrix} \\
&\longrightarrow \bigoplus_{U,V} X \otimes U \otimes W \otimes V^* \otimes \begin{bmatrix} V^* \\ W^*\,U^* \end{bmatrix} \\
&\longrightarrow \bigoplus_{U} X \otimes U \otimes W \otimes W^* \otimes U^* \\
&\xrightarrow{\widehat{w}_{kl}} \bigoplus_{U} X \otimes U \otimes U^* \\
&\xrightarrow{\text{pairing}} X.
\end{aligned}
$$

We now compute how the operation works on vector spaces:

$$
\begin{aligned}
v \otimes v^* &\mapsto \sum_m v \otimes w_m^* \otimes w_m \otimes v^* \\
&\mapsto \sum_{m,T,i} \bigl\langle (Tu_i)^*, v \otimes w_m^* \bigr\rangle Tu_i \otimes w_m \otimes v^* \\
&\mapsto \sum \bigl\langle (Tu_i)^*, v \otimes w_m^* \bigr\rangle u_i \otimes w_m \otimes \widetilde{T} v^* \\
&\mapsto d(W) \sum_{T,i} \bigl\langle (Tu_i)^*, v \otimes w_l^* \bigr\rangle \bigl\langle u_i \otimes w_k, \widetilde{T} v^* \bigr\rangle \\
&= d(W) \sum \bigl\langle u_i^*, T^*\bigl(v \otimes w_l^*\bigr) \bigr\rangle \bigl\langle u_i \otimes w_k, \widetilde{T} v^* \bigr\rangle \\
&= d(W) \sum_T \bigl\langle T^*\bigl(v \otimes w_l^*\bigr) \otimes w_k, \widetilde{T} v^* \bigr\rangle.
\end{aligned}
$$

Here the families $\{T : U \to V \otimes W^*\}_T$, $\{T^* : V \otimes W^* \to U\}_T$ are chosen so that $S^* T = \delta_{S,T} 1_U$ and set $\overline{T} = {}^t T^*$. Note that, if we denote by $\{u_i^*\}$ the dual basis of $\{u_i\}_i$, then the family $\{\overline{T} u_i^*\}$ is the dual basis of the basis $\{Tu_i\}_{T,i}$ of $V \otimes W^*$.

By the relation

$$
\sum_T {}^t \widetilde{T}(T^* \otimes 1) = \sum_T (1_V \otimes \epsilon_{W^*})(TT^* \otimes 1_W) = 1_V \otimes \epsilon_{W^*},
$$

the above operation on vector spaces ends up with

$$
d(W)\langle v, v^* \rangle \epsilon_{W^*}(w_l^* \otimes w_k) = d(W)\delta_{kl}\langle v, v^* \rangle = \widetilde{w}_{kl}(1)\langle v, v^* \rangle.
$$

Since the morphism λ is associated to the pairing

$$
v \otimes v^* \mapsto \langle v, v^* \rangle,
$$

at the last stage of composition, the above formula gives the result.

To see the $\mathcal{A}$-linearity, we again use the functoriality of trivializing morphisms and the problem is reduced to check the commutativity of

$$\begin{tikzcd}
\bigoplus_V U \otimes V \otimes V^* \arrow[rr] \arrow[d] & & U \\
\bigoplus_{V,W} W \otimes \begin{bmatrix} & W & \\ U & & V \end{bmatrix} \otimes V^* \arrow[r] & \bigoplus_W W \otimes W^* \otimes U, \arrow[ur] &
\end{tikzcd}$$

i.e., $(1 \otimes \epsilon_V)(f \otimes 1_{V^*}) = (\epsilon_W \otimes 1)(1_W \otimes \tilde{f})$ for $f \in \mathrm{Hom}(W, U \otimes V)$ with $\tilde{f} \in \mathrm{Hom}(V^*, W^* \otimes U)$ its Frobenius transform, which is an immediate consequence of rigidity identities. $\square$

By the covariance just checked, the morphism $\lambda : \mathbb{A} \otimes X \to X$ can be interpreted as defining a morphism ${}_{\mathcal{A}}\mathbb{A} \otimes {}_{\mathcal{A}}X \to {}_{\mathcal{A}}X$, which is denoted by l_X.

To see the invertibility of l_X, consider the morphism $\mu : X \to \mathbb{A} \otimes X$ defined by

$$X \to \bigoplus_V X \otimes V \otimes V^* \to \bigoplus_V F(V) \otimes X \otimes V^* = \mathbb{A} \otimes X,$$

where the first morphism is associated to the copairing

$$\bigoplus_V \mu_V \sum_i v_i \otimes v_i^*$$

and the weight $\{\mu_V\}$ will be specified soon after.

Now the composition $\pi(\widetilde{w}_{kl}) \circ \mu$ is given by

$$\begin{aligned}
X &\to \bigoplus_V X \otimes V \otimes V^* \xrightarrow{\delta_W} \bigoplus_V X \otimes V \otimes W^* \otimes W \otimes V^* \\
&\to \bigoplus_{U,V} X \otimes U \otimes \begin{bmatrix} & U & \\ V & & W^* \end{bmatrix} \otimes W \otimes V^* \\
&\to \bigoplus_{U,V} X \otimes U \otimes \begin{bmatrix} & V^* & \\ W^* & & U^* \end{bmatrix} \otimes W \otimes V^* \\
&\to \bigoplus_U X \otimes U \otimes W \otimes W^* \otimes U^* \xrightarrow{\widehat{w}_{kl}} \bigoplus_U X \otimes U \otimes U^* \\
&\to \bigoplus_U F(U) \otimes X \otimes U^*,
\end{aligned}$$

which we expect to be equal to $d(W)\delta_{kl}\mu$.

To see this, we work with operations on vector spaces:

$$\begin{aligned}
&\sum_{V,i} \mu_V v_i \otimes v_i^* \\
&\quad\mapsto \sum_{V,i,j} \mu_V v_i \otimes w_j^* \otimes w_j \otimes v_i^* \\
&\quad= \sum_{V,i,j} \sum_{U,T,a} \mu_V \langle (Tu_a)^*, v_i \otimes w_j^* \rangle Tu_a \otimes w_j \otimes v_i^* \\
&\quad\mapsto \sum \mu_V \langle (Tu_a)^*, v_i \otimes w_j^* \rangle u_a \otimes w_j \otimes \widetilde{T} v_i^* \\
&\quad= d(W) \sum_{V,i} \sum_{U,T} \sum_{a,b} \mu_V \langle (Tu_a)^*, v_i \otimes w_l^* \rangle \langle u_b \otimes w_k, \widetilde{T} v_i^* \rangle u_a \otimes u_b^* \\
&\quad= d(W) \sum_{U,V,T,b} \mu_V T^* \bigl({}^t\widetilde{T}(u_b \otimes w_k) \otimes w_l^* \bigr) \otimes u_b^*.
\end{aligned}$$

If we set $S = {}^t\widetilde{T} : U \otimes W \to V$ and let $S^* : V \to U \otimes W$ be the Frobenius transform of $T^* : V \otimes W^* \to U$, then the last expression takes the form

$$d(W) \sum_{U,V,S,b} \mu_V (1 \otimes \epsilon_W) \bigl(S^* S(u_b \otimes w_k) \otimes w_l^* \bigr) \otimes u_b^*.$$

Applying the formula

$$\sum_{V,S} d(V) S^* S = d(U) 1_{U \otimes W}$$

for the choice $\mu_V = d(V)$, the above summation is further reduced to

$$d(W) \sum_{U,b} (1 \otimes \epsilon_W) \bigl(u_b \otimes w_k \otimes w_l^* \bigr) \otimes u_b^* = d(W) \delta_{kl} \sum_{U,b} d(U) u_b \otimes u_b^*.$$

Thus, with the choice $\mu_V = d(V)$, we have

$$\pi(a^*) \circ \mu = a^*(1) \mu$$

for $a^* \in A^*$.

Lemma 4.2. *We now claim that*

$$\lambda \circ \mu = \left(\sum_V d(V)^2 \right) 1_X, \qquad \mu \circ \lambda = (\dim A) e_{\mathcal{A}} = \sum_V \sum_i \pi(\widehat{v}_{ii}).$$

Proof. The first relation is obvious from definitions.

On the tensor product $\mathbb{A} \otimes X$, the morphism $\pi(\widehat{w}_{ll})$ is, if the trivialization isomorphism $\mathbb{A} \otimes X \cong \bigoplus_V X \otimes V \otimes V^*$ is applied, given by

$$\bigoplus_V V \otimes V^* \to \bigoplus_V X \otimes V \otimes W^* \otimes W \otimes V^*$$

$$\begin{aligned}
&\to \bigoplus_{U,V} U \otimes \begin{bmatrix} U \\ V\,W^* \end{bmatrix} \otimes W \otimes V^* \\
&\to \bigoplus_{U,V} U \otimes \begin{bmatrix} V^* \\ W^*\,U^* \end{bmatrix} \otimes W \otimes V^* \\
&\to \bigoplus_{U} U \otimes W \otimes W^* \otimes U^* \\
&\xrightarrow{\widehat{w}_{ll}} \bigoplus_{U} U \otimes U^*.
\end{aligned}$$

According to this sequence of morphisms, we compute $(\dim A)e_{\mathcal{A}}$ as follows:

$$\begin{aligned}
v \otimes v^* \ &\mapsto \ \sum_{W,k} v \otimes w_k^* \otimes w_k \otimes v^* \\
&\mapsto \ \sum \big\langle (Tu_a)^*, v \otimes w_k^* \big\rangle Tu_a \otimes w_k \otimes v^* \\
&\mapsto \ \sum \big\langle (Tu_a)^*, v \otimes w_k^* \big\rangle u_a \otimes w_k \otimes \widetilde{T} v^* \\
&= \ \sum d(W) \big\langle (Tu_a)^*, v \otimes w_l^* \big\rangle \big\langle u_b \otimes w_l, \widetilde{T} v^* \big\rangle u_a \otimes u_b^* \\
&= \ \sum d(W) \big\langle u_b \otimes w_l, \widetilde{T} v^* \big\rangle T^*(v \otimes w_l^*) \otimes u_b^* \\
&= \ \sum d(W) \langle w_l \otimes v^*, Tu_b \rangle T^*\big(v \otimes w_l^*\big) \otimes u_b^* \\
&= \ \sum d(W) T^*\big(v \otimes w_l^*\big) \otimes {}^tT(w_l \otimes v^*) \\
&= \ \sum d(W) \big(T^* \otimes {}^tT\big)(1 \otimes \delta_W \otimes 1)(v \otimes v^*).
\end{aligned}$$

Now, letting $S : V^* \otimes U \to W^*$ and $S^* : W^* \to V^* \otimes U$ be Frobenius transforms of T and T^*, respectively, we have

$$\begin{aligned}
&\sum_{W,T} d(W)\big(T^* \otimes {}^tT\big)(1_V \otimes \delta_W \otimes 1_{V^*}) \\
&\quad = \sum_{W,S} d(W)(\epsilon_V \otimes 1_{UU^*})(1_V \otimes S^*S \otimes 1_{U^*})(1_{VV^*} \otimes \delta_{U^*}) \\
&\quad = d(U)(\epsilon_V \otimes \delta_{U^*})
\end{aligned}$$

because of

$$\sum_{W,S} d(W) S^*S = d(U) 1_{V^* \otimes U}.$$

Thus we have

$$\sum d(W)T^*\big(v \otimes w_l^*\big) \otimes {}^tT(w_l \otimes v^*) = \sum d(U)\epsilon_V(v \otimes v^*)\epsilon_{U^*},$$

which gives rise to the morphism $\mu \circ \lambda$. $\square$

By symmetry, we may expect for the right unit constraint as well. Explicit computations are as follows: Define a morphism $\rho : X \otimes \mathbb{A} \to X$ by the composition

$$\bigoplus_V X \otimes F(V) \otimes V^* \to \bigoplus_V V \otimes X \otimes V^* = \bigoplus_V X \otimes V \otimes V^* \to X,$$

where the last evaluation is specified by $v \otimes v^* \mapsto \langle v, v^* \rangle$. The inner morphism $\pi(\widehat{w}_{kl})$ is then given by

$$\begin{aligned}
\bigoplus_V X \otimes F(V) \otimes V^* &\to \bigoplus_V X \otimes F(W^*) \otimes F(W) \otimes F(V) \otimes V^* \\
&\to \bigoplus_{U,V} X \otimes W^* \otimes F(U) \otimes \begin{bmatrix} U \\ W\ V \end{bmatrix} \otimes V^* \\
&\to \bigoplus_{U,V} X \otimes W^* \otimes F(U) \otimes U^* \otimes W \\
&\to X \otimes F(U) \otimes U^* \\
&= X \otimes \mathcal{A}.
\end{aligned}$$

By trivializing the functor F, the composition of $\pi(\widetilde{w}_{kl})$ with the morphism $X \otimes \mathcal{A} \to X$ is associated to the composition

$$\begin{aligned}
\bigoplus_V V \otimes V^* \otimes X &\to \bigoplus_V W^* \otimes W \otimes V \otimes V^* \otimes X \\
&\to \bigoplus_{U,V} W^* \otimes U \otimes \begin{bmatrix} U \\ W\ V \end{bmatrix} \otimes V^* \otimes X \\
&\to \bigoplus_{U,V} W^* \otimes U \otimes U^* \otimes W \otimes X \\
&\to U \otimes U^* \otimes X \\
&\to X.
\end{aligned}$$

Now an explicit formula is obtained by working with vector spaces:

$$\begin{aligned}
v \otimes v^* &\mapsto \sum w_j^* \otimes w_j \otimes v \otimes v^* \\
&\mapsto \sum \bigl\langle (Tu_a)^*, w_j \otimes v \bigr\rangle w_j^* \otimes Tu_a \otimes v^* \\
&\mapsto \sum \bigl\langle (Tu_a)^*, w_j \otimes v \bigr\rangle w_j^* \otimes u_a \otimes \widetilde{T} v^* \\
&\mapsto d(W) \sum \bigl\langle (Tu_a)^*, w_k \otimes v \bigr\rangle \bigl\langle \bigl(u_b^* \otimes w_l\bigr)^*, \widetilde{T} v^* \bigr\rangle u_a \otimes u_b^* \\
&= d(W) \sum \bigl\langle w_l^* \otimes u_b, \widetilde{T} v^* \bigr\rangle T^*(w_k \otimes v) \otimes u_b^*.
\end{aligned}$$

Here we shall use the identity

$$\begin{aligned}\langle w_l^* \otimes u_b, \widetilde{T} v^* \rangle &= \langle w_l^* \otimes \epsilon_V, T u_b \otimes v^* \rangle \\ &= \sum \langle v_j^* \otimes w_l^*, T u_b \rangle \langle v_j, v^* \rangle \\ &= \langle v^* \otimes w_l^*, T u_b \rangle\end{aligned}$$

to obtain the expression

$$\begin{aligned}&= d(W) \sum \langle v^* \otimes w_l^*, T u_b \rangle T^*(w_k \otimes v) \otimes u_b^* \\ &= d(W) \sum \langle {}^t T\bigl(v^* \otimes w_l^*\bigr), u_b \rangle T^*(w_k \otimes v) \otimes u_b^* \\ &= d(W) \sum T^*(w_k \otimes v) \otimes {}^t T\bigl(v^* \otimes w_l^*\bigr) \\ &\to d(W) \sum \epsilon_U\bigl(T^* \otimes {}^t T\bigr)\bigl(w_k \otimes v \otimes v^* \otimes w_l^*\bigr) \\ &= d(W)\epsilon_{WV}(T T^* \otimes 1)\bigl(w_k \otimes v \otimes v^* \otimes w_l^*\bigr) \\ &= d(W)\epsilon_{WV}\bigl(w_k \otimes v \otimes v^* \otimes w_l^*\bigr) \\ &= d(W)\delta_{kl}\langle v, v^* \rangle.\end{aligned}$$

Thus $\rho \circ \pi(\widetilde{w}_{kl})$ is equal to $\widetilde{w}_{kl}(1)\rho$ and hence ρ induces a morphism $r_X : X \otimes_{\mathcal{A}} \mathbb{A} \to X$.

For the reverse morphism, we have

$$X \to \bigoplus_V X \otimes V \otimes V^* = \bigoplus_V V \otimes X \otimes V^* \to \bigoplus_V X \otimes F(V) \otimes V^*$$

with the first morphism given by

$$\bigoplus_V \sum_i d(V) v_i \otimes v_i^*.$$

Now the composition $X \to X \otimes \mathbb{A} \to X$ is equal to

$$\Bigl(\sum_V \dim(V)^2\Bigr) 1_X$$

whereas $X \otimes \mathbb{A} \to X \to X \otimes \mathbb{A}$ is given by

$$\Bigl(\sum_V \dim(V)^2\Bigr) e_{\mathcal{A}}.$$

Thus $r_X : X \otimes_{\mathcal{A}} \mathbb{A} \to X$ is an isomorphism of $\mathcal{A}$-$\mathcal{A}$ bimodules.

Remark. If we use the perturbed trivialization by $\alpha \in \mathrm{Aut}(\mathbb{A})$ for the $\mathcal{A}$-$\mathcal{A}$ action on $\mathbb{A}$, then λ, μ, and ρ are perturbed into $\lambda(\alpha \otimes 1)$, $(\alpha^{-1} \otimes 1)\mu$, and $\rho(1 \otimes \alpha)$, respectively.

In particular, if $\mathcal{T}$ is a C*-tensor category, we obtain unitary constraints by taking $\alpha = \{\sqrt{d(V)} 1_{V^*}\}_V$, i.e., they are associated to the pairing (copairing)

$$V \otimes V^* \ni v \otimes v^* \mapsto \sqrt{\delta(V)}\, \langle v, v^* \rangle, \qquad \sqrt{d(V)} \sum_i v_i \otimes v_i^* \in V \otimes V^*.$$

5. Triangle identities

We shall now check the triangle identity for $\{l_X, r_X\}$, i.e., given $\mathcal{A}$-modules $X_{\mathcal{A}}$ and ${}_{\mathcal{A}}Y$, the idempotent $e_{\mathcal{A}} \in \mathrm{End}(X \otimes Y)$ equalizes $\rho \otimes 1$ and $1 \otimes \lambda$ as

$$X \otimes \mathbb{A} \otimes Y \underset{1\otimes\lambda}{\overset{\rho\otimes 1}{\longrightarrow}} X \otimes Y \xrightarrow{e_{\mathcal{A}}} X \otimes Y.$$

By the formula

$$e_{\mathcal{A}} = \frac{1}{\dim A} \sum_{U,i} \pi(\widehat{u}_{ii}),$$

we need to consider the composition of

$$\begin{tikzcd}
X \otimes F(V) \otimes V^* \otimes Y \arrow[r] \arrow[d, equal] & V \otimes X \otimes V^* \otimes Y \arrow[r] & X \otimes Y \arrow[d, equal] \\
X \otimes F(V) \otimes V^* \otimes Y \arrow[r] & X \otimes V^* \otimes Y \otimes V \arrow[r] & X \otimes Y
\end{tikzcd}$$

with

$$XY \xrightarrow{\oplus 1\otimes\delta_{F(W)}\otimes 1} \bigoplus_W XF(W^*)F(W)Y \longrightarrow \bigoplus_W W^*XYW \xrightarrow{\sum \widehat{w}_{kk}} XY$$

(the tensor product symbol $\otimes$ being omitted to save space here and in what follows).

By the associativity of trivialization, we are faced to comparing

$$XF(V)Y \longrightarrow \bigoplus_W XF(V)F(W^*)F(W)Y \longrightarrow \bigoplus_W VW^*XYW \xrightarrow{\sum \widehat{w}_{kk}} VXY \tag{1}$$

and

$$XF(V)Y \longrightarrow \bigoplus_U XF(U)F(U^*)F(V)Y \longrightarrow \bigoplus_U UXYU^*V \xrightarrow{\sum \widehat{u}_{ii}} XYV \tag{2}$$

with the identification $V \otimes X \otimes Y = X \otimes Y \otimes V$.

To this end, we consider the diagram

$$\begin{tikzcd}
F(V) \arrow[r, "\oplus 1\otimes\delta_{F(V)}"] \arrow[d, "\delta_{F(U)^*}\otimes 1"'] & \bigoplus_W F(V) \otimes F(W^*) \otimes F(W) \arrow[d] \\
\bigoplus_U F(U) \otimes F(U^*) \otimes F(V) & \bigoplus_{U,W} F(U) \otimes \begin{bmatrix} U \\ V\ W^* \end{bmatrix} \otimes F(W), \arrow[l]
\end{tikzcd}$$

where the right vertical arrow is given by an irreducible decomposition

$$\left\{F(U) \overset{T}{\to} F(V) \otimes F(W^*) \overset{T^*}{\to} F(U)\right\}$$

and the bottom line by an irreducible decomposition

$$\left\{F(W) \xrightarrow{S} F(U^*) \otimes F(V) \xrightarrow{S^*} F(W)\right\}.$$

The diagram turns out to be commutative if S and T are related so that

$$S = \frac{d(W)}{d(U)} \widetilde{T}$$

with $\widetilde{T}$ the Frobenius transform of T. In fact, the relation ensures the identity

$$\sum_T (T^* \otimes S)(1_V \otimes \delta_W) = \delta_{U^*} \otimes 1_V.$$

By sandwiching the above diagram by $X \otimes \cdot \otimes Y$ and then applying trivialization isomorphisms, we obtain the commutative diagram

$$\begin{array}{ccccc}
XF(V)Y \longrightarrow & \bigoplus_W XF(V)F(W^*)F(W)Y & \longrightarrow & \bigoplus_{U,W} XF(U)\begin{bmatrix} & U & \\ V & & W^* \end{bmatrix}F(W)Y \\
& \downarrow & & \downarrow \\
& \bigoplus_W VW^*XYW & \longrightarrow & \bigoplus_{U,W} UX\begin{bmatrix} & U & \\ V & & W^* \end{bmatrix}YW
\end{array}$$

$$\begin{array}{ccccc}
\bigoplus_{U,W} XF(U)\begin{bmatrix} & U & \\ V & & W^* \end{bmatrix}F(W)Y & \longrightarrow & \bigoplus_U XF(U)F(U^*V)Y & & \\
\downarrow & & \downarrow & & \\
\bigoplus_{U,W} UX\begin{bmatrix} & U & \\ V & & W^* \end{bmatrix}YW & \longrightarrow & \bigoplus_U UXYU^*V & \longrightarrow & XYV,
\end{array}$$

where the upper route is exactly the morphism (1).

To identify the lower route, we inspect the morphism

$$\bigoplus_W VW^*W \to \bigoplus_{U,W} U\begin{bmatrix} & U & \\ V & & W^* \end{bmatrix}W \to \bigoplus_U UU^*V \to V,$$

which is given by

$$\begin{aligned}
v \otimes w^* \otimes w &\mapsto \sum \left\langle (Tu_i)^*, v \otimes w^* \right\rangle Tu_i \otimes w \\
&\mapsto \sum T^*(v \otimes w^*) \otimes Sw \\
&\mapsto \sum d(U)(\epsilon_U \otimes 1)(T^* \otimes S)(v \otimes w^* \otimes w).
\end{aligned}$$

The last summation is computed with the help of the relation

$$\begin{aligned}\sum_{U,T} d(U)(\epsilon_U \otimes 1)(T^* \otimes S) &= \sum d(W)(\epsilon_U \otimes 1)\bigl(T^* \otimes \widetilde{T}\bigr) \\ &= d(W) \sum (1 \otimes \delta_W)(TT^* \otimes 1_W) \\ &= d(W) 1_V \otimes \delta_W\end{aligned}$$

to get $\langle w^*, w\rangle v$, which is equal to

$$\sum_{U,i} \langle \widetilde{u}_{ii}, w^* \otimes w\rangle v.$$

Thus the bottom route turns out to be the composition

$$XF(V)Y \to VXY \to \bigoplus_W VXF(W^*)F(W)Y \to \bigoplus_W VW^*XYW \xrightarrow{\sum \widehat{w}_{kk}} VXY,$$

showing the equality of the morphisms (1) and (2).

To summarize the results obtained so far, we here introduce the following usage of terminology: by a Tannaka dual realized inside a tensor category $\mathcal{T}$, we shall mean a monoidal imbedding F of the Tannaka dual $\mathcal{A}$ of a finite-dimensional semisimple Hopf algebra A into the tensor category $\mathcal{T}$, which is fully faithful in the sense that the linear maps

$$F : \mathrm{Hom}(V, W) \to \mathrm{Hom}\bigl(F(V), F(W)\bigr)$$

on hom-vector spaces are bijective.

Now we have the following except for the semisimplicity of $\mathcal{M}(\mathcal{T})$, which will be proved after the rigidity result in Section 6.

Proposition 5.1. *Given a semisimple tensor category $\mathcal{T}$, we have constructed the semisimple bicategory $\mathcal{M}(\mathcal{T})$ indexed by Tannaka duals of finite-dimensional semisimple Hopf algebras realized in $\mathcal{T}$. More precisely, given a family $\{\omega_A\}$ of weights indexed by Hopf algebras realized inside $\mathcal{T}$, the pair (l_X, r_X) with $X = {}_{\mathcal{A}}X_{\mathcal{B}}$ gives unit constraints.*

Remark. Given a Tannaka dual $\mathcal{A}$ in $\mathcal{T}$, it is not obvious, at first glance, how big is the tensor category ${}_{\mathcal{A}}\mathcal{M}(\mathcal{T})_{\mathcal{A}}$ of $\mathcal{A}$-$\mathcal{A}$ bimodules.

It turn out in Section 7 to be large enough to recover the initial tensor category because $\mathcal{T}$ is realized as the tensor category of $\mathcal{B}$-$\mathcal{B}$ bimodules in ${}_{\mathcal{A}}\mathcal{M}(\mathcal{T})_{\mathcal{A}}$ with the Tannaka dual $\mathcal{B}$ of the dual Hopf algebra A^* being imbedded into ${}_{\mathcal{A}}\mathcal{M}(\mathcal{T})_{\mathcal{A}}$ (see Theorem 7.5).

Lemma 5.2. *Let A be a finite-dimensional semisimple Hopf algebra with $\mathcal{A}$ the tensor category of finite-dimensional A-modules. Given an imbedding $F : \mathcal{A} \to \mathcal{T}$ of $\mathcal{A}$ into a semisimple tensor category $\mathcal{T}$, let $\mathbb{A} = \bigoplus_V F(V) \otimes V^*$ be the associated object, where the direct sum is taken over all isomorphism classes* *of irreducible A-modules V. Then both of ${}_{\mathcal{A}}\mathbb{A}$ and $\mathbb{A}_{\mathcal{A}}$ are irreducible as $\mathcal{A}$-modules.*

Proof. Let

$$\phi = \bigoplus_V \phi_{V^*} \in \bigoplus_V \mathcal{B}(V^*) = \mathrm{End}(\mathbb{A})$$

belong to $\mathrm{End}({}_{\mathcal{A}}\mathbb{A})$, i.e.,

$$\begin{tikzcd} F(U)\otimes\mathbb{A} \arrow[r] \arrow[d, "1\otimes\phi"'] & \mathbb{A}\otimes U \arrow[d, "\phi\otimes 1"] \\ F(U)\otimes\mathbb{A} \arrow[r] & \mathbb{A}\otimes U \end{tikzcd}$$

for any U. The commutativity is then equivalent to

$$\begin{tikzcd} \bigoplus_{V,W} F(W)\otimes\begin{bmatrix} & W & \\ U & & V\end{bmatrix}\otimes V^* \arrow[r] \arrow[d] & \bigoplus_W F(W)\otimes W^*\otimes U \arrow[d] \\ \bigoplus_{V,W} F(W)\otimes\begin{bmatrix} & W & \\ U & & V\end{bmatrix}\otimes V^* \arrow[r] & \bigoplus_W F(W)\otimes W^*\otimes U. \end{tikzcd}$$

Removing the $F(W)$ factor, we have

$$\begin{tikzcd} \begin{bmatrix} & V^* & \\ W^* & & U\end{bmatrix}\otimes V^* \arrow[r] \arrow[d, "1\otimes\phi"'] & W^*\otimes U \arrow[d, "1\otimes\phi"] \\ \begin{bmatrix} & V^* & \\ W^* & & U\end{bmatrix}\otimes V^* \arrow[r] & W^*\otimes U \end{tikzcd}$$

for any U, V, and W, which means the equality

$$T\phi_{V^*} = (\phi_{W^*}\otimes 1_U)T$$

for any $T : V^* \to W^*\otimes U$.

If we take $V = \mathbb{C}$ and $U = W$ with $T = \delta_W$, then the condition is reduced to

$$\phi_{\mathbb{C}} \sum_k w_k^* \otimes w_k = \sum_k \phi_{W^*} w_k^* \otimes w_k,$$

which is equivalent to $\phi_{\mathbb{C}} w_k^* = \phi_{W^*} w_k^*$ for any k, i.e., $\phi_{W^*} = \phi_{\mathbb{C}} 1_{W^*}$ for any W. Thus, it is proportional to the identity morphism $1_{\mathbb{A}}$. $\square$

Remark. The triangle identities are satisfied for perturbed $\mathcal{A}$-$\mathcal{A}$ actions on $\mathbb{A}$ as well. Particularly, when $\mathcal{T}$ is a C*-tensor category, the unitary constraints for the choice $\theta = \{\sqrt{d(V)}\, 1_{V^*}\}$ of perturbation satisfy the triangle identity and hence give rise to unit objects, i.e., $\mathcal{M}(\mathcal{T})$ is a C*-bicategory.

Finally we record here that, other than the perturbation for actions, there remains somewhat trivial freedom for the choice of unit constraints: given a family $\{\omega_A\}_A$ of non-zero scalars, the unit constraints $l_X : {}_{\mathcal{A}}\mathbb{A} \otimes_{\mathcal{A}} X_{\mathcal{B}} \to {}_{\mathcal{A}}X_{\mathcal{B}}$ and $r_X : {}_{\mathcal{A}}X \otimes_{\mathcal{B}} \mathbb{B}_{\mathcal{B}} \to {}_{\mathcal{A}}X_{\mathcal{B}}$ are modified by multiplying ω_A and ω_B, respectively.

6. Rigidity

Let ${}_{\mathcal{A}}X_{\mathcal{B}}$ be an $\mathcal{A}$-$\mathcal{B}$ module in $\mathcal{T}$ and suppose that X admits a dual object X^* with a rigidity pair $\epsilon_X : X \otimes X^* \to I$, $\delta_X : I \to X^* \otimes X$. On the image of $\mathcal{A}$ in $\mathcal{T}$, we have the natural choice of dual objects (and rigidity pairs), which enables us to define rigidity pairs such as $\epsilon_{F(V)X} = \epsilon_{F(V)}(1 \otimes \epsilon_X \otimes 1)$, $\delta_{F(V)X} = (1 \otimes \delta_{F(V)} \otimes 1)\delta_X$. Note here that the rigidity for $F(V)$ satisfies the Frobenius duality and we can freely use the relation such as $F(V)^{**} = F(V)$ while we should be more careful when the object X is involved because there is no privileged identification.

Our task here is to check the rigidity of ${}_{\mathcal{A}}X_{\mathcal{B}}$. This being admitted, we can show the semisimplicity of $\mathcal{M}(\mathcal{T})$ as follows: Let ${}_{\mathcal{A}}Y_{\mathcal{B}}$ be an $\mathcal{A}$-$\mathcal{B}$ module. Since $\mathbb{A}$ and $\mathcal{B}$ are rigid as objects in $\mathcal{T}$, we have

$$\operatorname{End}({}_{\mathcal{A}}\mathbb{A} \otimes Y \otimes \mathbb{B}_{\mathcal{B}}) \cong \operatorname{Hom}(Y, \mathbb{A} \otimes Y \otimes \mathbb{B}) \cong \bigoplus_{V,W} \operatorname{End}(Y) \otimes \mathcal{L}(V) \otimes \mathcal{L}(W)$$

($\mathcal{L}$ indicating the algebra of linear operators) and hence $\operatorname{End}({}_{\mathcal{A}}Y_{\mathcal{B}}) = \operatorname{End}({}_{\mathcal{A}}\mathbb{A} \otimes_{\mathcal{A}} Y \otimes_{\mathcal{B}} \mathbb{B}_{\mathcal{B}})$ is semisimple as a diagonal corner of the semisimple $\operatorname{End}({}_{\mathcal{A}}\mathbb{A} \otimes Y \otimes \mathbb{B}_{\mathcal{B}})$.

Now we return to the rigidity proof. By applying the operation of taking transposed morphisms, we make X^* into a $\mathcal{B}$-$\mathcal{A}$ module: the trivializing isomorphism $G(W) \otimes X^* \otimes F(V) \to V \otimes X^* \otimes W$ is defined to be the transposed morphism of the isomorphism $\phi : W^* \otimes X \otimes V^* \to F(V^*) \otimes X \otimes G(W^*)$ with respect to the duality pairing $\epsilon_{F(V^*)XG(W^*)}$ (tensor product symbols being omitted in the suffix):

$$\begin{aligned}&(1_{VX^*W} \otimes \epsilon_{F(V^*)XG(W^*)})(1_{VX^*W} \otimes \phi \otimes 1_{G(W)X^*F(V^*)})\\ &\quad \times (\delta_{W^*XV^*} \otimes 1_{G(W)X^*F(V^*)}).\end{aligned}$$

Lemma 6.1. *We have the commutative diagrams*

$$\begin{tikzcd} X \otimes X^* \otimes F(V) \arrow[r] \arrow[d, "\epsilon\otimes 1"'] & X \otimes V \otimes X^* \arrow[d] \\ F(V) & F(V) \otimes X \otimes X^*, \arrow[l, "1\otimes\epsilon"] \end{tikzcd}$$

$$\begin{array}{ccc} G(W)\otimes X^*\otimes X & \longrightarrow & X^*\otimes W\otimes X \\ {\scriptstyle 1\otimes\delta}\uparrow & & \downarrow \\ G(W) & \xrightarrow[\delta\otimes 1]{} & X^*\otimes X\otimes G(W). \end{array}$$

Proof. The composite morphism $X \otimes X^* \otimes F(V) \to X \otimes V \otimes X^* \to F(V) \otimes X \otimes X^* \to F(V)$ is given by

$$(1_{F(V)} \otimes \epsilon_X \otimes \epsilon_{F(V^*)X})\big(\varphi_V^{-1} \otimes 1_{X^*} \otimes \varphi_{V^*}^{-1}\big)(1_X \otimes \delta_{XV^*}),$$

where the rigidity identity is used to get the expression

$$(1_{F(V)} \otimes \epsilon_{F(V^*)X})\big(1_{F(V)} \otimes \varphi_{V^*}^{-1} \otimes 1\big)\big(\varphi_V^{-1} \otimes 1\big)(1_X \otimes \delta_{V^*} \otimes 1_{X^*F(V)}).$$

Now we apply the associativity of φ, $\varphi_{V\otimes V^*} = (\varphi_V \otimes 1)(1 \otimes \varphi_{V^*})$, to obtain

$$(1_{F(V)} \otimes \epsilon_{F(V^*)X})(\delta_{F(V^*)} \otimes 1_{XX^*F(V)}) = \epsilon_X \otimes 1_{F(V)}.$$

Similarly for the second diagram. $\square$

Corollary 6.2. *The following diagrams commute*:

$$\begin{array}{ccccc} X\otimes X^* & \longrightarrow & X\otimes V\otimes V^*\otimes X^* & \longrightarrow & F(V)\otimes X\otimes X^*\otimes V^* \\ \downarrow & & & & \downarrow \\ X\otimes V^*\otimes V\otimes X^* & \longrightarrow & V^*\otimes X\otimes X^*\otimes F(V) & \longrightarrow & V^*\otimes F(V)=F(V)\otimes V^*, \end{array}$$

$$\begin{array}{ccccc} W^*\otimes G(W)=G(W)\otimes W^* & \longrightarrow & G(W)\otimes X^*\otimes X\otimes W^* & \longrightarrow & X^*\otimes W\otimes W^*\otimes X \\ \downarrow & & & & \downarrow \\ W^*\otimes X^*\otimes X\otimes G(W) & \longrightarrow & X^*\otimes W^*\otimes W\otimes X & \longrightarrow & X^*\otimes X. \end{array}$$

Define the morphism

$$\epsilon : X \otimes X^* \to \mathbb{A} = \bigoplus_V F(V) \otimes V^*$$

by the weighted summation of the above morphisms over $[V]$ with weight $\dim V$. Similarly we introduce the morphism

$$\delta : \mathbb{B} = \bigoplus_W G(W) \otimes W^* \to X^* \otimes X$$

by taking the summation on $[W]$ without weights.

Lemma 6.3. *The morphism $\epsilon : X \otimes X^* \to \mathbb{A}$ is $\mathcal{A}$-$\mathcal{A}$ linear, whereas $\delta : \mathbb{B} \to X^* \otimes X$ is $\mathcal{B}$-$\mathcal{B}$ linear.*

Proof. Consider the commutativity of the diagram

$$\begin{array}{ccc} F(U)\otimes X\otimes X^* & \longrightarrow & F(U)\otimes \mathbb{A} \\ \downarrow & & \downarrow \\ X\otimes X^*\otimes U & \longrightarrow & \mathbb{A}\otimes U. \end{array}$$

The composite morphism $F(U)\otimes X\otimes X^* \to F(U)\otimes \mathbb{A} \to \mathbb{A}\otimes U$ is given by

$$\begin{aligned} F(U)\otimes X\otimes X^* &\to \bigoplus_V F(U)\otimes X\otimes V\otimes V^*\otimes X^* \\ &\to \bigoplus_V F(U)\otimes F(V)\otimes X\otimes X^*\otimes V^* \\ &\to \bigoplus_V F(U)\otimes F(V)\otimes V^* \\ &\to \bigoplus_{V,W} F(W)\otimes \begin{bmatrix} W \\ U\ V \end{bmatrix}\otimes V^* \\ &\to \bigoplus_W F(W)\otimes W^*\otimes U. \end{aligned}$$

By the naturality of the trivialization $F(\cdot)\otimes X \to X\otimes(\cdot)$, this composition can be described by

$$\begin{aligned} F(U)\otimes X\otimes X^* &\to X\otimes U\otimes X^* \\ &\to \bigoplus_V X\otimes U\otimes V\otimes V^*\otimes X^* \\ &\to \bigoplus_{V,W} X\otimes W\otimes \begin{bmatrix} W \\ U\ V \end{bmatrix}\otimes V^*\otimes X^* \\ &\to \bigoplus_W X\otimes W\otimes W^*\otimes U\otimes X^* \\ &\to \bigoplus_W F(W)\otimes X\otimes X^*\otimes W^*\otimes U \\ &\to \bigoplus_W F(W)\otimes W^*\otimes U, \end{aligned}$$

whence the problem is reduced to showing

$$\begin{array}{ccc} U & \longrightarrow & \bigoplus_V U\otimes V\otimes V^* \\ \downarrow & & \downarrow \\ \bigoplus_W W\otimes W^*\otimes U & \longleftarrow & \bigoplus_{V,W} W\otimes \begin{bmatrix} W \\ U\ V \end{bmatrix}\otimes V^*. \end{array}$$

The commutativity of the last diagram is then a routine work of Frobenius transforms: Choosing bases $\{v_j\}$, $\{w_k\}$ of V, W, respectively, and a basis $\{T\}$ of $\mathrm{Hom}(W, U \otimes V)$, the longer circuit is given by

$$\begin{aligned} u &\mapsto \sum_{V,j} d(V) u \otimes v_j \otimes v_j^* \\ &\mapsto \sum_{T,W,k} d(V)\big\langle (Tw_k)^*, u \otimes v_j\big\rangle Tw_k \otimes v_j^* \\ &\mapsto \sum d(V)\big\langle w_k^*, T^*(u \otimes v_j)\big\rangle w_k \otimes \widetilde{T} v_j^* \\ &= \sum d(V) T^*(u \otimes v_j) \otimes \widetilde{T} v_j^*. \end{aligned}$$

Here $\{(Tw_k)^*\}_{W,T,k}$ denotes the dual basis associated to the basis $\{Tw_k\}_{W,T,k}$ of the vector space $U \otimes V$.

By replacing the summation indices T and T^* with their Frobenius transforms $S : U^* \otimes W \to V$ and $S^* : V \to U^* \otimes W$ (i.e., $\{S\}$ and $\{S^*\}$ denote bases in $\mathrm{Hom}(U^* \otimes W, V)$ and $\mathrm{Hom}(V, U^* \otimes W)$, respectively, which are obtained from $\{T\}$ and $\{T^*\}$ by applying the natural isomorphisms $\mathrm{Hom}(W, U \otimes V) \to \mathrm{Hom}(U^* \otimes W, V)$ and $\mathrm{Hom}(U \otimes V, W) \to \mathrm{Hom}(V, U^* \otimes W)$), we have (use $S = {}^{tt}S$)

$$\begin{aligned} &\sum_{T,V} d(V)\big(T^* \otimes \widetilde{T}\big)(1_U \otimes \delta_{V^*}) \\ &\quad = \sum_{S,V} d(V)(\epsilon_U \otimes 1_{WW^*U})(1_U \otimes S^* S \otimes 1_{W^*U})(1_U \otimes \delta_{W^*U}) \\ &\quad = d(W)(\epsilon_U \otimes 1_{WW^*U})(1_U \otimes \delta_{W^*U}) \\ &\quad = d(W)\delta_{W^*} \otimes 1_U, \end{aligned}$$

which is used to get

$$\sum d(V) T^*(u \otimes v_j) \otimes \widetilde{T} v_j^* = \sum_{W,k} d(W) w_k \otimes w_k^* \otimes u.$$

A bit of care is needed for the right action: the commutativity of the diagram

$$\begin{array}{ccc} X \otimes X^* \otimes F(U) & \longrightarrow & \mathbb{A} \otimes F(U) \\ \big\downarrow & & \big\downarrow \\ U \otimes X \otimes X^* & \longrightarrow & U \otimes \mathbb{A}. \end{array}$$

By using the previous lemma, the composite morphism $X \otimes X^* \otimes F(U) \to \mathbb{A} \otimes F(U) \to U \otimes \mathbb{A}$ is given by

$$X \otimes X^* \otimes F(U) \to \bigoplus_V X \otimes V^* \otimes V \otimes X^* \otimes F(U)$$

$$\begin{aligned}
&\to \bigoplus_V V^* \otimes X \otimes X^* \otimes F(V) \otimes F(U) \\
&\to \bigoplus_V V^* \otimes F(V) \otimes F(U) \\
&\to \bigoplus_{V,W} V^* \otimes F(W) \otimes \begin{bmatrix} W \\ V\,U \end{bmatrix} \\
&\to \bigoplus_W U \otimes W^* \otimes F(W).
\end{aligned}$$

By the naturality of trivialization, this is equal to

$$\begin{aligned}
X \otimes X^* \otimes F(U) &\to X \otimes U \otimes X^* \\
&\to \bigoplus_V X \otimes V^* \otimes V \otimes U \otimes X^* \\
&\to \bigoplus_{V,W} X \otimes V^* \otimes W \otimes \begin{bmatrix} W \\ V\,U \end{bmatrix} \otimes X^* \\
&\to \bigoplus_W X \otimes U \otimes W^* \otimes W \otimes X^* \\
&\to \bigoplus_W U \otimes W^* \otimes X \otimes X^* \otimes F(W) \\
&\to \bigoplus_W U \otimes W^* \otimes F(W).
\end{aligned}$$

If we compare this with the other composite morphism

$$\begin{aligned}
X \otimes U \otimes X^* &\to \bigoplus_W X \otimes U \otimes W^* \otimes W \otimes X^* \\
&\to \bigoplus_W U \otimes W^* \otimes X \otimes X^* \otimes F(W) \\
&\to \bigoplus_W U \otimes W^* \otimes F(W),
\end{aligned}$$

then the problem is reduced to the commutativity of

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & \bigoplus_V V^* \otimes V \otimes U \arrow[d] \\
\bigoplus_W U \otimes W^* \otimes W & \bigoplus_{V,W} V^* \otimes W \otimes \begin{bmatrix} W \\ V\,U \end{bmatrix} \arrow[l]
\end{tikzcd},$$

which is now easily checked as before.

A similar computation works for the $\mathcal{B}$-$\mathcal{B}$ linearity. For example, the commutativity of

$$\begin{tikzcd} G(W) \otimes \mathbb{B} \arrow[r] \arrow[d] & G(W) \otimes X^* \otimes X \arrow[d] \\ \mathbb{B} \otimes W \arrow[r] & X^* \otimes X \otimes W \end{tikzcd}$$

is reduced to that of

$$\begin{tikzcd} W \otimes V \otimes V^* \arrow[r] \arrow[d] & \bigoplus_U U \otimes \begin{bmatrix} U \\ W\ V \end{bmatrix} \otimes V^* \arrow[d] \\ W \arrow[r] & \bigoplus_U U \otimes U^* \otimes W, \end{tikzcd}$$

which holds if we define the morphism $\mathbb{B} \to X^* \otimes X$ without weights. $\square$

Lemma 6.4. *The morphisms $\epsilon : X \otimes X^* \to \mathbb{A}$ and $\delta : \mathbb{B} \to X^* \otimes X$ are supported by $e_{\mathcal{B}}$ and $e_{\mathcal{A}}$, respectively, i.e., $\epsilon \circ e_{\mathcal{B}} = \epsilon$ and $e_{\mathcal{A}} \circ \delta = \delta$.*

Proof. We shall check $\epsilon \circ e_{\mathcal{B}} = \epsilon$. By the commutativity of left and right actions on X, we see that the composition $\sum_k \epsilon \circ \pi(\widehat{w}_{kk})$ is given by

$$\begin{aligned} X \otimes X^* &\longrightarrow \bigoplus_V X \otimes V \otimes V^* \otimes X^* \\ &\longrightarrow \bigoplus_V F(V) \otimes X \otimes X^* \otimes V^* \\ &\longrightarrow \bigoplus_V F(V) \otimes X \otimes G(W)^* \otimes G(W) \otimes X^* \otimes V^* \\ &\longrightarrow \bigoplus_V F(V) \otimes W^* \otimes X \otimes X^* \otimes W \otimes V^* \\ &\xrightarrow{\epsilon_X} \bigoplus_V F(V) \otimes W^* \otimes W \otimes V^* \\ &\xrightarrow{\sum \pi(\widehat{w}_{kk})} \bigoplus_V F(V) \otimes V^*. \end{aligned}$$

From the definition of $G(W) \otimes X^* \to X^* \otimes W$, the morphism

$$X \otimes X^* \xrightarrow{1 \otimes \delta_{G(W)} \otimes 1} X \otimes G(W)^* \otimes G(W) \otimes X^* \longrightarrow W^* \otimes X \otimes X^* \otimes W \xrightarrow{\epsilon_{W^* X}} I$$

is equal to $d(W)\epsilon_X$. Since $\sum_k \pi(\widetilde{w}_{kk}) = d(W)(1 \otimes \epsilon_{W^*} \otimes 1)$, we obtain the relation

$$\sum_k \epsilon \circ \pi(\widehat{w}_{kk}) = d(W)^* \epsilon$$

and hence $\epsilon \circ e_{\mathcal{B}} = \epsilon$ by taking the summation over the set $\{[W]\}$. $\square$

We shall now compute

$$X \xrightarrow{\omega_B^{-1}} X \otimes_{\mathcal{B}} \mathbb{B} \xrightarrow{1 \otimes \delta} X \otimes_{\mathcal{B}} X^* \otimes_{\mathcal{A}} X \xrightarrow{\epsilon \otimes 1} \mathbb{A} \otimes_{\mathcal{A}} X \xrightarrow{\omega_A} X.$$

As ϵ, δ, and (λ, ρ) are supported by $e_{\mathcal{A}}$ or $e_{\mathcal{B}}$, the problem is equivalent to working with

$$X \xrightarrow{\omega_B^{-1}} X \otimes \mathbb{B} \xrightarrow{1 \otimes \delta} X \otimes X^* \otimes X \xrightarrow{\epsilon \otimes 1} \mathbb{A} \otimes X \xrightarrow{\omega_A} X.$$

From definition, the composition $X \to X \otimes \mathbb{B} \to X \otimes X^* \otimes X$ is given by

$$\begin{aligned} X &\xrightarrow{\text{weight}} \bigoplus_W W^* \otimes W \otimes X \to \bigoplus_W X \otimes W^* \otimes G(W) \\ &\longrightarrow \bigoplus_W X \otimes W^* \otimes X^* \otimes X \otimes G(W) \\ &\longrightarrow \bigoplus_W X \otimes W^* \otimes X^* \otimes W \otimes X \to X \otimes X^* \otimes X, \end{aligned}$$

where weight $= d(W)\omega_B^{-1} \dim(B)^{-1}$. By Lemma 6.1, this is equivalent to

$$\begin{aligned} X &\xrightarrow{\text{weight}} \bigoplus_W W^* \otimes W \otimes X \to \bigoplus_W W^* \otimes X \otimes G(W) \\ &\longrightarrow \bigoplus_W W^* \otimes X \otimes G(W) \otimes X^* \otimes X \\ &\longrightarrow \bigoplus_W W^* \otimes X \otimes X^* \otimes W \otimes X \to X \otimes X^* \otimes X. \end{aligned}$$

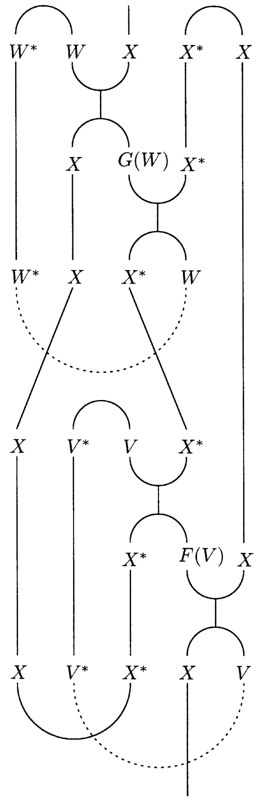
Similarly, the composition $X \otimes X^* \otimes X \to \mathbb{A} \otimes X \to X$ is given by

$$\begin{aligned} XX^*X &\xrightarrow{\text{weight}} \bigoplus_V XV^*VX^*X \to \bigoplus_V XV^*X^*F(V)X \\ &\longrightarrow \bigoplus_V XV^*X^*XV \to XX^*X \xrightarrow{\epsilon \otimes 1} X \end{aligned}$$

with weight $= d(V)\omega_A$.

Note here that by the commutativity $\mathcal{T} \otimes \mathcal{V} = \mathcal{V} \otimes \mathcal{T}$, the position of vector spaces such as V can be freely moved left and right, which is pictorially reflected in crossing lines (cf. Fig. 2).

Now, combining these two expressions and then applying the definition of the trivialization isomorphisms $G(W) \otimes X^* \to X^* \otimes W$, $V \otimes X^* \to X^* \otimes F(V)$, we have the morphism


W^*
W
X
X^*
X
$G(W)$
X^*
W^*
X
X^*
W
X
V^*
V
X^*
X^*
$F(V)$
X
X
V^*
X^*
X
V

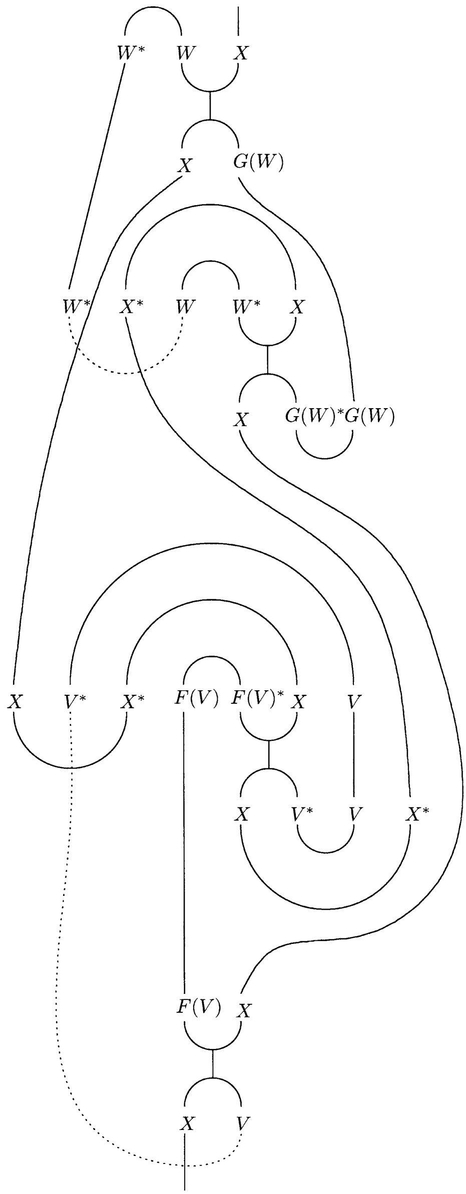

W^*
W
X
X
$G(W)$
W^*
X^*
W
W^*
X
X
$G(W)^*G(W)$
X
V^*
X^*
$F(V)$
$F(V)^*$
X
V
X
V^*
V
X^*
$F(V)$
X
X
V

Fig. 2.

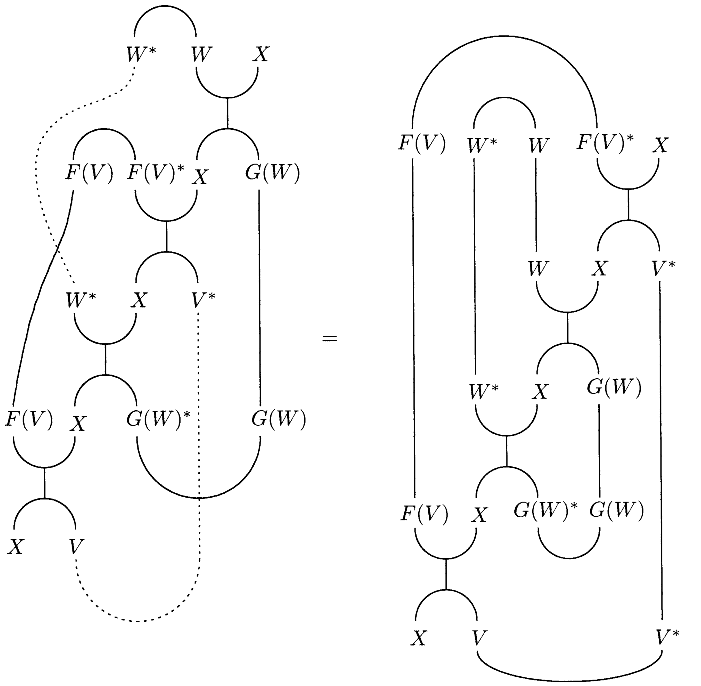


Fig. 3.

$$
\begin{aligned}
X &\to W^*WX \to W^*XG(W) \to W^*F(V)F(V)^*XG(W)\\
&\to F(V)W^*XV^*G(W) \to F(V)XG(W^*)V^*G(W)\\
&\to XVV^*G(W)^*G(W) \to X,
\end{aligned}
$$

which is summed over $[V]$ and $[W]$ with the weight $d(V)d(W)\omega_A/\omega_B \dim(A)$ multiplied (Fig. 2). By the commutativity of left and right actions, we can replace the part $F(V)^*WX \to XV^*G(W)$ with

$$WF(V^*)X \to WXV^* \to XG(W)V^*$$

to get the expression (Fig. 3)

$$
\begin{aligned}
X &\to F(V)W^*WF(V)^*X \to F(V)W^*WXV^* \to F(V)W^*XG(W)V^*\\
&\to F(V)XG(W^*)G(W)V^* \to XVG(W)^*G(W)V^* \to X.
\end{aligned}
$$

By the associativity of the right action on X, the last local morphism is reduced to

$$X \to F(V)F(V)^*X \to F(V)XV^* \to XVV^* \to X$$

multiplied by $d(W)$, which is further reduced to $d(V)d(W)1_X$ by the associativity of the left action on X.

In total, the morphism $X \to XX^*X \to X$ in question amounts to the scalar multiple of 1_X by

$$\sum_{V,W} \frac{d(V)^2 d(W)^2}{\dim B} \frac{\omega_A}{\omega_B} = \dim(A) \frac{\omega_A}{\omega_B}.$$

Similarly, we compute the composition

$$X^* \xrightarrow{\omega_B^{-1}} \mathbb{B} \otimes X^* \xrightarrow{\delta \otimes 1} X^* \otimes X \otimes X^* \xrightarrow{1 \otimes \epsilon} X^* \otimes \mathbb{A} \xrightarrow{\omega_A} X^*$$

to find that it is a scalar multiple of 1_{X^*} by the same scalar.

Proposition 6.5. *Let $\mathcal{T}$ be a rigid semisimple tensor category. Then the bicategory $\mathcal{M}(\mathcal{T})$ is rigid as well. More precisely, if the unit constraints are specified by a function $\{\omega_A\}_A$ indexed by finite-dimensional Hopf algebras realized inside $\mathcal{T}$, then a rigidity pair for an $\mathcal{A}$-$\mathcal{B}$ module X is given by $(\epsilon, c\delta)$ with $c = \dim(A)\omega_A/\omega_B$, where ϵ and δ are defined above.*

We now present results related to the notion of quantum dimension in tensor categories. Although there are several equivalent formulations for (quantum) dimension of objects in (rigid) tensor categories (see [1,4,28] for example, cf. also [25]), we here use the one introduced in [40,43]: By an involution, we shall mean a contravariant functor $*$ from $\mathcal{T}$ into $\mathcal{T}$ itself (the operation on morphisms being denoted by ${}^t f : Y^* \to X^*$ instead of f^* here) with natural families of isomorphisms $\{c_{X,Y} : Y^* \otimes X^* \to (X \otimes Y)^*\}$ (anticommutativity) and $\{d_X : X \to (X^*)^*\}$ (duality) satisfying the commutativity of the diagrams

$$\begin{tikzcd}
(X^* \otimes Y^*) \otimes Z^* \arrow[r, "c\otimes 1"] \arrow[d, "a"'] & (Y \otimes X)^* \otimes Z^* \arrow[r, "c"] & (Z \otimes (Y \otimes X))^* \arrow[d, "{}^t a"] \\
X^* \times (Y^* \otimes Z^*) \arrow[r, "1\otimes c"'] & X^* \otimes (Z \otimes Y)^* \arrow[r, "c"'] & ((Z \otimes Y) \otimes X)^*,
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes Y \arrow[r, "d\otimes d"] \arrow[d, "d"'] & X^{**} \otimes Y^{**} \arrow[d, "c"] \\
(X \otimes Y)^{**} \arrow[r, "{}^t c"'] & (Y^* \otimes X^*)^*,
\end{tikzcd}$$

and the equality ${}^t d_X = d_{X^*}^{-1} : X^{***} \to X^*$. (The naturality means ${}^t(f \otimes g) \overset{c}{\sim} {}^t g \otimes {}^t f$ and $f \overset{d}{\sim} {}^t({}^t f)$.) There is a coherence result on tensor categories with involution $(*, t, c, d)$, which enables us to restrict ourselves to strict involutions without losing generality [1,13].

A Frobenius duality in a tensor category then consists of a strict involution $(*, t, c, d)$ and a family of morphisms $\{\epsilon_X : X \otimes X^* \to I\}$, which satisfies

(i) $\epsilon_{X\otimes Y} = \epsilon_X(1_X \otimes \epsilon_Y \otimes 1_{X^*})$,
(ii) $\epsilon_Y(f \otimes 1) = \epsilon_X(1 \otimes {}^t f)$ for $f : X \to Y$,
(iii) the map $\mathrm{Hom}(X, Y) \ni f \mapsto \epsilon_Y(f \otimes 1) \in \mathrm{Hom}(X \otimes Y^*, I)$ being injective and
(iv) $\epsilon_X(f \otimes 1)^t \epsilon_X = \epsilon_{X^*}(1 \otimes f)^t \epsilon_{X^*}$ for $f \in \mathrm{End}(X)$ (the operation of taking dual objects being assumed to be strict here for simplicity, see [43] for details).

If the tensor category $\mathcal{T}$ is furnished with a Frobenius duality $\{\epsilon_X : X \otimes X^* \to I\}$, it is natural to use the following normalization for the trivializing isomorphisms of the unit object $\mathbb{A}$: Let the trivializing isomorphisms be chosen by taking $\theta = \{\sqrt{d(V)}\, 1_{V^*}\}$ as gauge in the remark after Lemma 3.2. The morphisms $\epsilon : X \otimes X^* \to \mathbb{A}$ and $\delta : \mathbb{B} \to X^* \otimes X$ are then changed into the ones associated to the pairing

$$V \otimes V^* \ni v \otimes v^* \mapsto \sqrt{d(V)}\langle v, v^* \rangle$$

or its dualized copairing

$$\sqrt{d(V)} \sum_i v_i \otimes v_i^* \in V \otimes V^*.$$

Proposition 6.6. *Suppose that the semisimple tensor category $\mathcal{T}$ is furnished with a Frobenius duality $\{\epsilon_X\}$ and let the unit constraint $\mathbb{A} \otimes X \to X$ be normalized by the factor $\omega_A = |A|^{-1/2}$ for each A with $|A| = \dim A$. Then the renormalized family $\{|A|^{-1/4}|B|^{-1/4}\epsilon\}$ gives a Frobenius duality in the bicategory $\mathcal{M}(\mathcal{T})$.*

Corollary 6.7 (Dimension formula). *For an $\mathcal{A}$-$\mathcal{B}$ module ${}_{\mathcal{A}}X_{\mathcal{B}}$, its dimension is calculated by*

$$\dim({}_{\mathcal{A}}X_{\mathcal{B}}) = \frac{\dim(X)}{|A|^{1/2}|B|^{1/2}}.$$

Here $\dim(X)$ *denotes the dimension of X as an object of $\mathcal{T}$.*

7. Duality for orbifolds on tensor categories

Let H be an object in a rigid semisimple bicategory and assume that H satisfies the condition (referred to as the absorbing property in what follows)

$$H \otimes H^* \otimes H \cong H \oplus \cdots \oplus H.$$

Given an object H of this type, we can associate a Hopf algebra B so that its Tannaka dual $\mathcal{B}$ is equivalent to the tensor category generated by $H^* \otimes H$ [41, Appendix C]. More explicitly, we can construct a monoidal functor E, which assigns the finite-dimensional vector space $E(X)$ to each object X in $(H^* \otimes H)^n$ with n a positive integer, where $E(X)$ is defined by

$$E(X) = \mathrm{Hom}(H, H \otimes X)$$

and the multiplicativity isomorphism $E(X) \otimes E(Y) \to E(X \otimes Y)$ is given by

$$E(X) \otimes E(Y) \ni x \otimes y \mapsto (x \otimes 1_Y)y \in E(X \otimes Y).$$

Example 7.1. Consider the Tannaka dual $\mathcal{A}$ of a finite-dimensional Hopf algebra A realized in a semisimple tensor category $\mathcal{T}$ and let $\mathbb{A}$ be the associated unit object for $\mathcal{A}$-$\mathcal{A}$ modules.

Then, by forgetting the left $\mathcal{A}$-module structure, the right $\mathcal{A}$-module $H = \mathbb{A}_{\mathcal{A}}$ satisfies the above condition as an object in an "off-diagonal piece" in the bicategory

$$\begin{pmatrix} \mathcal{T} & \mathcal{M}_{\mathcal{A}} \\ {}_{\mathcal{A}}\mathcal{M} & {}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}} \end{pmatrix}.$$

In fact, we have

$$H \otimes_{\mathcal{A}} H^* = \mathbb{A} = \bigoplus_V V^* \otimes F(V)$$

and therefore

$$\bigoplus_V V^* \otimes F(V) \otimes \mathbb{A}_{\mathcal{A}} \cong \bigoplus_V V^* \otimes \mathbb{A}_{\mathcal{A}} \otimes V = \bigoplus_V V^* \otimes V \otimes \mathbb{A}_{\mathcal{A}}$$

is isomorphic to a direct sum of H's.

Moreover we can identify the associated Hopf algebra with A: Given an object V in $\mathcal{A}$, the vector space $E(F(V)) = \mathrm{Hom}(H, F(V) \otimes H)$ is naturally isomorphic to V by the trivialization isomorphism $F(V) \otimes H \cong H \otimes V$ and the simplicity of $H_{\mathcal{A}}$. Moreover, we have the commutative diagram

$$\begin{tikzcd} V \otimes W \arrow[r, equal] \arrow[d] & V \otimes W \arrow[d] \\ E(F(V)) \otimes E(F(W)) \arrow[r] & E(F(V \otimes W)) \end{tikzcd}$$

and the monoidal functor E is naturally isomorphic to the identity functor in $\mathcal{A}$. Thus the associated Hopf algebra is naturally isomorphic to A, whereas the object $H^* \otimes H$ generates the tensor category monoidally equivalent to the Tannaka dual of the dual Hopf algebra $B = A^*$.

Proposition 7.2. *The construction of Hopf algebras from objects of absorbing property is universal, i.e., any finite-dimensional semisimple Hopf algebra arises this way.*

Returning to the initial case of this section, the obvious identification

$$H \otimes X \to E(X) \otimes H$$

can be interpreted as giving a right action of $\mathcal{B}$ on H.

Consider the composite isomorphism

$$H^* \otimes H \to \bigoplus_X X \otimes \mathrm{Hom}(X, H^* \otimes H) \to \bigoplus_X X \otimes E(X^*) = \mathbb{B}.$$

We shall show that this isomorphism is $\mathcal{B}$-$\mathcal{B}$ linear, i.e., the commutativity of

$$\begin{array}{ccc} X \otimes H^* \otimes H \otimes Y & \longrightarrow & X \otimes \mathbb{B} \otimes Y \\ \downarrow & & \downarrow \\ H^* \otimes E(X) \otimes E(Y) \otimes H & \longrightarrow & E(Y) \otimes \mathbb{B} \otimes E(X) \end{array}$$

or equivalently, by applying the functor $\mathrm{Hom}(Z, \cdot)$ with Z a simple object, we have the commutative diagram of vector spaces. For simplicity, letting $X = I$ (the letter X will be used as a dummy index), the relevant isomorphisms are given by

$$\begin{array}{ccccc} \begin{bmatrix} Z \\ H^*HY \end{bmatrix} & \longrightarrow & \displaystyle\bigoplus_X \begin{bmatrix} X \\ H^*H \end{bmatrix} \otimes \begin{bmatrix} Z \\ X\,Y \end{bmatrix} & \longrightarrow & \displaystyle\bigoplus_X \begin{bmatrix} H \\ H\,X^* \end{bmatrix} \otimes \begin{bmatrix} X^* \\ Y\,Z^* \end{bmatrix} \\ \downarrow & & & & \downarrow \\ \begin{bmatrix} H \\ H\,Y \end{bmatrix} \otimes \begin{bmatrix} Z \\ H^*\,H \end{bmatrix} & \longrightarrow & \begin{bmatrix} H \\ H\,Y \end{bmatrix} \otimes \begin{bmatrix} H \\ H\,Z^* \end{bmatrix} & \longrightarrow & \begin{bmatrix} H \\ HYZ^* \end{bmatrix}. \end{array}$$

To check the commutativity, let us start with a vector

$$x \otimes T \in \begin{bmatrix} H \\ H\,X^* \end{bmatrix} \otimes \begin{bmatrix} X^* \\ Y\,Z^* \end{bmatrix}.$$

The upper horizontal line is then described by

$$(\widetilde{x} \otimes 1)\widetilde{T} \mapsto \widetilde{x} \otimes \widetilde{T} \mapsto x \otimes T,$$

while the right and the left vertical lines are presented by $x \otimes T \mapsto (1 \otimes T)x$ and

$$(\widetilde{x} \otimes 1)\widetilde{T} \mapsto \sum_{j,k} \big\langle z_k^*\big(1 \otimes y_j^*\big), (\widetilde{x} \otimes 1)\widetilde{T}\big\rangle y_j \otimes z_k$$

with $\{y_j, y_j^*\}$ and $\{z_k, z_k^*\}$ in the duality relation ($y_j^* y_j = 1_H$ and $z_k^* z_k = 1_Z$ particularly). Finally, the bottom line is given by

$$\sum_{j,k} c_{jk} y_j \otimes z_k \mapsto \sum_{j,k} c_{jk} y_j \otimes \widetilde{z}_k \mapsto \sum_{j,k} c_{jk}(y_j \otimes 1)\widetilde{z}_k.$$

To identify the last summation with $(1 \otimes T)x$, we rewrite c_{jk} as follows:

$$
\begin{aligned}
&d(Z)c_{jk}\\
&\quad = \epsilon_Z\big(z_k^* \otimes 1\big)\big(1 \otimes y_j^* \otimes 1\big)\big(\widetilde{x} \otimes 1\big)\big(\widetilde{T} \otimes 1\big)\delta_{Z^*}\\
&\quad = \epsilon_Z\big(z_k^* \otimes 1\big)\big(1 \otimes y_j^* \otimes 1\big)(1 \otimes \epsilon_{X^*} \otimes 1)\big(1 \otimes x \otimes \widetilde{T} \otimes 1\big)(\delta_H \otimes \delta_{Z^*})\\
&\quad = \epsilon_{H^*}\big(1 \otimes \widetilde{z_k^*}\big)\big(1 \otimes y_j^* \otimes 1\big)(1 \otimes \epsilon_{X^*} \otimes 1)\big(1 \otimes x \otimes \widetilde{T} \otimes 1\big)(1 \otimes \delta_{Z^*})\delta_H,
\end{aligned}
$$

which yields the relation

$$
\frac{d(Z)}{d(H)}c_{jk}1_H = \widetilde{z_k^*}\big(y_j^* \otimes 1\big)(1 \otimes \epsilon_{X^*} \otimes 1)\big(x \otimes \widetilde{T} \otimes 1\big)(1_H \otimes \delta_{Z^*}).
$$

Now this formula is used to get

$$
\begin{aligned}
\sum_k c_{jk}\widetilde{z_k} &= \sum_k c_{jk}\widetilde{z_k}1_H\\
&= \sum_k \frac{d(H)}{d(Z)}\widetilde{z_k}\widetilde{z_k^*}\big(y_j^* \otimes 1\big)(1 \otimes \epsilon_{X^*} \otimes 1)\big(x \otimes \widetilde{T} \otimes 1\big)(1_H \otimes \delta_{Z^*}).
\end{aligned}
$$

From the relation

$$
\big\langle \widetilde{z_k^*}\widetilde{z_k}\big\rangle = \epsilon_Z\big(z_k^* \otimes 1\big)(z_k \otimes 1)\delta_{Z^*} = d(Z),
$$

we see that $\widetilde{z_k^*}\widetilde{z_k} = d(Z)/d(H)1_H$ and hence

$$
(\widetilde{z_k})^* = \frac{d(H)}{d(Z)}\widetilde{z_k^*}.
$$

Feeding this back into the above summation, we have

$$
\sum_k c_{jk}\widetilde{z_k} = \big(y_j^* \otimes 1\big)(1 \otimes \epsilon_{X^*} \otimes 1)\big(x \otimes \widetilde{T} \otimes 1\big)(1_H \otimes \delta_{Z^*})
$$

and then

$$
\begin{aligned}
\sum_{j,k} c_{jk}(y_j \otimes 1)\widetilde{z_k} &= \sum_j \big(y_j y_j^* \otimes 1\big)(1 \otimes \epsilon_{X^*} \otimes 1)\big(x \otimes \widetilde{T} \otimes 1\big)(1_H \otimes \delta_{Z^*})\\
&= (1 \otimes \epsilon_{X^*} \otimes 1)\big(x \otimes \widetilde{T} \otimes 1\big)(1_H \otimes \delta_{Z^*})\\
&= (1 \otimes T)x.
\end{aligned}
$$

Lemma 7.3. *We have*

$$
{}_{\mathcal{B}}H^* \otimes H_{\mathcal{B}} \cong {}_{\mathcal{B}}\mathbb{B}_{\mathcal{B}}, \qquad H \otimes_{\mathcal{B}} H^* \cong I.
$$

Proof. We have just checked the former relation. By Frobenius reciprocity (see [16] for example), this implies

$$
\begin{aligned}
\dim \mathrm{End}(H \otimes_{\mathcal{B}} H^*) &= \dim \mathrm{Hom}(H_{\mathcal{B}}, H \otimes_{\mathcal{B}} H^* \otimes H_{\mathcal{B}})\\
&= \dim \mathrm{End}({}_{\mathcal{B}}H^* \otimes H_{\mathcal{B}}) = 1
\end{aligned}
$$

and hence $H \otimes_{\mathcal{B}} H^* = I$ by semisimplicity. $\square$

Since bimodules with the similar property are referred to as imprimitivity bimodules in connection with Mackey's imprimitivity theorem on induced representations [10,30], we call an object M in a rigid bicategory an *imprimitivity object* if both of $M \otimes M^*$ and $M^* \otimes M$ are isomorphic to unit objects. In a tensor category, this is nothing but saying that M is an invertible object.

The following observation, though obvious, is the essence of duality for orbifold constructions.

Lemma 7.4. *Let*

$$\begin{pmatrix} \mathcal{T} & \mathcal{M} \\ \mathcal{M}^* & \mathcal{S} \end{pmatrix}$$

be a rigid semisimple bicategory and M be an imprimitivity object in $\mathcal{M}$.

Then two tensor categories $\mathcal{S}$ and $\mathcal{T}$ are equivalent. More precisely,

$$X \mapsto M \otimes X \otimes M^*, \qquad Y \mapsto M^* \otimes Y \otimes M$$

gives the monoidal equivalence between $\mathcal{S}$ and $\mathcal{T}$.

Given a monoidal imbedding $F : \mathcal{A} \to \mathcal{T}$ of the Tannaka dual $\mathcal{A}$ of a finite-dimensional semisimple Hopf algebra A into a rigid semisimple tensor category $\mathcal{T}$, let $H = \mathbb{A}_{\mathcal{A}}$ be an off-diagonal object in the bicategory

$$\begin{pmatrix} \mathcal{T} & \mathcal{M}_{\mathcal{A}} \\ {}_{\mathcal{A}}\mathcal{M} & {}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}} \end{pmatrix}.$$

Here $\mathcal{M}_{\mathcal{A}}$ denotes the category of right $\mathcal{A}$-modules in $\mathcal{T}$ and similarly for others.

Then H meets the absorbing property and the tensor subcategory of ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ generated by $H^* \otimes H = {}_{\mathcal{A}}\mathbb{A} \otimes \mathbb{A}_{\mathcal{A}}$ is isomorphic to the Tannaka dual $\mathcal{B}$ of the dual Hopf algebra of A. Let $G : \mathcal{B} \to {}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ be the accompanied monoidal imbedding. Recall here that the Tannaka dual $\mathcal{A}$ of A is the one associated to $H \otimes H^*$ as seen in the above example.

Thus we can talk about $\mathcal{B}$-modules in $\mathcal{M}$: Let $\mathcal{M}_{\mathcal{B}}$ (respectively ${}_{\mathcal{B}}\mathcal{M}$) be the category of right (respectively left) $\mathcal{B}$-modules in $\mathcal{M}_{\mathcal{A}}$ (respectively ${}_{\mathcal{A}}\mathcal{M}$) and ${}_{\mathcal{B}}\mathcal{M}_{\mathcal{B}}$ be the category of $\mathcal{B}$-$\mathcal{B}$ bimodules in ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$. Then these, together with the starting tensor category $\mathcal{T}$, form a bicategory

$$\begin{pmatrix} \mathcal{T} & \mathcal{M}_{\mathcal{B}} \\ {}_{\mathcal{B}}\mathcal{M} & {}_{\mathcal{B}}\mathcal{M}_{\mathcal{B}} \end{pmatrix}.$$

Thanks to the previous discussions, the object $H = \mathbb{A}_{\mathcal{A}}$ in $\mathcal{M}_{\mathcal{A}}$ admits a structure of right $\mathcal{B}$-module, which gives rise to an imprimitivity object $M_{\mathcal{B}}$ in $\mathcal{M}_{\mathcal{B}}$. Then the above lemma shows that the tensor category ${}_{\mathcal{B}}\mathcal{M}_{\mathcal{B}}$ is isomorphic to the original tensor category.

To extract the meaning of this, we first introduce the notation $\mathcal{T} \rtimes_F \mathcal{A}$ for the tensor category ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$, which is interpreted as the crossed product of $\mathcal{T}$ by F. Then the monoidal imbedding $G : \mathcal{B} \to \mathcal{T} \rtimes_F \mathcal{A}$ describes the dual symmetry in $\mathcal{T} \rtimes_F \mathcal{A}$ and we can construct the second crossed product $(\mathcal{T} \rtimes_F \mathcal{A}) \rtimes_G \mathcal{B}$.

Theorem 7.5. *With the notation described above, we have the duality for crossed products*: *the second crossed product tensor category* $(\mathcal{T} \rtimes_F \mathcal{A}) \rtimes_G \mathcal{B}$ *is monoidally equivalent to the original tensor category* $\mathcal{T}$ *in a canonical way.*

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