JOURNAL OF ALGEBRA **179,** 751–774 (1996)
ARTICLE NO. 0035

Conjugacy Separability of Amalgamated Free Products of Groups

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Communicated by Alexander Lubotsky

Received August 23, 1994

INTRODUCTION

A group G is said to be *conjugacy separable* if any two elements x and y of G, whose images are conjugate in every finite quotient of G, are conjugate in G. The importance of this notion was pointed out by Mal'cev, who proved in [14] that if a finitely presented group G is conjugacy separable, then G has a solvable conjugacy problem, that is, there exists an algorithm to decide whether or not any two given elements of G are conjugate.

In this paper we are concerned with the conjugacy separability of certain amalgamated free products of groups.

Blackburn [3] proved that finitely generated torsion-free nilpotent groups are conjugacy separable; and Baumslag [2] showed that free groups are conjugacy separable. The result of Blackburn was extended by Remeslennikov [15] and Formanek [9] to polycyclic-by-finite groups. Dyer [7] proved that free-by-finite groups are also conjugacy separable. However, it is not

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0021-8693/96 $12.00

known in general whether a finite extension of a conjugacy separable group is conjugacy separable.

In [21] Stebe and in [16] Remeslennikov show that the free product of conjugacy separable groups is conjugacy separable. The extension of this result to amalgamated free products is complicated for several reasons. Note that if a group is conjugacy separable, then it must be residually finite. Let $G = G_1 *_H G_2$ be an amalgamated free product of the groups G_1 and G_2 amalgamating a common subgroup H. The first problem that one encounters is that the residual finiteness of G_1 and G_2 does not imply in general that G is residually finite. Baumslag [1] proved that if G_1 and G_2 are either both free or both torsion-free finitely generated nilpotent groups, and H is cyclic, then $G = G_1 *_H G_2$ is residually finite. It is probably known that, if G_1 and G_2 are free-by-finite or finitely generated nilpotent-by-finite groups (not necessarily both of the same type) and H is cyclic, then $G = G_1 *_H G_2$ is residually finite (we do not know an explicit reference for this result, but we prove it in Proposition 3.2).

In [8] Dyer shows that if G_1 and G_2 are either both free or both finitely generated nilpotent groups, and H is cyclic, then $G = G_1 *_H G_2$ is conjugacy separable. Tang [24] generalized this theorem of Dyer. To explain the results of Tang, we need first some terminology. One says that a group R has the *unique root property* if whenever x and y are elements of R of infinite order, and $x^n = y^n$ (for some natural number n), then $x = y$. A subgroup K of a group R is said to be *isolated* in R if whenever $x \in R$, and $x^n \in K$ (for some $n = 1, 2, \ldots$), then $x \in K$. Tang proves that $G = G_1 *_H G_2$ is conjugacy separable under the following conditions: (i) G_1 and G_2 are either both free-by-finite or both finitely generated nilpotent-by-finite groups; (ii) H is cyclic; and either (iii) G_1 and G_2 have the unique root property, or (iii)$'$ H is isolated in G_1 and G_2.

In this paper we strengthen Tang's theorems to prove that if G_1 and G_2 are free-by-finite or finitely generated nilpotent-by-finite groups and H is cyclic, then $G = G_1 *_H G_2$ is conjugacy separable (see Theorem 3.8). The methods we use to prove our results are quite different from those used in the papers mentioned above. If R is a residually finite group, and x and y are elements of R, then the images of x and y are conjugate in every finite quotient of R if and only if x and y are conjugate in the profinite completion $\hat{R}$ of R. In the cases we are concerned with, namely if G_1 and G_2 are free-by-finite or finitely generated nilpotent-by-finite groups and H is cyclic, the profinite completion $\hat{G}$ of $G = G_1 *_H G_2$ is the amalgamated free product $\hat{G} = \hat{G}_1 \amalg_{\hat{H}} \hat{G}_2$ of $\hat{G}_1$ and $\hat{G}_2$ amalgamating $\hat{H}$, in the category of profinite groups. This simple observation allows us to use the theory of profinite groups acting on profinite trees (cf. [11, 26]). According to the Bass–Serre theory of groups acting on trees (cf. [19]), $G = G_1 *_H G_2$ acts in a natural way on a tree $\mathscr{S}(G)$ associated with this amalgamated product. Similarly, $\hat{G}$ acts continuously on a profinite tree $\mathscr{S}(\hat{G})$ associated with $\hat{G}_1 \amalg_{\hat{H}} \hat{G}_2$. It turns out that, under our hypotheses G_1, G_2, and H are closed in the profinite topology of G; this implies that $\mathscr{S}(G)$ is embedded naturally in $\mathscr{S}(\hat{G})$. The methods of proof in the results of this paper are based on the study of the connections between $\mathscr{S}(G)$ and $\mathscr{S}(\hat{G})$.

Finally, we point out that Shirvani [20] has also investigated conjugacy separability in fundamental groups of graphs of groups, using techniques similar to ours.

2. NOTATION AND TERMINOLOGY

Let G be a group, H and K subgroups of G, and $h, k \in G$. Then, as usual, we define $h^k = khk^{-1}$; $H^K = \{h^k \mid h \in H,\ k \in K\}$; and $h^K = \{h^k \mid k \in K\}$. $\mathfrak{N}_G(h)$ denotes the normalizer of $\langle h \rangle$ in G, i.e., $\mathfrak{N}_G(h) = \{g \in G \mid g\langle h \rangle g^{-1} = \langle h \rangle\}$.

Recall that a *profinite group* is an inverse limit of finite groups (each of them endowed with the discrete topology), i.e., a compact, Hausdorff, totally disconnected topological group. Let R be a residually finite group. If x and y are elements of R, we use the notation $x \sim_R y$ to indicate that x and y are conjugate in R. Denote by $\hat{R}$ the *profinite completion* of R, that is, $\hat{R} = \neg \varprojlim R/U$, where U runs through the collection $\mathfrak{U}$ of all normal subgroups R of finite index. Then R is naturally embedded in $\hat{R}$. Now, $\hat{R}$ is a profinite group and it induces a topology on R that is called the *profinite topology* of R. An element $x \in R$ is called *conjugacy distinguished* if the conjugacy class of x in R is closed in the profinite topology of R. One says that R is *conjugacy separable* if every element of R is conjugacy distinguished. Equivalently, R is conjugacy separable if and only if for any two elements $x, y \in R$, $x \sim_{\hat{R}} y$ implies that $x \sim_R y$. Or, in other words, R is conjugacy separable if and only if for any two elements $x, y \in R$, $x \sim_{R/N} y$ for all $N \in \mathfrak{U}$, implies that $x \sim_R y$. If X is a subset of R, then $\mathrm{Cl}(X)$ and $\overline{X}$ denote the topological closures of X in R (where R is endowed with its profinite topology) and $\hat{R}$, respectively.

Let G_1 and G_2 be residually finite groups, and assume that H is a common subgroup of G_1 and G_2. We denote by $G = G_1 *_H G_2$ the amalgamated free product of G_1 and G_2 amalgamating H. G. Baumslag has established the following result, which gives sufficient conditions for G to be residually finite.

PROPOSITION 1.1 [1, Proposition 2]. *The group $G = G_1 *_H G_2$ is residually finite if there exist families $\{N_{1\lambda} \mid \lambda \in \Lambda\}$ and $\{N_{2\lambda} \mid \lambda \in \Lambda\}$ of normal* *subgroups of finite index in* G_1 *and* G_2 *respectively, such that* $N_{1\lambda} \cap H = N_{2\lambda} \cap H$, *for all* $\lambda \in \Lambda$, $\bigcap N_{1\lambda} = \bigcap N_{2\lambda} = 1$, *and* $\bigcap HN_{1\lambda} = \bigcap HN_{2\lambda} = H$.

Let Γ_1 and Γ_2 be profinite groups with a common closed subgroup Δ. One says that the profinite amalgamated free product of Γ_1 and Γ_2 with amalgamated subgroup Δ exists if the canonical homomorphisms of Γ_1 and Γ_2 into the push-out of Γ_1 and Γ_2 over Δ in the category of profinite groups are monomorphisms. If it exists, we denote by $\Gamma = \Gamma_1 \amalg_\Delta \Gamma_2$ the amalgamated free product of Γ_1 and Γ_2 amalgamating Δ, in the category of profinite groups. One has the following criterion.

Criterion [17, Theorem 1.2]. The profinite amalgamated free product $\Gamma = \Gamma_1 \amalg_\Delta \Gamma_2$ exists if and only if there exist families $\{N_{1\lambda} \mid \lambda \in \Lambda\}$ and $\{N_{2\lambda} \mid \lambda \in \Lambda\}$ of open normal subgroups in Γ_1 and Γ_2, respectively, such that $N_{1\lambda} \cap \Delta = N_{2\lambda} \cap \Delta$, for all $\lambda \in \Lambda$, and $\bigcap N_{1\lambda} = \bigcap N_{2\lambda} = 1$.

One can easily see that if $\Gamma = \Gamma_1 \amalg_\Delta \Gamma_2$ is the profinite amalgamated free product of Γ_1 and Γ_2 amalgamating Δ, then Γ is the completion of $\Gamma_1 *_\Delta \Gamma_2$ with respect to the topology consisting of those normal subgroups N of finite index in $\Gamma_1 *_\Delta \Gamma_2$ such that $N \cap \Gamma_i$ is open in Γ_i $(i = 1, 2)$ (cf. [17]). Moreover this topology of $\Gamma_1 *_\Delta \Gamma_2$ is Hausdorff, and so one can think of $\Gamma_1 *_\Delta \Gamma_2$ as a dense subgroup of Γ.

Next we recall some basic notions in the Bass–Serre theory of groups acting on trees [19, 5]. A *graph* $\mathscr{X}$ is a set with a distinguished subset of vertices $V = V(\mathscr{X})$, and a set of *edges* $E = E(\mathscr{X}) = \mathscr{X} \setminus V(\mathscr{X})$, together with two maps, d_0, d_1: $\mathscr{X} \to V$, that are the identity on V. If $e \in E$, $d_0(e)$ and $d_1(e)$ are the *initial* and *terminal* vertices of e, respectively. If $\mathscr{X}$ and $\mathscr{X}'$ are graphs, a morphism φ: $\mathscr{X} \to \mathscr{X}'$ is a mapping such that $d_i\varphi = \varphi d_i$ $(i = 0, 1)$. If $\mathscr{X}$ is a graph, for each edge e we introduce formal symbols e^1 $(= e)$ and e^{-1}, to be thought of as traveling along e one way or the opposite way. We set $d_0(e^{-1}) = d_1(e)$ and $d_1(e^{-1}) = d_0(e)$. If v, w are vertices in a graph $\mathscr{X}$, a path, $p_{v,w}$ joining v to w is a finite sequence of vertices and symbols $e^{\pm 1}, v = v_0, e_1^{\varepsilon(1)}, v_1, e_2^{\varepsilon(2)}, v_2, \dots, e_n^{\varepsilon(n)} v_n = w$, where $\varepsilon(i) = \pm 1$, $v_{i-1} = d_0(e^{(\varepsilon(i))})$, $v_i = d_1(e^{\varepsilon(i)})$ $(i = 1, \dots, n)$. If for every i, $e_i^{\varepsilon(i)} \neq e_{i+1}^{-\varepsilon(i+1)}$, we say that this path is *reduced*. A reduced path $p_{v,w}$ is a *cycle* if $v = w$. A graph $\mathscr{X}$ is *connected* if any two vertices in Γ are joined by a path. A graph $\mathscr{X}$ is a tree if it is connected, and it contains no cycles. A *graph of groups* $(\mathfrak{G}, \mathscr{X})$ consists of a graph $\mathscr{X}$, and a family of groups $\mathfrak{G}(\mathscr{X}) = \{\mathfrak{G}(x) \mid x \in \mathscr{X}\}$; in addition, for each edge e there is a pair of monomorphisms α_e^i: $\mathfrak{G}(e) \to \mathfrak{G}(d_i(e))$, $(i = 0, 1)$. Consider a graph of groups $(\mathfrak{G}, \mathscr{X})$, where $\mathscr{X}$ is a connected graph. Choose a maximal subtree $\mathscr{T}$ of $\mathscr{X}$. Following Dicks [5], we say that a $\mathscr{T}$*-specialization* of $(\mathfrak{G}, \mathscr{X})$ to a group H is a system of homomorphisms ψ_v: $\mathfrak{G}(v) \to H$, and elements $h_e \in H$, where $v \in V(\mathscr{X})$ and $e \in E(\mathscr{X})$, with the following properties:

(1) $h_e = 1$, for $e \in E(\mathscr{T})$,

(2) $\psi_v(\alpha_e^0(g)) = h_e \psi_w(\alpha_e^1(g))h_e^{-1}$, for every edge $e \in E(\mathscr{X})$ with initial vertex v and terminal vertex w, and each $g \in \mathfrak{G}(e)$.

The *fundamental group* of the graph of groups $(\mathfrak{G}, \mathscr{X})$ is a group $\pi_1(\mathfrak{G}, \mathscr{X})$ having a $\mathscr{T}$-specialization $\{\varphi_v, t_e\}$ with the following universal property: for every $\mathscr{T}$-specialization $\{\psi_v, h_e\}$ to a group H, there exists a unique homomorphism ω: $\pi_1(\mathfrak{G}, \mathscr{X}) \to H$ such that $\omega(t_e) = h_e$ and $\omega\varphi_v = \psi_v$, for all $v \in V(\mathscr{X})$ and $e \in E(\mathscr{X})$. We observe that the fundamental group $\pi_1(\mathfrak{G}, \mathscr{X})$ is unique up to isomorphism and it is independent of the choice of $\mathscr{T}$ [5, Theorem 2.4]. Next we recall the definition of the *standard tree* $\mathscr{S} = \mathscr{S}(\mathfrak{G}, \mathscr{X})$ associated with the fundamental group $G = \pi_1(\mathfrak{G}, \mathscr{X})$ of the graph of groups $(\mathfrak{G}, \mathscr{X})$. Define $G(v) = \varphi_v(\mathfrak{G}(v))$, for $v \in V(X)$, and $G(e) = \varphi_{d_0(e)}\alpha_e^0(\mathfrak{G}(e))$, for $e \in E(X)$. Then $\mathscr{S} = \bigcup_{x \in \mathscr{X}} G/G(x)$. Define $V(\mathscr{S}) = \bigcup_{v \in V} G/G(v)$, $d_0(gG(e)) = gG(d_0(e))$ and $d_1(gG(e)) = gt_eG(d_1(e))$. The group $G = \pi_1(\mathfrak{G}, \mathscr{X})$ acts on $\mathscr{S}$ in a natural way, and the quotient graph $\mathscr{S}/G$ is $\mathscr{X}$. Note that if $G = G_1 *_H G_2$, then G is the fundamental group of a graph of groups $(\mathfrak{G}, \mathscr{X})$, where $\mathscr{X}$ consists of one edge e and two vertices v_1, v_2, with $\mathfrak{G}(e) = H$, $\mathfrak{G}(v_1) = G_1$ and $\mathfrak{G}(v_2) = G_2$. It follows that there is a corresponding standard tree $\mathscr{S}(G) = \mathscr{S}(G_1 *_H G_2)$, with a natural action of G on $\mathscr{S}(G)$. Consider now a graph $\mathscr{X}$ consisting of one edge e and one vertex v, and a graph of groups $(\mathfrak{G}, \mathscr{X})$; set $\mathfrak{G}(v) = G_1$, $\alpha_e^1(\mathfrak{G}(e)) = H_2$ and $\alpha_e^0(\mathfrak{G})) = H_1$. Then the fundamental group $\pi_1(\mathfrak{G}, X)$ is an HNN extension $\mathrm{HNN}(G_1, H_1, H_2, t)$ (cf. I.1.5 in [19]).

Let G be a group that acts on a connected graph $\mathscr{X}$; denote by $\mathscr{Y}$ the quotient graph $\mathscr{X}/G$, and let ψ: $X \to \mathscr{X}/G$ be the canonical epimorphism of graphs. A connected transversal Σ of ψ consists of a subtree $\mathscr{L}$ of $\mathscr{X}$ that ψ maps isomorphically to a maximal subtree $\mathscr{D}$ of $\mathscr{X}/G$, together with a set of edges with initial vertices in $\mathscr{L}$ such that ψ maps $\Sigma \setminus \mathscr{L}$ bijectively to $E(\mathscr{X}/G) \setminus \mathscr{D}$. We denote by $\mathscr{X}^G$ the subgraph of fixed points of $\mathscr{X}$ under the action of G.

A *profinite graph* (or *Boolean graph*) $\mathscr{X}$ is a profinite space (or Boolean space, i.e., a compact, Hausdorff, totally disconnected topological space) with a distinguished closed subset of vertices $V = V(\mathscr{X})$ and a subspace of (oriented) edges $E = E(\mathscr{X}) = \mathscr{X} \setminus V(\mathscr{X})$, together with two continuous maps, $d_0(e), d_1(e)$: $\mathscr{X} \to V$, that are the identity on V. If $e \in E$, $d_0(e)$ and $d_1(e)$ are the *initial* and *terminal* vertices of e, respectively. If $\mathscr{X}$ and $\mathscr{X}'$ are graphs, a morphism φ: $\mathscr{X} \to \mathscr{X}'$ is a continuous mapping such that

$d_i\varphi = \varphi d_i$ $(i = 0, 1)$. Observe that a profinite graph is in particular a graph, and that a finite graph is profinite. It is easy to see that every profinite graph is a projective limit of finite graphs. We say that a profinite graph is *connected* if every finite epimorphic image of it is connected in the usual sense. To explain the notion of profinite tree we need some additional notation. Let $\hat{\mathbb{Z}}$ be the profinite completion of the group of integers. (Observe that $\hat{\mathbb{Z}}$ is a topological ring; in fact, $\hat{\mathbb{Z}} = \prod \mathbb{Z}_p$, where p runs through the set of prime numbers, and $\mathbb{Z}_p$ denotes the ring of p-adic integers.) Let $(S, *)$ denote a *profinite pointed space*, that is, a profinite space S with a distinguished point $*$; and let us assume, as usual, that a map of profinite pointed spaces $\varphi: (S, *) \to (S', *')$ is a continuous map such that $\varphi(*) = *'$. If A is a profinite abelian group, we consider it as pointed space by thinking of zero as its distinguished point. If $(T, *)$ is a finite pointed space, define $\hat{\mathbb{Z}}[T, *]$ to be the direct sum of copies of $\hat{\mathbb{Z}}$ indexed by T-$\{*\}$; we think of $(T, *)$ as being embedded in $\hat{\mathbb{Z}}[T, *]$ by identifying each $t \in T$-$\{*\}$ with 1 in the corresponding copy of $\hat{\mathbb{Z}}$, and by identifying $*$ with zero. Every profinite pointed space $(S, *)$ can be expressed as a projective limit $(S, *) = \varprojlim (S_i, *)$ of finite pointed spaces $(S_i, *)$. Then the groups $\hat{\mathbb{Z}}[S_i, *]$ form, in a natural way, a projective system of abelian profinite groups, and we denote by $\mathbb{Z}[\![S, *]\!]$ or by $\mathfrak{A}[S, *]$ the abelian profinite group

$$\mathfrak{A}[S, *] = \varprojlim \mathbb{Z}[S_i, *].$$

It is not hard to see that $\mathfrak{A}[S, *]$ is well-defined, and that $(S, *)$ is naturally embedded in $\mathfrak{A}[S, *]$. In fact, $\mathfrak{A}[S, *]$ is the so-called free profinite abelian group on the pointed space $(S, *)$, and it is characterized by the following universal property: whenever A is an abelian profinite group and $\theta: (S, *) \to A$ is a continuous mapping of pointed spaces, there exists a unique continuous homomorphism $\bar{\theta}: \mathfrak{A}[S, *] \to A$ extending θ. Next, let $\mathscr{X}$ be a nonempty profinite graph, and denote by $(E^*(\mathscr{X}), *)$ the quotient space $E^*(\mathscr{X}) = \mathscr{X}/V(\mathscr{X})$ obtained from $\mathscr{X}$ by collapsing the set of vertices $V(\mathscr{X})$ to a distinguished point $*$. Consider the following sequence of abelian profinite groups and continuous homomorphisms,

$$0 \to \mathfrak{A}[E^*(\mathscr{X}), *] \xrightarrow{d} \mathfrak{A}[V(\mathscr{X})] \xrightarrow{\varepsilon} \hat{\mathbb{Z}} \to 0,$$

where d and ε are the continuous homomorphisms defined by $d(x) = d_1(x) - d_0(x)$, for each $x \in \mathscr{X} - V(\mathscr{X})$, $d(*) = 0$, and $\varepsilon(v) = 1$, for each $v \in V(\mathscr{X})$. One says that the graph $\mathscr{X}$ is a *profinite tree* if the above sequence is exact. It is easily seen that a profinite tree is a connected profinite graph in the sense mentioned above, in fact a profinite graph is connected if and only if the above sequence is exact at $\mathfrak{A}[V(\mathscr{X})]$. (For more information about profinite graphs the reader may consult [11] or [26].)

Next we consider a finite graph of profinite groups $(\mathfrak{G},\mathscr{X})$ (see Section 3 in [26] for details about the statements and concepts in this paragraph). This is defined as for abstract groups, with the additional requirements that the vertex and edge groups be profinite and the monomorphisms α_e^i be continuous. The same universal property considered above, but in the category of profinite groups, serves to define the profinite fundamental group $\Pi_1(\mathfrak{G},\mathscr{X})$ of the finite graph of profinite groups $(\mathfrak{G},\mathscr{X})$. In the same manner one defines the standard profinite tree $\mathscr{S}$ associated with $\Pi_1(\mathfrak{G},\mathscr{X})$; and $\Pi_1(\mathfrak{G},\mathscr{X})$ acts continuously on $\mathscr{S}$ in a natural way. When $\Gamma = \Gamma_1 \amalg_\Delta \Gamma_2$ is a profinite amalgamated free product, Γ can be similarly interpreted as the profinite fundamental group of the graph of profinite groups over a graph consisting of one edge and two distinct vertices. It follows that there is a corresponding standard profinite tree $\mathscr{S}(\Gamma) = \mathscr{S}(\Gamma_1 \amalg_\Delta \Gamma_2)$, with a natural action of Γ on $\mathscr{S}(\Gamma)$.

Assume that the amalgamated free product of abstract groups $G = G_1 *_H G_2$ is residually finite. Consider the family $\{N_\lambda \mid \lambda \in \Lambda\}$ of all normal subgroups of finite index of G. Define the families $\{N_{1\lambda} = N_\lambda \cap G_1 \mid \lambda \in \Lambda\}$ and $\{N_{2\lambda} = N_\lambda \cap G_2 \mid \lambda \in \Lambda\}$ of normal subgroups of finite index of G_1 and G_2, respectively. Note that $N_{1\lambda} \cap H = N_{2\lambda} \cap H$, for all $\lambda \in \Lambda$. Define profinite groups $\Gamma_1 = \varprojlim G_1/N_{1\lambda}$, $\Gamma_2 = \varprojlim G_2/N_{2\lambda}$, and $\Delta = \varprojlim H/(H \cap N_{1\lambda})$. Then Δ can be thought of as a common closed subgroup of Γ_1 and Γ_2. One can apply the above criterion to see that the profinite amalgamated free product $\Gamma = \Gamma_1 \amalg_\Delta \Gamma_2$ exists. In fact Γ is the profinite completion of G. Using the above notation, we consider the standard tree $\mathscr{S}(G)$ associated with $G = G_1 *_H G_2$, and the standard profinite tree $\mathscr{S}(\Gamma)$ associated with $\Gamma = \Gamma_1 \amalg_\Delta \Gamma_2$. Since $\Gamma = \varprojlim G/N_\lambda$, it follows that $\mathscr{S}(\Gamma) = \varprojlim \mathscr{S}(G)/N_\lambda$. Finally, it is easily seen that if G_1, G_2, and H are closed in the profinite topology of G, $\mathscr{S}(G)$ is naturally embedded in $\mathscr{S}(\Gamma)$.

More generally, suppose $G = \pi_1(\mathfrak{G},\mathscr{X})$ is the fundamental group of a finite graph of groups, and assume that G is residually finite. Then $\Gamma = \hat{G}$ is the profinite fundamental group $\Pi_1(\hat{\mathfrak{G}},\mathscr{X})$ of a graph of profinite groups $(\hat{\mathfrak{G}},\mathscr{X})$, where $\hat{\mathfrak{G}}(v)$ and $\hat{\mathfrak{G}}(e)$ are the completions of $\mathfrak{G}(v)$ and $\mathfrak{G}(e)$, with respect to the topologies induced from the profinite topology of G. It follows that $\hat{\mathfrak{G}}(v)$ and $\hat{\mathfrak{G}}(e)$ are embedded in $\Gamma = \hat{G} = \Pi_1(\hat{\mathfrak{G}},\mathscr{X})$. Let $\mathscr{S}(G)$ denote the standard tree associated with $G = \pi_1(\mathfrak{G}, X)$, and $\mathscr{S}(\Gamma)$ the standard profinite tree associated with $\Gamma = \hat{G} = \Pi_1(\hat{\mathfrak{G}},\mathscr{X})$. Since $\Gamma = \varprojlim G/N$, it follows that $\mathscr{S}(\Gamma) = \varprojlim \mathscr{S}(G)/N$, where N runs through the family of all normal subgroups of finite index of G. One easily checks that if the vertex subgroups $\mathfrak{G}(v)$ and edge subgroups $\mathfrak{G}(e)$ are closed in the profinite topology of G, $\mathcal{S}(G)$ is naturally embedded in $\mathcal{S}(\Gamma)$.

2. PRELIMINARY RESULTS

In this section we establish some general results about abstract and profinite groups acting on trees.

LEMMA 2.1. (i) *Let $\mathcal{X}$ be a profinite graph, and let $\mathcal{Y}$ be a profinite connected subgraph of $\mathcal{X}$. Consider the profinite quotient graph $\mathcal{X}/\mathcal{Y}$ obtained by collapsing $\mathcal{Y}$ to a single vertex* (*see Lemma* 1.4 *in* [26]). *Then the preimage $\mathcal{Z}$ of a profinite connected subgraph $\mathcal{R}$ of $\mathcal{X}/\mathcal{Y}$ is connected.*

(ii) *Let Γ be a profinite group that acts on a profinite tree $\mathcal{T}$. Assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint Γ-invariant profinite subtrees of $\mathcal{T}$. Then the sets of fixed points $\mathcal{T}_1^{\Gamma}$ and $\mathcal{T}_2^{\Gamma}$ are not empty.*

Proof. (i) The result is obvious if $\mathcal{X}$ is a finite graph. Express $\mathcal{X}$ as an inverse limit of finite graphs, $\mathcal{X} = \varprojlim \mathcal{X}_i$. Denote by $\mathcal{Y}_i$ and $\mathcal{Z}_i$ the canonical images of $\mathcal{Y}$ and $\mathcal{Z}$ in $\mathcal{X}_i$, respectively. Clearly $\mathcal{X}/\mathcal{Y} = \varprojlim \mathcal{X}_i/\mathcal{Y}_i$. Let $\mathcal{R}_i$ be the image of $\mathcal{R}$ in $\mathcal{X}_i/\mathcal{Y}_i$. Then $\mathcal{Z}_i$ is the preimage of $\mathcal{R}_i$ in $\mathcal{X}_i$. Since $\mathcal{R}_i$ is connected, so is $\mathcal{Z}_i$; hence, $\mathcal{Z} = \varprojlim \mathcal{Z}_i$ is also connected.

(ii) It suffices to prove that $\mathcal{T}_1^{\Gamma} \neq \emptyset$. Consider the profinite graph $\mathcal{T}''$ obtained from $\mathcal{T}$ by collapsing $\mathcal{T}_1$ to a vertex denoted v_1, and $\mathcal{T}_2$ to a vertex denoted v_2. Then by Proposition 1.17 in [25], $\mathcal{T}''$ is a profinite tree. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are Γ-invariant, there is a natural action of Γ on $\mathcal{T}''$ induced by the action on $\mathcal{T}$. By Theorem 2.8 in [26], the subgraph $\mathcal{T}''^{\Gamma}$ of fixed points of $\mathcal{T}''$ under the action of Γ is also a profinite tree with two distinct vertices, and hence it contains an edge. It follows that $\mathcal{T}^{\Gamma}$ is not empty. Next consider the profinite graph $\mathcal{T}'$ obtained from $\mathcal{T}$ by collapsing $\mathcal{T}_1$ to a vertex denoted v_1. Again by Theorem 2.8 in [26], the subgraph $\mathcal{T}'^{\Gamma}$ of fixed points of $\mathcal{T}'$ under the action of Γ is a profinite tree. Observe that the preimage of $\mathcal{T}'^{\Gamma} \cap \mathcal{T}_1 \neq \emptyset$, for otherwise the quotient graph of $\mathcal{T}^{\Gamma} \cup \mathcal{T}_1$ obtained by collapsing $\mathcal{T}^{\Gamma}$ to a vertex denoted v_1, and $\mathcal{T}_1$ to a vertex denoted v_2, would not be connected. ∎

The next result provides a criterion for the uniqueness of minimal G-invariant subtrees of a tree on which a group G acts, in both the abstract and the profinite setting.

LEMMA 2.2. (i) *Let G be an abstract group that acts on a tree $\mathcal{T}$ in such a way that $\mathcal{T}/G$ is a finite graph. Then there exists a minimal G-invariant subtree $\mathcal{T}_G$ of $\mathcal{T}$. Moreover, $\mathcal{T}_G$ is unique if and only if G does not fix any edge of $\mathcal{T}$.*

(ii) *Let* Γ *be a profinite group that acts on a profinite tree* $\mathscr{T}$. *Then there is a minimal* Γ*-invariant profinite subtree* $\mathscr{T}_\Gamma$ *of* $\mathscr{T}$. *Moreover,* $\mathscr{T}_\Gamma$ *is unique if and only if* Γ *does not fix any edge of* $\mathscr{T}$.

Proof. (i) To show the existence of minimal G-invariant subtrees it suffices to prove that one can apply Zorn's lemma to the collection of all nonempty G-invariant subtrees of $\mathscr{T}$ ordered by inclusion: just observe that if $\mathscr{T}_i$ $(i \in I)$ is a chain of such (nonempty) subtrees, then $\bigcap \mathscr{T}_i$ is a nonempty subtree; for let t' be an element in the intersection of all the images $\mathscr{T}'_i$ in $\mathscr{T}/G$ of each $\mathscr{T}_i$ (such a t' exists since $\mathscr{T}/G$ is finite); let $t_i \in T_i$ map to t'; then clearly all the t_i $(i \in I)$ are in the same orbit, and so Gt_i is a subset of every $\mathscr{T}_j$ $(j \in I)$. Suppose that $\mathscr{T}_1$ and $\mathscr{T}_2$ are two such minimal G-invariant subtrees. It suffices to show that $\mathscr{T}_1 \cap \mathscr{T}_2 \neq \emptyset$. Suppose $\mathscr{T}_1 \cap \mathscr{T}_2 = \emptyset$. Consider the tree $\mathscr{T}'$ obtained from $\mathscr{T}$ by collapsing $\mathscr{T}_1$ to a point t_1 and $\mathscr{T}_2$ to a point t_2. Then G acts on $\mathscr{T}'$ with fixed points t_1 and t_2. Hence G fixes all edges of the path in $\mathscr{T}'$ from t_1 to t_2. Now the preimage of an edge of $\mathscr{T}'$ consists of a single edge of $\mathscr{T}$. Thus a preimage of a fixed edge of $\mathscr{T}'$ must be an edge of $\mathscr{T}$ fixed by G, contradicting the assumption of the lemma.

(ii) Observe that in this case $\mathscr{T}$ is a compact space, and so the intersection of a chain of nonempty profinite subtrees of $\mathscr{T}$ is nonempty. Hence by Zorn's lemma, there exists a minimal Γ-invariant profinite subtree $\mathscr{T}_\Gamma$ of $\mathscr{T}$. If Γ does not fix any vertex of $\mathscr{T}$, then $|\mathscr{T}_\Gamma| > 1$, and so $\mathscr{T}_\Gamma$ is unique by Lemma 1.5 in [25]. Let v be a vertex of $\mathscr{T}$ fixed by Γ. Then $\{v\}$ is a minimal Γ-invariant profinite subtree. By Lemma 2.1, any minimal Γ-invariant profinite subtree must consist of only one vertex, say $\{w\}$. By Theorem 2.8 in [26], the subgraph of $\mathscr{T}^\Gamma$ of fixed points is also a profinite tree. Hence, if $v \neq w$ then there exists an edge in $\mathscr{T}^\Gamma$, a contradiction. ▮

LEMMA 2.3. (i) *Let* $\Gamma = \Pi_1(\mathfrak{G}, \mathscr{X})$ *be the profinite fundamental group of a finite graph of profinite groups* $(\mathfrak{G}, \mathscr{X})$, *such that the natural homomorphism* $\pi_1(\mathfrak{G}, \mathscr{X}) \to \Pi_1(\mathfrak{G}, \mathscr{X})$ *is an embedding. Let* v_1 *and* v_2 *be two different vertices of* $\mathscr{X}$, *and assume that* $\gamma_1 \in \mathfrak{G}(v_1)$ *and* $\gamma_2 \in \mathfrak{G}(v_2)$, *but* $\gamma_1 \notin \mathfrak{G}(e)^{\mathfrak{G}(v_1)}$ *and* $\gamma_2 \notin \mathfrak{G}(e)^{\mathfrak{G}(v_2)}$. *Then* γ_1 *and* γ_2 *are not conjugate in* Γ.

(ii) *Let* $\Gamma = \Gamma_1 \amalg_\Delta \Gamma_2$ *be the profinite amalgamated free product of two profinite groups* Γ_1 *and* Γ_2 *amalgamating a common closed subgroup* Δ. *Let* $\gamma_1 \in \Gamma_1 \setminus \Delta^{\Gamma_1}$ *and* $\gamma_2 \in \Gamma_2 \setminus \Delta^{\Gamma_2}$. *Then* γ_1 *and* γ_2 *are not conjugate in* Γ.

(iii) *Let* $\Gamma = \Pi_1(\mathfrak{G}, \mathscr{X})$ *be the profinite fundamental group of a finite graph of profinite groups* $(\mathfrak{G}, \mathscr{X})$, *such that each edge group* $\mathfrak{G}(e)$ *is finite. Let* v_1 *and* v_2 *be two different vertices of* $\mathscr{X}$, *and assume that* $\gamma_1 \in \mathfrak{G}(v_1)$ *and* $\gamma_2 \in \mathfrak{G}(v_2)$, *but* γ_1 *and* γ_2 *do not belong to a conjugate, in* $\mathfrak{G}(v_1)$ *and* $\mathfrak{G}(v_2)$ *respectively, of any edge group* $\mathfrak{G}(e)$. *Then* γ_1 *and* γ_2 *are not conjugate in* Γ.

Proof. Clearly (ii) and (iii) are consequences of (i). To prove (i), let $\mathfrak{U}$ be the collection of all open normal subgroups U of Γ. Consider the fundamental group (respectively, the profinite fundamental group) $G(U) = \pi_1(\mathfrak{G}(U), \mathscr{X})$ (respectively, $\Gamma(U) = \Pi_1(\mathfrak{G}(U), \mathscr{X})$) of the finite graph of groups $(\mathfrak{G}_U, \mathscr{X})$, where $\mathfrak{G}_U(x) = \mathfrak{G}(x)/(\mathfrak{G}(x) \cap U)$, for $x \in \mathscr{X}$). Then

$$\Gamma(U) = \widehat{G(U)} \qquad \text{and} \qquad \Gamma = \varprojlim \Gamma(U).$$

Denote by $\gamma_1(U)$ and $\gamma_2(U)$ the images of γ_1 and γ_2 in $\mathfrak{G}(v_1)/(\mathfrak{G}(v_1) \cap U)$ and $\mathfrak{G}(v_2)/(\mathfrak{G}(v_2) \cap U)$, respectively. Choose U so that $\gamma_1(U)\gamma_2(U) \notin (\mathfrak{G}(e)/(\mathfrak{G}(e) \cap U))^{\mathfrak{G}(v_2)}$, for $e \in E(\mathscr{X})$. It is easy to see that $\gamma_1(U)$ and $\gamma_2(U)$ are not conjugate in $G(U)$. Since $G(U)$ is a free-by-finite group, one has that $G(U)$ is conjugacy separable (cf. Theorem 3 in [7]). Therefore $\gamma_1(U)$ and $\gamma_2(U)$ are not conjugate in $\Gamma(U)$. It follows that γ_1 and γ_2 are not conjugate in Γ. ∎

LEMMA 2.4. (i) *Let R be a residually finite group and let C be a cyclic subgroup of R such that the topology induced on C by the profinite topology of R coincides with the profinite topology of C. Let C_1 and C_2 be subgroups of C which are conjugate in R. Then $C_1 = C_2$.*

(ii) *Let Γ be a profinite group, and Δ a closed procyclic subgroup of Γ. Let Δ_1 and Δ_2 be closed subgroups of Δ which are conjugate in Γ. Then $\Delta_1 = \Delta_2$.*

Proof. (i) Let N be any normal subgroup of R of finite index. Then C_1N/N and C_2N/N are conjugate in R/N, and so they have the same order. Since they are subgroups of the finite cyclic group CN/N, it follows that $C_1N/N = C_2N/N$. One deduces that the closures of C_1 and C_2 in the profinite topology of R (and hence of C) coincide. Since C is cyclic, C_1 and C_2 are closed in the profinite topology of C, and thus $C_1 = C_2$.

(ii) The proof in this case is similar to the one used in (i). ∎

In the following proposition we study the normalizer of a subgroup of a vertex group of a fundamental group of a graph of groups, in both the abstract and the profinite situation.

PROPOSITION 2.5. (1) *Let G be a group that acts on a tree $\mathscr{S}$, such that the stabilizer G_e is a cyclic group for each edge e. Let H be a subgroup of G_v for some $v \in V(\mathscr{S})$. Assume that either* (i) *each G_e is finite, or* (ii) *the profinite topology of G induces on each G_e its full profinite topology. Then G can be represented as a fundamental group of a graph of groups $(\mathfrak{G}, \mathscr{X})$ such that $\mathscr{X} = \mathscr{S}/G$, $\mathfrak{G}(x) = G_s$, where $Gs = x$, and the normalizer $\mathfrak{N}_G(H) = \pi_1(\mathfrak{G}', \mathscr{Y})$, with $\mathscr{Y}$ a subgraph of $\mathscr{X}$, and $\mathfrak{G}'(y) = \mathfrak{N}_{\mathfrak{G}(y)}(H)$, for all $y \in \mathscr{Y}$.*

(2) *Let Γ be a profinite group that acts continuously on a profinite tree $\mathscr{S}$ so that $\mathscr{S}/\Gamma$ is finite. Suppose in addition that the stabilizer Γ_e is a*

procyclic group for each edge e. Let Δ *be a closed subgroup of* Γ_v *for some* $v \in V(\mathscr{S})$. *Then* Γ *can be represented as a profinite fundamental group of a graph of groups* $(\mathfrak{G}, \mathscr{X})$ *such that* $\mathscr{X} = \mathscr{S}/\Gamma$, $\mathfrak{G}(x) = \Gamma_s$, *where* $\Gamma s = x$, *and* $\mathfrak{N}_\Gamma(\Delta) = \Pi_1(\mathfrak{G}', \mathscr{Y})$, *with* $\mathscr{Y}$ *a subgraph of* $\mathscr{X}$ *and* $\mathfrak{G}'(y) = \mathfrak{N}_{\mathfrak{G}(y)}(\Delta)$, *for all* $y \in \mathscr{Y}$.

Proof. (1) Let $\mathscr{T} = \mathscr{S}^H$ be the subtree of fixed points of H in $\mathscr{S}$. Consider the natural epimorphism of graphs $\varphi: \mathscr{S} \to \mathscr{S}/G$. First, we prove that the normalizer $\mathfrak{N}_G(H)$ acts on $\mathscr{T}$, and that the natural mapping $\psi: \mathscr{T} \to \mathscr{T}/\mathfrak{N}_G(H)$ is the restriction $\varphi_{|\mathscr{T}}$ of φ to $\mathscr{T}$.

Let $g \in \mathfrak{N}_G(H)$ and let $t \in \mathscr{T}$; then $hgt = gh't = gt$, where $h, h' \in H$; so $gt \in \mathscr{T}$. Therefore $\mathfrak{N}_G(H)$ acts on $\mathscr{T}$. Next, consider the case when H does not stabilize any edge of $\mathscr{S}$, i.e., $\mathscr{T} = \{v\}$. Then $\mathfrak{N}_G(H)$ is contained in G_v, and so $\psi = \varphi_{|\mathscr{T}}$. Now, consider the case when H stabilizes some edge of $\mathscr{S}$, say e. We need to show that if for some $g \in G$, $ge \in \mathscr{T}$, then $g \in \mathfrak{N}_G(H)$. First, observe that the stabilizer of ge is precisely gG_eg^{-1}. Hence H and gHg^{-1} are subgroups of gG_eg^{-1}. If G_e is finite cyclic, then $H = gHg^{-1}$, since H and gHg^{-1} have the same order. If G_e is infinite, let K be any finite quotient of G; then by the above argument, the images of H and gHg^{-1} in K coincide; hence, by the assumption (ii) in the statement we get $H = gHg^{-1}$. This proves that $\psi = \varphi_{|\mathscr{T}}$.

Consider a maximal subtree $\mathscr{D}$ of $\mathscr{T}/\mathfrak{N}_G(H)$; extend $\mathscr{D}$ to a maximal subtree $\mathscr{D}'$ of $\mathscr{S}/G$; let Σ be a connected transversal of ψ; then there is a connected transversal Σ' of φ such that $\Sigma' \cap \mathscr{T} = \Sigma$. Note that $\varphi(\Sigma') = \mathscr{X}$. Define $\mathscr{Y} = \psi(\Sigma)$. Then according to Theorem I.13 in [19], $G = \pi_1(\mathfrak{G}, \mathscr{X})$, where $\mathfrak{G}(x) = G_s$, with $Gs = x$ and $s \in \Sigma'$; and $\mathfrak{N}_G(H) = \pi_1(\mathfrak{G}', \mathscr{Y})$, where

$$\mathfrak{G}'(x) = \big(\mathfrak{N}_G(H)\big)_s = \mathfrak{N}_{G_s}(H),$$

with $Gs = x$ and $s \in \Sigma$.

(2) Let $\mathscr{T} = \mathscr{S}^\Delta$ be the profinite subtree of fixed points of Δ in $\mathscr{S}$ (cf. Theorem 2.8 in [26]). Consider the natural epimorphism of graphs $\varphi: \mathscr{S} \to \mathscr{S}/\Gamma$. First we prove that the normalizer $\mathfrak{N}_\Gamma(\Delta)$ acts on $\mathscr{T}$ continuously, and the natural mapping $\psi: \mathscr{T} \to \mathscr{T}/\mathfrak{N}_\Gamma(\Delta)$ is the restriction $\varphi_{|\mathscr{T}}$ of φ to $\mathscr{T}$.

Let $\gamma \in \mathfrak{N}_\Gamma(\Delta)$ and let $t \in \mathscr{T}$; then $\delta\gamma t = \gamma\delta' t = \gamma t$, where $\delta, \delta' \in \Delta$; so $\delta t \in \mathscr{T}$. Therefore $\mathfrak{N}_\Gamma(\Delta)$ acts on $\mathscr{T}$. Next, consider the case when Δ does not stabilize any edge of $\mathscr{S}$, i.e., $\mathscr{T} = \{v\}$. Then $\mathfrak{N}_\Gamma(\Delta)$ is contained in Γ_v, and so $\psi = \varphi_{|\mathscr{T}}$. Now consider the case when Δ stabilizes some edge of $\mathscr{S}$, say e. We need to show that if for some $\gamma \in \Gamma$, $\gamma e \in \mathscr{T}$, then $\gamma \in \mathfrak{N}_\Gamma(\Delta)$. First, observe that the stabilizer of γe is precisely $\gamma\Gamma_e\gamma^{-1}$. Hence Δ and $\gamma\Delta\gamma^{-1}$ are subgroups of $\gamma\Gamma_e\gamma^{-1}$. If Γ_e is finite cyclic, then Δ and $\gamma\Delta\gamma^{-1}$ have the same order, and so $\Delta = \gamma\Delta\gamma^{-1}$. If Γ_e is infinite, let $\mathbf{K}$ be any finite quotient of Γ; then by the above argument, the images of Δ and $\gamma\Delta\gamma^{-1}$ in $\mathbf{K}$ coincide; hence we get $H = gHg^{-1}$. This proves that $\psi = \varphi_{|\mathcal{T}}$.

Consider a maximal subtree $\mathcal{D}$ of $\mathcal{T}/\mathfrak{N}_\Gamma(\Delta)$; extend $\mathcal{D}$ to a maximal subtree $\mathcal{D}'$ of $\mathcal{S}/\Gamma$; let Σ be a connected transversal of ψ (observe that Σ exists since $\mathcal{S}/\Gamma$ is finite); then there is a connected transversal Σ' of φ such that $\Sigma' \cap \mathcal{T} = \Sigma$. Note that $\varphi(\Sigma') = \mathcal{X}$. Define $\mathcal{Y} = \psi(\Sigma)$. Then according to Proposition 4.4 in [27], $\Gamma = \Pi_1(\mathfrak{G}, \mathcal{X})$, where $\mathfrak{G}(x) = \Gamma_s$, with $\Gamma s = x$ and $s \in \Sigma'$; and $\mathfrak{N}_\Gamma(\Delta) = \Pi_1(\mathfrak{G}', \mathcal{Y})$, where

$$\mathfrak{G}'(x) = (\mathfrak{N}_\Gamma(\Delta))_s = \mathfrak{N}_{\Gamma_s}(\Delta), \qquad \text{with } \Gamma s = x \quad \text{and} \quad s \in \Sigma. \qquad \blacksquare$$

Remark 2.6. If, in the statement (2) of the above proposition, $\mathcal{S}/\Gamma$ is infinite, the conclusion and proof are still valid, if one adopts the appropriate definition of a profinite fundamental group of a profinite graph of profinite groups (cf. [27]).

COROLLARY 2.7. (i) *Let G_1, G_2 be residually finite groups with a common cyclic subgroup H. Assume that the profinite topology of G_i induces the full profinite topology of H $(i = 1, 2)$. Let $G = G_1 *_H G_2$ be their amalgamated free product. Let $h \in H$. Then $\mathfrak{N}_G(h) = \mathfrak{N}_{G_1}(h) *_H \mathfrak{N}_{G_2}(h)$, where $\mathfrak{N}_G(h)$ denotes the normalizer of $\langle h \rangle$ in G.*

(ii) *Let Γ_1 and Γ_2 be profinite groups with a common closed procyclic subgroup Δ, such that $\Gamma = \Gamma_1 \amalg_\Delta \Gamma_2$ exists. Let $\delta \in \Delta$. Then $\mathfrak{N}_\Gamma(\delta) = \mathfrak{N}_{\Gamma_1}(\delta) \amalg_\Delta \mathfrak{N}_{\Gamma_2}(\delta)$.*

Proof. This follows from the proof of the proposition: it corresponds to the case in the proof when $\mathcal{T}$ contains an edge. $\blacksquare$

Let G be a group acting on a tree $\mathcal{S}$. One says that an element g of G is *hyperbolic* if g does not fix any vertex of $\mathcal{S}$. If G is the fundamental group $\pi_1(\mathfrak{G}, \mathcal{X})$ of a graph of groups $(\mathfrak{G}, \mathcal{X})$, one has that G acts on the standard tree $\mathcal{S}(G)$ associated with $G = \pi_1(\mathfrak{G}, \mathcal{X})$; then we say that an element of G is hyperbolic, if it is hyperbolic with respect to this action.

LEMMA 2.8. *Suppose that G is the fundamental group $\pi_1(\mathfrak{G}, \mathcal{X})$ of a finite graph of groups $(\mathfrak{G}, \mathcal{X})$. Assume that G is residually finite and that $\mathfrak{G}(v)$ and $\mathfrak{G}(e)$ are closed in the profinite topology of G. Let a and b be two elements of G which are conjugate in $\hat{G}$. Then a is hyperbolic if and only if b is hyperbolic (as elements of G).*

Proof. Let $\Gamma = \hat{G}$. Since G is the fundamental group $\pi_1(\mathfrak{G}, \mathcal{X})$ of a finite graph of groups $(\mathfrak{G}, \mathcal{X})$ and the vertex and edge groups are closed in the profinite topology of G, then $\mathcal{S}(G)$ is embedded in $\mathcal{S}(\Gamma)$, as was indicated in the last paragraph of Section 1.

Let $\gamma \in \Gamma$ be such that $b = \gamma a \gamma^{-1}$. Assume that a fixes some vertex of $\mathcal{S}(G)$, say v_1. We need to prove that b also fixes a vertex of $\mathcal{S}(G)$.

Suppose not. Then by a theorem of Tits (cf., Prop. I.24 in [19]), there is an infinite straight line $\mathscr{T}_b$ in $\mathscr{S}(G)$, defined as follows: let $m = \min\{l(v, bv) \mid v \in V(\mathscr{S}(G))\}$, where $l(v, bv)$ denotes the distance from v to bv in the tree $\mathscr{S}(G)$; then $\mathscr{T}_b$ is the subtree of $\mathscr{S}(G)$ whose set of vertices is $\{v \in V(\mathscr{S}(G)) \mid l(v, bv) = m\}$. Moreover b acts freely on $\mathscr{T}_b$ as a translation of length m. Let $\mathscr{T}$ be a segment of $\mathscr{T}_b$ of length m. Then $\mathscr{T}_b = \langle b \rangle \mathscr{T}$. As mentioned in the last paragraph of Section 1, when G is the fundamental group $\pi_1(\mathfrak{G}, \mathscr{X})$ of a finite graph of groups $(\mathfrak{G}, \mathscr{X})$, one has $\Gamma = \hat{G} = \Pi_1(\hat{\mathfrak{G}}, \mathscr{X})$. Consider the closure $\overline{\mathscr{T}_b} = \overline{\langle b \rangle} \mathscr{T}$ of $\mathscr{T}_b$ in the standard profinite tree $\mathscr{S}(\Gamma)$. Observe that b does not fix any vertex of $\overline{\mathscr{T}_b}$, for if $b'v \in \overline{\mathscr{T}_b}$ with $b' \in \overline{\langle b \rangle}$, $v \in \mathscr{T}$, and $bb'v = b'v$, then $bv = v$, since b and b' commute. Consider the vertex γv_1 of $\mathscr{S}(\Gamma)$. Note that b (and hence $\overline{\langle b \rangle}$) fixes γv_1, and so $\gamma v_1 \in \overline{\mathscr{T}_b}$. It follows by Lemma 2.1(ii) that b fixes some vertex in $\overline{\mathscr{T}_b}$, which is a contradiction. ∎

PROPOSITION 2.9. *Let $G = \pi_1(\mathfrak{G}, \mathscr{X})$ be the fundamental group of a finite graph of groups $(\mathfrak{G}, \mathscr{X})$, and assume that G is residually finite, and that $\mathfrak{G}(x)$ is closed in the profinite topology of G, for all x in X. Let $b \in G$. If b is hyperbolic, then $\overline{\langle b \rangle}$ acts freely on the standard tree $\mathscr{S}(\Gamma)$ of $\Gamma = \hat{G} = \Pi_1(\hat{\mathfrak{G}}, \mathscr{X})$.*

Proof. Using the notation in the proof of Lemma 2.8, b acts on $\mathscr{T}_b$ as a translation of length m, and therefore the group $\langle b \rangle$ acts freely on $\mathscr{T}_b$. Moreover $\overline{\mathscr{T}_b} = \overline{\langle b \rangle} \mathscr{T}$. We claim that $\overline{\langle b \rangle}$ acts freely on $\overline{\mathscr{T}_b}$. Remark that if $b' \in \overline{\langle b \rangle}$ fixes one vertex of $\overline{T_b}$, then it fixes all the vertices of $\overline{T_b}$. Denote by K the closed subgroup of $\overline{\langle b \rangle}$ consisting of those elements that act trivially on $\overline{T_b}$; we need to show that $K = 1$. Clearly $K \neq \overline{\langle b \rangle}$, since $\langle b \rangle$ acts freely on $\overline{T_b}$. Now, $\overline{\langle b \rangle}/K$ acts freely on the profinite tree $\overline{T_b}$ with finite quotient graph $\overline{T_b}/(\overline{\langle b \rangle}/K)$ (for $T_b/\langle b \rangle$ is finite). Then, according to [11, Theorem 1.7], $\overline{\langle b \rangle}/K$ is a free prosolvable group, and since $\overline{\langle b \rangle}$ is procyclic and nontrivial, one has $\overline{\langle b \rangle}/K \cong \hat{\mathbb{Z}}$. Thus $K = 1$. This proves the claim.

If $\overline{\langle b \rangle} = \mathscr{S}(\Gamma)$, obviously $\overline{\langle b \rangle}$ acts freely on $\mathscr{S}(\Gamma)$. If, on the other hand, $\langle b \rangle \neq \mathscr{S}(\Gamma)$, our result follows from Lemma 2.1(ii). ∎

Remark 2.10. Let $\mathfrak{C}$ be a nonempty class of finite groups closed under subgroups, quotients, and extensions. Let G be an abstract group, and let $\mathfrak{U}$ be the collection of all normal subgroups U of G such that $G/U \in \mathfrak{C}$. If $\bigcap_{U \in \mathfrak{U}} U = 1$, we say that G is residually $\mathfrak{C}$. Then $\varprojlim_{U \in \mathfrak{U}} G/U$ is called the pro-$\mathfrak{C}$ completion of G. We say that G is conjugacy $\mathfrak{C}$-separable if for any two elements $x, y \in G$, $x \sim_{G/U} y$ for all $U \in \mathfrak{U}$, implies that $x \sim_G y$. There are corresponding natural notions of pro-$\mathfrak{C}$ trees, pro-$\mathfrak{C}$ fundamental group of a finite graph of pro-$\mathfrak{C}$ groups, and standard pro-$\mathfrak{C}$ tree. Then one can restate many of the results in this section in terms of pro-$\mathfrak{C}$ topology and pro-$\mathfrak{C}$ completions, rather than profinite topology and profinite completion. Specifically the results 2.1–2.3, 2.5(ii), 2.6, 2.8, and 2.9 can be so restated, and the proofs remain valid using essentially the same techniques that we have employed above.

3. THE MAIN RESULT

In this section we apply the techniques developed above to prove the conjugacy separability of an amalgamated free product of groups that are either free-by-finite or finitely generated nilpotent-by-finite, amalgamating a cyclic group.

A group G is called *subgroup separable* if every finitely generated subgroup of G is closed in the profinite topology of G. It is proved in [14, 12] that free groups and finitely generated nilpotent groups are subgroup separable; it easily follows that the groups considered in this paper, i.e., free-by-finite or finitely generated nilpotent-by-finite groups, are also subgroup separable.

Let x be an element in a group R. Following Tang (cf. [24]), one says that R is *quasi* $\langle x \rangle$*-potent* (in fact, Tang refers to this concept as weakly $\langle x \rangle$-potent) if there exists a positive integer r such that for every positive integer n, there exists a normal subgroup U_n of R of finite index such that in the quotient group G/U_n, the image of x has exactly order rn. The group R is said to be *quasi potent* if R is quasi $\langle x \rangle$-potent for every element x in R of infinite order. Observe that if R is quasi $\langle x \rangle$-potent, then the profinite topology of R induces on $\langle x \rangle$ its full profinite topology.

LEMMA 3.1 (C. Y. Tang [24], Lemma 3.2). *Let R be either a free-by-finite or a finitely generated nilpotent-by-finite group. Then R is quasi potent.* [*Consequently, if x is an element of infinite order in R, the closure of $\langle x \rangle$ in $\hat{R}$ is isomorphic to $\hat{\mathbb{Z}}$.*]

PROPOSITION 3.2. *Let G_1, G_2 be either free-by-finite or finitely generated nilpotent-by-finite groups, let H be a common cyclic subgroup of G_1 and G_2. Then $G = G_1 *_H G_2$ is residually finite, the profinite topology of G induces on each G_i the full profinite topology of G_i $(i = 1, 2)$, $\hat{H}$ is naturally embedded in $\hat{G}_1$ and $\hat{G}_2$, and $\hat{G} = \hat{G}_1 \amalg_{\hat{H}} \hat{G}_2$.*

Proof. The residual finiteness of G is probably known, but we do not have an explicit reference for this fact; in any case, we prove it here. As pointed out above, it follows from Lemma 3.1 that the profinite topology of G_i induces on H its full profinite topology $(i = 1, 2)$, i.e., $\hat{H}$ is naturally embedded in $\hat{G}_1$ and $\hat{G}_2$. Consider the family $\{N_\lambda \mid \lambda \in \Lambda\}$ of all normal subgroups of finite index of G. First, we shall prove that the family

$\{N_{i\lambda} = N_\lambda \cap G_i \mid \lambda \in \Lambda\}$ determines the full profinite topology of G_i $(i = 1, 2)$. For this it suffices to prove that for any given normal subgroup of finite index N_i in G_i, one can find a $\lambda \in \Lambda$ such that $N_{i\lambda} \leq N_i$ $(i = 1, 2)$. If H is finite, this clearly can be done (cf. Proposition II.12 in [19]). Suppose $H = \langle h \rangle$ is infinite. Let $\langle h^{t_1} \rangle = H \cap N_1$ and $\langle h^{t_2} \rangle = H \cap N_2$, for some natural numbers t_1 and t_2. Now, by Lemma 3.1, G_i is quasi potent. Hence there exist integers r_1 and r_2 such that for any given integers n_1 and n_2, there exist normal subgroups M_1 and M_2 of G_1 and G_2, respectively, such that the images of h^{t_1} and h^{t_2} in G_1/M_1 and G_2/M_2 have orders $n_1 r_1$ and $n_2 r_2$, respectively. Hence $M_1 \cap N_1 \cap H = \langle h^{t_1 n_1 r_1} \rangle$ and $M_2 \cap N_2 \cap H = \langle h^{t_2 n_2 r_2} \rangle$. Choose n_1 and n_2 such that $t_1 n_1 r_1 = t_2 n_2 r_2$. Then $M_1 \cap N_1 \cap H = M_2 \cap N_2 \cap H$. Next we show that there exists $\lambda \in \Lambda$ such that $M_1 \cap N_1 = N_{1\lambda}$ and $M_2 \cap N_1 = N_{2\lambda}$. For this we shall find a normal subgroup M of finite index in G such that $M \cap G_i = M_i \cap N_i$ $(i = 1, 2)$. Consider the natural epimorphism of groups

$$\varphi\colon G = G_1 *_H G_2 \to G_0 = G_1/(M_1 \cap N_1) *_{H/(M_1 \cap N_1 \cap H)} G_2/(M_2 \cap N_2).$$

Since G_0 is free-by-finite, there exists a free normal subgroup F of G_0 of finite index. Define $M = \varphi^{-1}(F)$. It is easy to see that M satisfies the required properties. This proves that the profinite topology of G induces on each G_i its full profinite topology $(i = 1, 2)$, so that we may assume that $\hat{G}_1$ and $\hat{G}_2$ are naturally embedded in $\hat{G}$, and clearly $\hat{G}$ is topologically generated by $\hat{G}_1$ and $\hat{G}_2$. Next, note that obviously $N_{1\lambda} \cap H = N_{2\lambda} \cap H$ for each $\lambda \in \Lambda$ and $\bigcap_\lambda N_{i\lambda} = 1$ $(i = 1, 2)$; furthermore, $\bigcap_\lambda N_{i\lambda} H = H$ $(i = 1, 2)$, since G_1 and G_2 are subgroup separable, as we pointed out at the beginning of this section. It follows then from Proposition 1.1 that G is residually finite.

To complete the proof it remains to show that

$$\begin{array}{ccccc} & & \hat{G}_1 & & \\ & \nearrow & & \searrow & \\ \hat{H} & & & & \hat{G} \\ & \searrow & & \nearrow & \\ & & \hat{G}_2 & & \end{array}$$

is a push-out diagram in the category of profinite groups. Let $\eta_i\colon \hat{G}_i \to A$ $(i = 1, 2)$ be homomorphisms into a finite group A such that $\eta_1(x) = \eta_2(x)$ for all $x \in \hat{H}$. Consider the natural embedding

$$\gamma\colon G = G_1 *_H G_2 \to \hat{G}$$

induced by the natural embeddings $G_1 \to \hat{G}_1$ and $G_2 \to \hat{G}_2$. Then there exists a unique homomorphism $\tau\colon G \to A$ extending $\eta_1|_{G_1}$ and $\eta_2|_{G_2}$.

Therefore τ induces a unique continuous homomorphism $\eta: \hat{G} \to A$ such that $\eta^{\gamma} = \tau$. It follows that η extends η_i $(i = 1, 2)$. ∎

LEMMA 3.3. *Let G_1, G_2 be either free-by-finite or finitely generated nilpotent-by-finite groups, and let H be a common cyclic subgroup of G_1 and G_2. Then G_1, G_2, and H are closed in the profinite topology of $G = G_1 *_H G_2$.*

Proof. By Proposition 3.2, $\hat{G}_1 *_H \hat{G}_2$ is a dense subgroup of $\hat{G}$. Let $k \in \hat{G}_1 \cap G$. Assume first that $k \in \hat{H} \cap G$. We need to prove that $k \in H$. If $k \notin H$, then k has a canonical representation as $k = w_1 w_2 w_3 \dots$, where $w_i \in (G_1 \cup G_2) \setminus H$, and if $w_i \in G_1$ (respectively, if $w_i \in G_2$) then $w_{i+1} \in G_2$ (respectively, then $w_{i+1} \in G_1$). It follows that $w_i \notin \hat{H}$, since $G_1 \cap \hat{H} = G_2 \cap \hat{H} = H$, because G_1 and G_2 are subgroup separable, as we pointed out at the beginning of this section. Hence $k = w_1 w_2 w_3 \dots$ is also a canonical representation as an element of $\hat{G}_1 *_{\hat{H}} \hat{G}_2$. Therefore $k \notin \hat{H}$, contradicting our hypothesis.

Suppose now that $k \in (\hat{G}_1 \cap G) \setminus \hat{H}$. Let $k = w_1 w_2 w_3 \dots$ be a canonical representation of k in $G = G_1 *_H G_2$ as above. Again this is also a canonical representation of k in $\hat{G}_1 *_{\hat{H}} \hat{G}_2$. Since $k \in \hat{G}_1$, the length of this representation must be 1. Thus $k = w_1 \in \hat{G}_1 \cap (G_1 \cup G_2) = (\hat{G}_1 \cap G_1) \cup (\hat{G}_1 \cap G_2) = G_1 \cup (\hat{H} \cap G_2) = G_1 \cup H = G_1$. So $\hat{G}_1 \cap G = G_1$. Similarly $\hat{G}_2 \cap G = G_2$. ∎

LEMMA 3.4. *Let F be a free group, and K an infinite cyclic subgroup of F. Then $\overline{\mathfrak{N}_F(K)} = \mathfrak{N}_{\hat{F}}(\bar{K})$, where $\overline{\mathfrak{N}_F(K)}$ denotes the closure of $\mathfrak{N}_F(K)$ in $\hat{F}$.*

Proof. Since K is contained in a finitely generated free factor of F, we may assume that F has finite rank. Let X be a basis of F. Let $\mathscr{S}$ be the Cayley graph of F with respect to X. Then $\mathscr{S}$ is a tree (cf. Proposition I.15 in [19]). Say $K = \langle k \rangle$. Consider the smallest K-invariant subtree $\mathscr{T}$ of $\mathscr{S}$ containing the vertex 1. Then

$$\mathscr{T} = \bigcup_{v \in V(\mathscr{X})} K[1, k],$$

where $[1, k]$ denotes the unique reduced path from 1 to k in $\mathscr{S}$. Clearly $\mathscr{T}$ is a minimal K-invariant subtree. Observe that $\mathscr{T}/K$ is finite. By Lemma 2.2(i), $\mathscr{T}$ is the unique K-invariant subtree of $\mathscr{S}$, since K does not fix any edges of $\mathscr{S}$. Note that $\mathscr{S}$ is naturally embedded in the profinite Cayley graph $\hat{\mathscr{S}}$ of free profinite group $\hat{F}$ with respect to X. Then the closure of $\mathscr{T}$ in $\hat{\mathscr{S}}$ is

$$\bar{\mathscr{T}} = \bigcup_{v \in V(\mathscr{X})} \bar{K}[1, k].$$

Observe that K is closed in the profinite topology of F, i.e., $\bar{K} \cap F = K$; therefore, $\bar{\mathscr{T}} \cap \mathscr{S} = \mathscr{T}$. We claim that $\bar{\mathscr{T}}$ is a minimal $\bar{K}$-invariant profinite subtree of $\hat{\mathscr{S}}$. For let R be a $\bar{K}$-invariant profinite subtree of $\bar{\mathscr{T}}$. Then $R \cap \mathscr{S}$ is a nonempty K-invariant subtree of $\mathscr{T}$, and so $R \cap \mathscr{S} = \mathscr{T}$; thus $R = \bar{\mathscr{T}}$, and hence we have proved the claim. One deduces from Lemma 2.2(ii) that $\bar{\mathscr{T}}$ is the unique minimal $\bar{K}$-invariant profinite subtree of $\hat{\mathscr{S}}$.

Now, $\mathfrak{N}_{\hat{F}}(\bar{K})$ acts naturally on $\bar{\mathscr{T}}$, for if $\gamma \in \mathfrak{N}_{\hat{F}}(\bar{K})$, then $\gamma\bar{\mathscr{T}}$ is obviously a minimal $\bar{K}$-invariant profinite subtree of $\hat{\mathscr{S}}$, and so $\bar{\mathscr{T}} = \gamma\bar{\mathscr{T}}$. Next consider the natural epimorphism of quotient graphs

$$\varphi: \mathscr{T}/\mathfrak{N}_F(K) = \bar{\mathscr{T}}/\overline{\mathfrak{N}_F(K)} \to \bar{\mathscr{T}}/\mathfrak{N}_{\hat{F}}(\bar{K}).$$

We shall show that φ is an isomorphism. If $t_1, t_2 \in V(\mathscr{T})$, and $\varphi(t_1) = \varphi(t_2)$, then there exists $\delta \in \mathfrak{N}_{\hat{F}}(\bar{K})$ such that $\delta t_1 = t_2$. Since t_1 and t_2 are elements of F, it follows that $\delta \in F$. So φ is an isomorphism, and hence $\overline{\mathfrak{N}_F(K)} = \mathfrak{N}_{\hat{F}}(\bar{K})$, as desired. ∎

LEMMA 3.5. *Let R be either a free-by-finite group or a finitely generated nilpotent-by-finite group. Let $r_1, r_2 \in R$ be such that $r_1^R \cap \langle r_2 \rangle = \emptyset$. Then $r_1^{\hat{R}} \cap \overline{\langle r_2 \rangle} = \emptyset$.*

Proof. Since R is quasi potent by Lemma 3.1, the closure $\overline{\langle g \rangle}$ in $\hat{R}$ of any infinite cyclic subgroup $\langle g \rangle$ of R is isomorphic to $\hat{\mathbb{Z}}$, and, in particular, torsion-free. If either r_1 or r_2 has finite order, and $r_1^{\hat{R}} \cap \overline{\langle r_2 \rangle} \neq \emptyset$, then both of them must have finite order; in this case the result follows from the fact that R is conjugacy separable (cf. [7], and [15] or [9]). Assume now that both r_1 and r_2 have infinite order. Let F be a normal subgroup of finite index in R such that F is either free or a finitely generated nilpotent group. Assume that $\gamma r_1 \gamma^{-1} = r_2^{\alpha} \in \overline{\langle r_2 \rangle}$, for some $\gamma \in \hat{R}$, $\alpha \in \hat{\mathbb{Z}}$. We shall show that then $r_1^R \cap \langle r_2 \rangle \neq \emptyset$, contradicting our hypothesis. Note that $\gamma = \delta r$ for some $r \in R$ and $\delta \in \bar{F} = \hat{F}$. Observe that $\bar{F} \cap \langle r_1 \rangle = F \cap \langle r_1 \rangle = \langle r_1^n \rangle$, and $\bar{F} \cap \langle r_2 \rangle = F \cap \langle r_2 \rangle = \langle r_2^k \rangle$, for some n, $k \in \mathbb{Z}$. So $\gamma r_1^n \gamma^{-1} \in \overline{\langle r_2^k \rangle}$. Hence $\delta(rr_1r^{-1})^n\delta^{-1} \in \overline{\langle r_2^k \rangle}$. Then for any normal subgroup N of F of finite index one has $\delta rr_1^n r^{-1}\delta^{-1}N \in \langle r_2^k \rangle N/N$, in F/N. Since $rr_1^n r^{-1} \in F$, it follows from Lemmas 6 and 8 in [8] that $(rr_1^n r^{-1})^F \cap \langle r_2^k \rangle \neq \emptyset$. Say $frr_1^n r^{-1} f^{-1} \in \langle r_2^k \rangle$, for some $f \in F$. Substituting r_1 by $frr_1 r^{-1} f^{-1}$ if necessary, we may assume that $r_1^n = r_2^m$, for some $m \in \mathbb{Z}$, and that $\gamma \in \bar{F} = \hat{F}$. Then $\gamma r_2^m \gamma^{-1} = \gamma r_1^n \gamma^{-1} = (r_2^n)^{\alpha}$. It follows from Lemma 2.4 that the closed subgroup generated by r_2^m coincides with the closed subgroup generated by $(r_2^n)^{\alpha}$; i.e., γ normalizes $\overline{\langle r_2^m \rangle}$. Next we claim that either γ centralizes r_2^m or γ inverts r_2^m. To see this we consider two cases. If R is free-by-finite, the claim follows from the fact that $\overline{\mathfrak{N}_F(\langle r_2^m \rangle)} = \mathfrak{N}_{\hat{F}}\overline{\langle r_2^m \rangle}$ (see Lemma 3.4). If R is finitely generated nilpotent-by-finite, observe that $\overline{\langle \gamma, r_2^m \rangle}$ is a nilpotent closed subgroup of $\hat{F}$; then, since γ

normalizes $\overline{\langle r_2^m \rangle}$ and since $\overline{\langle r_2 \rangle} \cong \hat{\mathbb{Z}}$ by Lemma 3.1, we deduce that γ centralizes r_2^m. This proves the claim. Thus, in any case, $(r_2^\alpha)^n = \gamma r_2^m \gamma^{-1} \in \langle r_2^m \rangle \leq \langle r_2 \rangle$. Now, since $\langle r_2 \rangle \cong \hat{\mathbb{Z}}$, and since $\hat{\mathbb{Z}}/\mathbb{Z}$ is torsion-free, one deduces that r_2^α is in $\langle r_2 \rangle$. Put $r' = r_2^\alpha$; then $\gamma \mathrm{r}_1 \gamma^{-1} = \mathrm{r}' \in \langle r_2 \rangle$. Since R is conjugacy separable (cf. [7, 15, 9]), one gets $\langle r_1 \rangle^R \cap \langle r_2 \rangle \neq \emptyset$, as desired. ∎

LEMMA 3.6. *Let G be a free-by-finite or a finitely generated nilpotent-by-finite group. Let $C_1 = \langle x_1 \rangle$ and $C_2 = \langle x_2 \rangle$ be cyclic subgroups of G. If $C_1 \cap C_2 = 1$, then $\overline{C}_1 \cap \overline{C}_2 = 1$, where $\overline{C}_i$ denotes the closure of C_i in $\hat{G}$ $(i = 1, 2)$.*

Proof. *Case* (i). G is free-by-finite. If either C_1 or C_2 is finite, then the result is clear. So, suppose C_1 and C_2 are infinite. Let F be a free normal subgroup of G of finite index. If $\overline{C}_1 \cap \overline{C}_2 \neq 1$, then $(\overline{F} \cap \overline{C}_1) \cap (\overline{F} \cap \overline{C}_2) \neq 1$. Hence we may assume that $G = F$ is free. Now, the subgroup $\langle C_1, C_2 \rangle = \langle x_1, x_2 \rangle$ is free on the basis $\{x_1, x_2\}$. By a theorem of M. Hall (cf. Theorem 1 in [4]), there exists a subgroup U of F of finite index such that U is free on a set that contains x_1 and x_2. So $\overline{U} = \hat{U}$ is a free profinite group on a set that contains x_1 and x_2. Therefore, $\overline{C}_1 \cap \overline{C}_2 = 1$.

Case (ii). G is finitely generated nilpotent-by-finite. Recall that G is residually finite, and that if H is a subgroup of G, then H is closed in the profinite topology of G (cf. [14]); moreover, this topology induces on H the full profinite topology of H (cf. Theorem 20B in [10]). In particular, if H is normal in G, then

$$\widehat{G/H} \cong \hat{G}/\hat{H} \cong \hat{G}/\overline{H}.$$

If either C_1 or C_2 is finite, the result is clear; hence from now on we assume that C_1 and C_2 are infinite cyclic. Let N be a subgroup of G of finite index. If $\overline{C}_1 \cap \overline{C}_2 \neq 1$, then $(\overline{C}_1 \cap \overline{N}) \cap (\overline{C}_2 \cap \overline{N}) \neq 1$. Now, $\overline{C}_1 \cap \overline{N} = \overline{C_1 \cap N}$ and $\overline{C}_2 \cap \overline{N} = \overline{C_2 \cap N}$, because C_1 and C_2 are cyclic, and the profinite topology of G induces on C_1 and C_2 their full profinite topologies. Hence $\overline{(C_1 \cap N)} \cap \overline{(C_2 \cap N)} \neq 1$. Therefore, since G contains a nilpotent subgroup N of finite index, we may assume that $G = N$ is nilpotent.

We proceed by induction on the nilpotency class of G. If G is abelian, the result is easily verified. Assume that the result holds for every nilpotent group of class at most n, and suppose that G is nonabelian, nilpotent of class $n + 1$. Next we claim that if A is a nontrivial abelian subgroup of G which is normalized by C_1 and C_2, and $A \cap C_1 \neq 1 \neq A \cap C_2$, then $\overline{C}_1 \cap \overline{C}_2 = 1$. To see this observe first that $\overline{(C_1 \cap A)} \cap \overline{(C_2 \cap A)} = 1$, since A is abelian and since the closure of $C_1 \cap A$ (respectively, $C_2 \cap A$) in G coincides with the closure of $C_1 \cap A$ (respectively, $C_2 \cap A$) in A.

Now, AC_i is a subgroup of G, and since $(C_i\colon A \cap C_i)$ is finite, one deduces, as above, that $\overline{C}_i \cap \overline{A} = \overline{C_i \cap A}$ $(i = 1, 2)$. It follows that $\overline{A} \cap \overline{C}_1 \cap \overline{C}_2 = 1$. Since $\overline{A}$ has finite index in $\overline{AC_i}$ $(i = 1, 2)$, it also has finite index in $\overline{AC_1} \cap \overline{AC_2}$; therefore, $\overline{A} \cap \overline{C}_1 \cap \overline{C}_2$ has finite index in $\overline{C}_1 \cap \overline{C}_2$. So $\overline{C}_1 \cap \overline{C}_2$ is finite, and thus $\overline{C}_1 \cap \overline{C}_2 = 1$. This proves the claim.

Consider the last nontrivial term L of the lower central series of G. Then L is in center of G. Set $T = LC_1 \cap LC_2$, and note that C_1 and C_2 centralize T; now, if $C_1 \cap T \neq 1 \neq C_2 \cap T$, then the result follows from the claim above. Hence we may assume that either $1 = C_1 \cap T = C_1 \cap LC_2$ or $1 = C_2 \cap T = C_2 \cap LC_1$. In either case it follows that $LC_1 \cap LC_2 = L$. Then from the induction hypothesis applied to G/L, one deduces that $\overline{LC}_1 \cap \overline{LC}_2 = \overline{L}$. Therefore, $\overline{C}_1 \cap \overline{C}_2 \leq \overline{L}$. Note that $LC_1 = L \times C_1$, and hence $\overline{LC}_1 = \overline{L} \times C_1$ (since G induces on LC_1 its full profinite topology); thus $\overline{L} \cap \overline{C}_1 = 1$, and similarly, $\overline{L} \cap \overline{C}_2 = 1$. Hence $\overline{C}_1 \cap \overline{C}_2 = \overline{C}_1 \cap \overline{C}_2 \cap \overline{L} = 1$. ∎

LEMMA 3.7. *Let G be either a finitely generated nilpotent-by-finite or a free-by-finite group. Then, for every $g \in G$ of infinite order and every $\gamma \in \hat{G}$ such that $\gamma\overline{\langle g\rangle}\gamma^{-1} = \overline{\langle g\rangle}$, we have that either $\gamma g \gamma^{-1} = g$ or $\gamma g \gamma^{-1} = g^{-1}$.*

Proof. Let H be a normal subgroup of finite index in G, where H is either a finitely generated torsion-free nilpotent group or a free group.

Let $0 \neq n \in \mathbb{Z}$. Observe that the result holds for g if it holds for g^n; for suppose it holds for g^n, and assume that $\gamma \in \hat{G}$ and $\gamma g \gamma^{-1} = g^{\alpha}$ for some $\alpha \in \hat{\mathbb{Z}}$. Since γ normalizes $\overline{\langle g^n\rangle}$, it follows that either $\gamma g^n \gamma^{-1} = g^n$ or $\gamma g^n \gamma^{-1} = g^{-n}$; hence either $\gamma g^{n\alpha}\gamma^{-1} = g^n$ or $\gamma g^{n\alpha}\gamma^{-1} = g^{-n}$; therefore, either $\alpha = 1$ or $\alpha = -1$.

Let $\gamma \in \hat{G}$ and let $g \in G$ be such that $\gamma g \gamma^{-1} = g^{\alpha}$, for some $\alpha \in \hat{\mathbb{Z}}$. We need to show that $\alpha = 1$ or $\alpha = -1$. As observed above, we may assume that $g \in H$.

Case 1. H is finitely generated torsion-free nilpotent.

We proceed by induction on the nilpotency class of H. If H is abelian, then $H \leq \mathfrak{C}_G(g) \leq \mathfrak{N}_G(g)$, and so $\mathfrak{C}_G(g)$ and $\mathfrak{N}_G(g)$ have finite index in G. Therefore $\mathfrak{C}_{\hat{G}}(g)$ and $\mathfrak{N}_{\hat{G}}(g)$ are the closures in $\hat{G}$ of $\mathfrak{C}_G(g)$ and $N_G(g)$ respectively; furthermore, $\mathfrak{N}_G(g)/C_G(g) = \mathfrak{N}_{\hat{G}}(g)/\mathfrak{C}_{\hat{G}}(g)$. For the general case, let Z be the center of H and assume $Z \neq H$. If $g^n \in Z$ for ssome natural number $n \neq 0$, we may assume $g \in Z$. Then again $H \leq \mathfrak{C}_G(g) \leq \mathfrak{N}_G(g)$, and we proceed as above. Finally, suppose that $\langle g\rangle \cap Z = 1$. Then $\langle g, Z\rangle = Z \times \langle g\rangle$. Since H/Z is a finitely generated nilpotent group, we have that $\overline{\langle gZ\rangle} \cong \hat{\mathbb{Z}}$. Therefore, $\overline{\langle g\rangle} \cap \overline{Z} = 1$. Hence, by the induction hypothesis applied to G/Z, either $\gamma g\overline{Z}\gamma^{-1} = g\overline{Z}$ or $\gamma g\overline{Z}\gamma^{-1} = g^{-1}\overline{Z}$. Thus, either $\gamma g \gamma^{-1} = g$ or $\gamma g \gamma^{-1} = g^{-1}$.

Case 2. H is free.

By a theorem of M. Hall (see [12]), $\langle g \rangle$ is a free factor of a subgroup L of finite index in H; since L has finite index in G, there exists a normal subgroup L_G of G of finite index such that $L_G \leq H$. It follows that for some natural number n, $\langle g^n \rangle$ is a free factor of L_G; as observed above, we may substitute g by g^n, and hence we may assume that $H = L_G$ and $\langle g \rangle$ is free factor of H. Since $H = \langle g \rangle * K$ for some $K \leq H$ and H has finite index in G, we have that $\hat{H} = \overline{H} = \overline{\langle g \rangle} \amalg \overline{K}$. By Corollary 3.13 of [26] one has that $\mathfrak{N}_{\hat{H}}(\overline{\langle g \rangle}) = \mathfrak{C}_{\hat{H}}(g) = \overline{\langle g \rangle}$. We deduce that $\mathfrak{N}_{\hat{G}}(\overline{\langle g \rangle}) \cap \overline{H} = \mathfrak{N}_{\overline{H}}(\overline{\langle g \rangle}) = \overline{\langle g \rangle} = \mathfrak{C}_{\overline{H}}(g) = \mathfrak{C}_{\hat{G}}(g) \cap \overline{H}$. Then $\mathfrak{N}_{\hat{G}}(\overline{\langle g \rangle}) / \overline{\langle g \rangle} = \mathfrak{N}_{\hat{G}}(\overline{\langle g \rangle}) / (\overline{H} \cap \mathfrak{N}_{\hat{G}}(\overline{\langle g \rangle})) \leq \hat{G}/\hat{H} \cong G/H$. Hence $\mathfrak{N}_{\hat{G}}(\overline{\langle g \rangle})$ is contained in $G\overline{\langle g \rangle}$. Now, since $\overline{\langle g \rangle}$ centralizes g, we deduce that $\mathfrak{N}_{\hat{G}}(\overline{\langle g \rangle}) = \overline{\langle g \rangle} \mathfrak{N}_G(\langle g \rangle)$. We infer that $\gamma \in \mathfrak{N}_{\hat{G}}(\overline{\langle g \rangle})$ either commutes with g or inverts g. ∎

THEOREM 3.8. *Let G_1 and G_2 be groups that are free-by-finite or finitely generated nilpotent-by-finite* (*not necessarily both of the same type*), *and assume that H is a common cyclic subgroup of G_1 and G_2. Let $G = G_1 *_H G_2$ be the amalgamated free product of G_1 and G_2 amalgamating H. Then G is conjugacy separable.*

Proof. By Proposition 3.2, G is residually finite, and $\hat{G} = \Gamma = \Gamma_1 \amalg_\Delta \Gamma_2$, where $\Gamma_1 = \hat{G}_1$, $\Gamma_2 = \hat{G}_2$ and $\Delta = \hat{H}$. Consider the standard tree $\mathscr{S}(G)$ associated with $G = G_1 *_H G_2$, and the standard profinite tree $\mathscr{S}(\Gamma)$ associated with $\Gamma = \Gamma_1 \amalg_\Delta \Gamma_2$. By Lemma 3.3, G_1, G_2, and H are closed in the profinite topology of G, and therefore $\mathscr{S}(G)$ is naturally embedded in $\mathscr{S}(\Gamma)$. Let a, b elements of G, and assume that $b = \gamma a \gamma^{-1}$, where γ is an element of Γ.

Case 1. The element a fixes a vertex of $\mathscr{S}(G)$ (i.e., a is not hyperbolic).

Then by Lemma 2.8, b also fixes some vertex of $\mathscr{S}(G)$. This means that a and b are conjugate in G to elements of G_1 or G_2. So we may assume that $a, b \in G_1 \cup G_2$. Then by Proposition 3.2 and Lemma 2.3, a and b are either both in Γ_1 or both in Γ_2, or one of them is conjugate to an element of Δ, and so by Lemma 3.5, to an element of H; hence we may assume that a and b are either both in Γ_1 or both in Γ_2, say both are in Γ_1. By Lemma 3.3, $G \cap \Gamma_1 = G_1$. Whence we may assume that $a, b \in G_1$. If a and b are conjugate in Γ_1, then they are conjugate in G_1 ([7], and [15] or [9]). So we may assume a and b are not conjugate in Γ_1.

Claim. We may assume that $a, b \in H$.

To prove this claim consider the vertex $v_1 = 1\Gamma_1$ in $\mathscr{S}(\Gamma)$. Obviously Γ_1 is the stabilizer of v_1 in Γ. Since $b = \gamma a \gamma^{-1}$, v_1 and γv_1 are fixed by b, and

v_1 and $\gamma^{-1}v_1$ are fixed by a. By Theorem 2.8 in [26], the subgraph of $\mathcal{S}(\Gamma)$ fixed by b is a profinite subtree. So there exists an edge e_1 in $\mathcal{S}(\Gamma)$ such that $d_0(e_1) = v_1$ and $be_1 = e_1$. Similarly, there exists an edge e_2 in $\mathcal{S}(\Gamma)$ such that $d_0(e_2) = v_1$ and $ae_2 = e_2$. Consider the edge $e = 1\Delta$ in $\mathcal{S}(\Gamma)$. Then there exist $\gamma_1, \gamma_1' \in \Gamma_1$ such that $\gamma_1 e_1 = e = \gamma_1' e_2$. So $\gamma_1 b \gamma_1^{-1} \in \Delta$ and $\gamma_1' a \gamma_1'^{-1} \in \Delta$, since Δ is the stabilizer of e in Γ. By Lemma 3.5, $a^{G_1} \cap H \neq 1$, and similarly $b^{G_1} \cap H \neq 1$. Therefore we may assume that $a, b \in H$. This proves the claim.

Now, by a slight variation of Lemma 2.4, $\langle a \rangle = \langle b \rangle$. Whence $\gamma \in \mathfrak{N}_{\Gamma}(\overline{\langle a \rangle}) = \mathfrak{N}_{\Gamma_1}(\overline{\langle a \rangle}) \amalg_{\Delta} \mathfrak{N}_{\Gamma_2}(\overline{\langle a \rangle})$, by Corollary 2.7. According to Lemma 3.7, there exist natural homomorphisms $\mathfrak{N}_{\Gamma_i}(\overline{\langle a \rangle}) \to \mathrm{Aut}(\langle a \rangle)$ $(i = 1, 2)$ into the finite group $\mathrm{Aut}(\langle a \rangle)$, that are trivial on Δ; these maps induce a homomorphism $\varphi\colon \mathcal{N}_{\Gamma}(\overline{\langle a \rangle}) = \mathfrak{N}_{\Gamma_1}(\overline{\langle a \rangle}) \amalg_{\Delta} \mathfrak{N}_{\Gamma_2}(\overline{\langle a \rangle}) \to \mathrm{Aut}(\langle a \rangle)$. Put $k = \varphi(\gamma)$. Since the image of φ is finite and $\mathfrak{N}_{\Gamma_1}(\overline{\langle a \rangle}) *_{\Delta} \mathfrak{N}_{\Gamma_2}(\overline{\langle a \rangle})$ is dense in $\mathfrak{N}_{\Gamma}(\overline{\langle a \rangle}) = \mathfrak{N}_{L\Gamma_1}(\overline{\langle a \rangle}) \amalg_{\Delta} \mathfrak{N}_{\Gamma_2}(\overline{\langle a \rangle})$, there exists an element $g' \in \mathcal{N}_{\Gamma_1}(\overline{\langle a \rangle}) *_{\Delta} \mathcal{N}_{\Gamma_2}(\overline{\langle a \rangle})$ such that $\varphi(g') = k$. Write $g' = w_1 w_2 \cdots w_n$, with each w_i in $\mathcal{N}_{\Gamma_1}(\overline{\langle a \rangle}) \cup \mathcal{N}_{\Gamma_2}(\overline{\langle a \rangle})$. Put $a_i = w_i a_{i+1} w_i^{-1}$ $(i = 1, \ldots, n;\ a_{n+1} = a)$. Note that $a_i \in \langle a \rangle \leq H$ (in fact $a_i = a^{\pm 1}$ if H is infinite, by Lemma 3.7). Since G_1 and G_2 are conjugacy separable (cf. [7], and [15] or [9]), there exist elements $g_1, g_2, \ldots, g_n$ in $G_1 \cup G_2$ such that $a_i = g_i a_{i+1} g_i^{-1}$ $(i = 1, \ldots, n)$. Put $g = g_1 \ldots g_n$. Then $g \in G$, and $gag^{-1} = g_1 \ldots g_n a g_n^{-1} \ldots g_1^{-1} = g' a g'^{-1} = b$. This completes the proof of Case 1.

Case 2. The element a does not fix a vertex of $\mathcal{S}(G)$ (in other words, a is hyperbolic).

By Lemma 2.8, b is also hyperbolic. By a theorem of Tits (cf. Prop. 24 in [19]), there are infinite straight lines $\mathcal{T}_a$ and $\mathcal{T}_b$ in $\mathcal{S}(G)$ defined as follows: let

$$m_1 = \min\{l(v, av) \mid v \in V(\mathcal{S}(G))\},$$

$$m_2 = \min\{l(v, bv) \mid v \in V(\mathcal{S}(G))\};$$

then $\mathcal{T}_a$ and $\mathcal{T}_b$ are the subtrees of $\mathcal{S}(G)$, whose vertices are $\{v \in V(\mathcal{S}(G)) \mid l(v, av) = m_1\}$ and $\{v \in V(\mathcal{S}(G)) \mid l(v, bv) = m_2\}$, respectively. Moreover a and b act freely on $\mathcal{T}_a$ and $\mathcal{T}_b$ as translations of lengths m_1 and m_2, respectively. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be segments of $\mathcal{T}_a$ and $\mathcal{T}_b$ of lengths m_1 and m_2, respectively. Then $\mathcal{T}_a = \langle a \rangle \mathcal{T}_1$, $\mathcal{T}_b = \langle b \rangle \mathcal{T}_2$.

Set $e = 1H$, the edge of $\mathcal{S}(G)$ stabilized by H. We claim that one may assume that $e \in \mathcal{T}_1 \cap \mathcal{T}_2$. To see this, consider $g_1, g_2 \in G$ such that $e \in g_1\mathcal{T}_1$ and $e \in g_2\mathcal{T}_2$. Set $a' = g_1 a g_1^{-1}$ and $b' = g_2 b g_2^{-1}$. Then a' and b' are also hyperbolic, and one has corresponding straight lines $\mathcal{T}_{a'} = g_1\mathcal{T}_a$, $\mathcal{T}_{b'} = g_2\mathcal{T}_b$. Define $\mathcal{T}'_1 = g_1\mathcal{T}_1$ and $\mathcal{T}'_2 = g_2\mathcal{T}_2$. Then clearly $\mathcal{T}_{a'} = \langle a' \rangle \mathcal{T}'_1$ and $\mathcal{T}_{b'} = \langle b' \rangle \mathcal{T}'_2$. Since a and b are conjugate in G if and only if a' and

b' are conjugate in G, the claim follows. So from now on we assume that $e \in \mathscr{T}_1 \cap \mathscr{T}_2$.

Consider the profinite subgraphs of $\mathscr{S}(\Gamma)$ defined as $\bar{\mathscr{T}}_a = \overline{\langle a\rangle}\mathscr{T}_1$ and $\bar{\mathscr{T}}_b = \overline{\langle b\rangle}\mathscr{T}_2$. By Proposition 2.9, $\overline{\langle a\rangle}$ and $\overline{\langle b\rangle}$ act freely on $\bar{\mathscr{T}}_a$ and $\bar{\mathscr{T}}_b$, respectively. Since $\gamma a \gamma^{-1} = b$, $\overline{\langle b\rangle}$ also acts freely on $\gamma\bar{\mathscr{T}}_a$. By Lemma 2.2(ii), $\gamma\bar{\mathscr{T}}_a = \bar{\mathscr{T}}_b$. Then $\gamma e \in \bar{\mathscr{T}}_b$. Choose $b' \in \overline{\langle b\rangle}$ such that $b'\gamma e \in \mathscr{T}_2$. Then $b'\gamma e = ge$, for some $g \in G$. Hence $b'\gamma = g\delta$, for some $\delta \in \Delta$. Now, $a = \gamma^{-1}b'^{-1}bb'\gamma = \delta^{-1}g^{-1}bg\delta$. Therefore, using $g^{-1}bg$ instead of b, we can assume that γ $(= \delta)$ is in Δ.

Since a and b are hyperbolic, they can be written in $G = G_1 *_H G_2$, as $a = v_1 v_2 \ldots v_n$, $b = w_1 w_2 \ldots w_m$ $(n, m \geq 2)$, where $v_i, w_i \in (G_1 \cup G_2) \setminus H$, and if v_i, $w_i \in G_1$ (respectively, if v_i, $w_i \in G_2$) then v_{i+1}, $w_{i+1} \in G_2$ (respectively, then v_{i+1}, $w_{i+1} \in G_1$). Recall that by Lemma 3.3, $\Delta \cap G_1 = \Delta \cap G_2 = H$; therefore the above expressions of a and b are also representations of a and b in $R = \Gamma_1 *_\Delta \Gamma_2$, with $v_i, w_i \notin \Delta$. Since a and b are conjugate in R by an element of Δ, we get $n = m$. By Lemma 8 in [22], there exist elements $\delta_0, \ldots, \delta_n \in \Delta$ such that

$$w_1 = \delta_0^{-1} v_1 \delta_1, \quad w_2 = \delta_1^{-1} v_2 \delta_2, \ldots, w_n = \delta_{n-1}^{-1} v_n \delta_n, \quad \text{with } \delta_0 = \delta_n. \quad (*)$$

Subcase (i). There is some i $(= 1, \ldots, n)$ such that $v_i \Delta v_i^{-1} \cap \Delta = 1$.

Then, since $w_i = \delta_{i-1}^{-1} v_i \delta_i$, we obtain $w_i \in \Delta v_i \Delta$. By Theorem 2.1 in [18] and Theorem 1 in [23], the product of two finitely generated subgroups of G_k is closed in the profinite topology of G_k $(k = 1, 2)$. So $\Delta v_i \Delta \cap G_k = H v_i H$ $(k = 1$ or $2)$. Therefore, $w_i = h_{i-1}^{-1} v_i h_i$, for some $h_{i-1}, h_i \in H$. Then $\delta_{i-1} h_{i-1}^{-1} = v_i \delta_i h_i^{-1} v_1^{-1}$. From the assumption in this subcase, it follows that $\delta_i = h_i$ and $\delta_{i-1} = h_{i-1}$. One deduces then from $(*)$ and Lemma 3.3 that $\delta_j \in H$ for all $j = 0, 1, \ldots, n$. Thus by Lemma 8 in [22], a and b are conjugate in G.

Subcase (ii). For every i $(= 1, \ldots, n)$, $v_i \Delta v_i^{-1} \cap \Delta \neq 1$.

By Lemma 3.6, $v_i H v_i^{-1} \cap H \neq 1$, and so $v_i H v_i^{-1} \cap H$ has finite index in H. It follows that $v_i \Delta v_i^{-1} \cap \Delta$ has finite index in Δ. Therefore,

$$\Lambda = \bigcap_{i=1}^{n} v_i \Delta v_i^{-1} \cap \Delta$$

is an open subgroup of Δ. Hence $\gamma = \lambda h$, for some $h \in H$, $\lambda \in \Lambda$. So we may assume that $\gamma = \lambda$.

Define $L = \langle a, \Lambda \cap H\rangle$. Next we claim that Λ and $\Lambda \cap H$ are normalized by a. First, we prove that Λ is normalized by each v_i. Indeed, Λ and $v_i^{-1}\Lambda v_i$ are conjugate subgroups of the cyclic group Δ; hence by Lemma 2.4(ii), $\Lambda = v_i^{-1}\Lambda v_i$. Since $v_i \in G_1 \cup G_2$, and $(G_1 \cup G_2) \cap \Delta = H$ (cf.

Lemma 3.3), it follows that $\Lambda \cap H = v_i(\Lambda \cap H)v_i^{-1}$. Since $a = v_1 v_2 \dots v_n$, the claim follows.

Note now that since Λ is open in Δ, $\Lambda \cap H$ is dense in Λ (in fact $\Lambda = \overline{\Lambda \cap H}$); hence $\overline{L} = \overline{\langle a, \Lambda \rangle}$. Next observe that $b \in L$. For, since $b = \gamma a \gamma^{-1}$, $\gamma \in \Lambda$, and Λ is normal in $\overline{L}$, then $b = a\lambda'$ for some $\lambda' \in \Lambda$; hence from $b = a\lambda' \in G$, one deduces that $\lambda' \in G$; thus $\lambda' \in H \cap \Lambda$, and so $b \in L$.

Since a is hyperbolic, $L = (\Lambda \cap H) \rtimes \langle a \rangle$. Hence $\overline{L} = \overline{\langle \Lambda, \langle a \rangle \rangle}$. By Proposition 2.9, $\overline{\langle a \rangle} \cap \Delta = 1$, and so $\overline{L} = \Lambda \rtimes \overline{\langle a \rangle} = (\widehat{\Lambda \cap H}) \rtimes \overline{\langle a \rangle} = \hat{L}$. If L is abelian, then $a = b$, because $\gamma \in \overline{L}$. If L is not abelian, let $\Delta \cap H = \langle x \rangle$. Then $axa^{-1} = x^{-1}$. Say $b = x^n a^m$ (some $n, m \in \mathbb{Z}$), and $\gamma = x^\alpha a^\beta$ (some $\alpha, \beta \in \hat{\mathbb{Z}}$). Then $x^n a^m = b = \gamma a \gamma^{-1} = x^{-2\alpha} a$. Hence $m = 1$, and $2\alpha = n \in \mathbb{Z}$. Therefore $\alpha = n/2 \in \mathbb{Z}$. Set $c = x^{n/2}$. Then $c \in \Delta \cap H \leq G$, and $cac^{-1} = b$. ∎

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