Spectra of symmetric tensors and $m$-root Finsler models

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Article history:
Received 17 February 2011
Accepted 20 June 2011
Available online 20 July 2011
Submitted by E. Tyrtyshnikov

AMS classification:
65F30
15A18
15A69
53B40
53C60

Keywords:
Recession vector
Degeneracy sets
Spectra
Eigenvalues
Asymptotic rays
Best rank-one approximation

The present work studies spectral properties of multilinear forms attached to the Berwald-Moor, Chernov and Bogoslovsky locally Minkowski Finsler geometric structures of $m$-root type. We determine eigenvalues and the corresponding eigenvectors (of type $Z$, $H$ and $E$) of these forms, in the framework of symmetric tensors and multivariate homogeneous polynomials. The geometric relevance of the spectral data is emphasized, and the existent relations between spectra, polyangles and Riesz-type associated 1-forms of the corresponding geometric models, are described. As well, the best rank-one approximation for the 4-dimensional Berwald-Moor and Chernov cases, is derived.

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1. Introduction

Recently, the class of $m$-root Finsler metrics provided challenging models for ecology [1], for Relativity (the Roxburgh spherical symmetric models [31]) and for HARDI (Higher Angular Resolution Diffusion Imaging, introduced by Astola and Florack [2]). Moreover, these metrics proved to provide alternative non-standard models for Special Relativity, thus becoming a fruitful subject of research of the last decade [12,21,25–27].

The present work deals with the super-symmetric tensors which are canonically determined by such structures, from the point of view of numerical multilinear algebra – a field which has highly developed in recent years. This new field of research involves computational topics regarding higher-
order tensors, using specific notions as: tensor decomposition, tensor rank, multi-way eigenvalues and
eigenvectors, lower-rank approximation of tensors, numerical stability and perturbation analysis of
tensor computation, etc. (e.g. [4,9,13–15,18–20,29,30]). Among the multiple applications of this field,
we mention: digital image restoration, multi-way data analysis, blind source separation and blind
source deconvolution in signal processing, higher-order statistics, etc (e.g. [32]).

Particularly, the blind source separation problem (BSS problem, or Independent Component Analy-
sis – ICA) from signal processing has, among its aims, both interception and classification in military
applications and surveillance of communications in the civil resort. The problem of blind separation of
cumulant symmetric tensor $T$ by another, rank-one, tensor. In our paper, we determine the one-rank approx-
imation for two of the investigated super-symmetric tensors. The present work relates the Berwald-
Moor [25,26], Chernov [17] and Bogoslovsky [11] locally Minkowski Finsler metrics of
$\mathbb{R}^4$ and has been recognized (e.g. [16]) as a good candidate model for Special Relativity.

1 Both $H_3$ and $H_4$ models represent the core of recently developed models for Relativity – the main subject of investigation of the
International Research Institute of Hypercomplex Systems in Geometry and Physics – Moscow, Russia [25–27].

2 The Bogoslovsky form has been proposed as Finsler relativistic metric (in one of its alternatives) in earlier seminal works [11],
and has been recognized (e.g. [16]) as a good candidate model for Special Relativity.

\[
F^{(m)}(y) = T(y, \ldots, y) \sim T = \sum_{i_1, \ldots, i_m \in T \cap} T_{i_1, \ldots, i_m} \cdot dx^{i_1} \otimes \cdots \otimes dx^{i_m},
\]

namely:

(a) the $H_4$ Berwald–Moor metric and corresponding symmetric tensor in $\mathbb{R}^4$:

\[
F_{BM_4}(y) = \sqrt{[y_1 y_2 y_3 y_4]}, \quad (1.1)
\]

\[
A_{ijkl} = \frac{1}{4!}, \quad \text{for } \{i, j, k, l\} = \{1, 2, 3, 4\}, \quad 0 \text{ otherwise}; \quad (1.2)
\]

and the $H_3$ Berwald–Moor metric and tensor$^1$:

\[
F_{BM_3}(y) = \sqrt{y_1 y_2 y_3}, \quad (1.3)
\]

\[
A_{ijk} = \frac{1}{3!}, \quad \text{for } \{i, j, k\} = \{1, 2, 3\}, \quad 0 \text{ otherwise}; \quad (1.4)
\]

(b) the Chernov metric and symmetric tensor in $\mathbb{R}^4$:

\[
F_{C_4}(y) = \sqrt{[y_1 y_2 y_3 + y_1 y_2 y_4 + y_1 y_3 y_4 + y_2 y_3 y_4]}, \quad (1.5)
\]

\[
B_{ijk} = \frac{1}{3!}, \quad \text{for distinct } \{i, j, k\} \subset \{1, 2, 3, 4\}, \quad 0 \text{ otherwise}; \quad (1.6)
\]

and in $\mathbb{R}^3$ (the Minkowski–Lorentz framework):

\[
F_{C_3}(y) = \sqrt{[y_1 y_2 + y_1 y_3 + y_2 y_3]}, \quad (1.7)
\]

\[
B_{ij} = \frac{1}{3!}, \quad \text{for distinct } \{i, j\} \subset \{1, 2, 3\}, \quad 0 \text{ otherwise}; \quad (1.8)
\]

(c) the Bogoslovsky metric and symmetric tensor in $\mathbb{R}^3$:$^2$

\[
F_{B_3}(y) = \sqrt{[y_1^2 y_2 y_3 + y_2^2 y_3 y_1 + y_3^2 y_1 y_2]}, \quad (1.9)
\]

\[
B_{ijkl} = \frac{1}{36}, \quad \text{for } \{i, j, k, l\} = \{1, 2, 3\}, \quad 0 \text{ otherwise}. \quad (1.10)
\]
In the following, using the tools developed in [24,29,30], we investigate spectral aspects of these five symmetric tensors, including, for the 4-dimensional case, the solution for the one-rank approximation problem. The emerging new geometric framework is tightly related to the hypercomplex polynomials theory and their applications [25,26]. This leads both to the enhancement of the algebraic subjacent theory due to the geometrical viewpoint, but also to the possibility of illustrating basic nontrivial and non-evident objects of the Berwald-Moor type approach by means of the relatively simple objects, such as the polynomials [25,26,17,27,33].

2. Eigenvalues and eigenvectors of symmetric tensors

2.1. Spectral data of Z, E and H type

Consider a symmetric tensor field $T \in T^0_m(\mathbb{R}^n)$ on the flat manifold $V = \mathbb{R}^n$.

**Definition 2.1.** We say that $\lambda \in \mathbb{R}$ is a Z-eigenvalue ($\lambda \in \sigma_Z(T)$), and that a vector $y \in T^1_0(\mathbb{R}^n) \equiv \mathbb{R}^n$ is an associated Z-eigenvector to $\lambda$, if they satisfy the system:

$$Ty^{m-1} = \lambda y, \quad g(y, y) = 1, \quad (2.1)$$

where we denoted $Ty^{m-1} = \sum_{i, i_2, ..., i_m} T_{i_2, ..., i_m} y_{i_2} \cdot \cdots \cdot y_{i_m} \cdot dx^i$. In the complex case, one simply calls $\lambda$ and $y$ eigenvalue and eigenvector, respectively. Recently, Qi [28,30] defined the following alternative spectral objects:

**Definition 2.2.** A real number $\lambda \in \mathbb{R}$ is an H-eigenvalue and a vector $y \in \mathbb{R}^n$ is an H-eigenvector associated to $\lambda$, if they satisfy the homogeneous polynomial system of order $m - 1$:

$$(Ty^{m-1})_k = \lambda (y_k)^{m-1}. \quad (2.2)$$

In the complex case, $\lambda$ and $y$ are called E-eigenvalue and E-eigenvector, respectively.

Regarding the spectra consistency, it is known that $\sigma_Z(T)$ and $\sigma_H(T)$ are nonempty for even symmetric tensors, and that a symmetric tensor $T$ is positive definite/semi-definite iff all its $H$ (or Z) eigenvalues are positive/non-negative.

2.2. Geometric considerations

In general, while considering an $m$-multilinear symmetric form $T$ defined on $V = \mathbb{R}^n$, we note that the definition of $Z$ and $H$ spectral data reveal certain relations between the polynomials [25–27] determined by the poly-scalar product $T$ and classic Euclidean and Riemann–Finsler geometric structures, as follows:

(a) Denoting by $\delta$ the Euclidean inner product, the Z-eigensystem (2.1) for $\lambda$ and $y$, can be written as:

$$T(y^{m-1}, z) = \lambda \delta(y, z), \quad \forall z \in \mathbb{R}^n, \|y\|_2 = 1,$$

i.e., the $(m - 1)$-polyangle determined by the poly-scalar product $T$ and the classic Euclidean inner product, based on Z-eigenvectors of $T$, are homothetic while applied to Euclidean unit vectors.

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3 In the positive-definite variational approach, a more relaxed requirement is the weak symmetry instead the stronger one of super-symmetry, namely: $\nabla(Ty^n) = my^{m-1}$. The two concepts substantially differ, since there exist weakly-symmetric tensors, like the 2-dimensional 4th order tensor $T(a, b, c, d) = 3a_1b_1c_1d_2 + a_2b_1c_1d_1$, $\forall a, b, c, d \in \mathbb{R}^2$, which are non-symmetric. Indeed, $Ty^3 = (3y_1^2y_2, y_1^2), Ty^4 = 4y_1^2y_2, \nabla(Ty^4) = (12y_1^3y_2, 4y_1^3) = 4Ty^3$. In our work, all the five considered tensors are symmetric, confining to the general $n$-way framework.
(b) Denoting by \( C \) the Riemann–Finsler multilinear symmetric form associated to the \( m \)-pseudonorm \( F_{RF}(y) = \sqrt[n]{y_1^m + \cdots + y_n^m} \), namely

\[
C = \sum_{i=1}^{n} \otimes^m dx^i = \sum \delta_{i_1,\ldots,i_m} dx^{i_1} \otimes \cdots \otimes dx^{i_m}, \tag{2.3}
\]

we note that the \( H \)-eigensystem (2.2) can be written as:

\[
T(y^{m-1}, z) = \lambda C(y^{m-1}, z), \quad \forall z \in \mathbb{R}^n,
\]

i.e., the \((m - 1, 1)\)-polyangles of the poly-scalar products \( T \) and \( C \), based on the \( H \)-eigenvectors of \( T \), are homothetic for Euclidean unit vectors.

### 2.3. Algebraic considerations

The eigenvalues defined by (2.1) and (2.2) can be characterized in terms of homothety of linear forms, as follows:

(a) Consider the mappings \( \delta_*, T_* : T_0^1(V) \to T_1^0(V) \), given by

\[
\delta_*(y) = \sum_{i \in \mathbb{T}, \bar{n}} y_i dx^i, \quad T_*(y) = \sum_{i,i_1,\ldots,i_{m-1} \in \mathbb{T}, \bar{n}} T_{i,i_1,\ldots,i_{m-1}} y_{i_1,\ldots,y_{i_{m-1}}} dx^i.
\]

Then \( \lambda \in \mathbb{R} \) is a \( Z \)-eigenvalue and \( y \in S^{n-1} \subset V = \mathbb{R}^n \) is an associated \( Z \)-eigenvector iff

\[ T_*(y) = \lambda \cdot \delta_*(y), \]

i.e., the two defined by \( y \) Riesz linear forms attached to \( T \) and \( \delta \) are homothetic with factor \( \lambda \).

(b) The extended Riemann–Finsler metric \( F_{RF} \) from (2.3) provides the associated mapping

\[
C_* : T_0^1(V) \to T_1^0(V), \quad C_*(y) = \sum_{i \in \mathbb{T}, \bar{n}} (y_i)^{m-1} dx^i.
\]

Then \( \lambda \in K = \mathbb{R} \) is an \( H \)-eigenvalue of \( T \) with associated \( H \)-eigenvector \( y \) iff

\[ T_*(y) = \lambda C_*(y), \]

i.e., the two Riesz-type linear forms attached to \( T \) and \( C \) defined by \( y \) are homothetic with factor \( \lambda \). Analogously, for \( K = \mathbb{C} \), the last property can be rephrased for \( E \)-spectra.

### 2.4. Relations between \( Z \), \( H \), \( E \) and \( B \)-spectra

The concept of \( B \)-eigenvalue/eigenvector embraces both the \( H \)- and \( E \)-siblings and, in the even-order case, the \( Z \)-ones. Namely, for two \( m \)-order \( n \)-dimensional symmetric tensors \( T, B \), we call \( B \)-eigenvalue and corresponding \( B \)-eigenvector the couple \((\lambda, y) \in K \times K^n \) which satisfies the conditions:

\[
\sum_{i_2,\ldots,i_{m-1}=1}^{n} (T_{ki_2,\ldots,i_{m}} - \lambda B_{ki_2,\ldots,i_{m}}) y_{i_2,\ldots, y_{i_{m}}} = 0, \quad \forall k \in \mathbb{T}, \bar{n}.
\]

Then the \( Z \)-eigenvalues are exactly the particular \( B \)-eigenvalues obtained for

\[ B_{i_1,\ldots,i_{m}} = \delta_{i_1,i_2} \cdots \delta_{i_{m-1},i_m} \]

\(^4\) We consider arbitrary \( n \geq m \geq 2 \), an extension of the metric firstly considered by Riemann, where \( m = n = 4 \).

\(^5\) The last sum is considered for all values for the indices \( i_1,\ldots,i_{m} \in \mathbb{T}, \bar{n} \) and \( \delta_{i_1,\ldots,i_{m}} \) is the \( m \)-Kronecker symbol (1 for equal indices, and 0 otherwise).

\(^6\) We set \( K = \mathbb{R} \) for the \( Z \) and \( H \) spectra, and \( K = \mathbb{C} \) for the \( E \) spectrum.
(for \(m\) even), and the \(H\)- and \(E\)-eigenvalues are the particular ones, for
\[
B_{i_1, \ldots, i_m} = \delta_{i_1, \ldots, i_m} = 1, \quad \text{for } i_1 = \cdots = i_m; \quad 0, \text{ otherwise.}
\]

3. Berwald-Moor and Chernov cases in \(\mathbb{R}^4\)

In the following we describe the \(Z, H\) and \(E\) spectral data for the Berwald-Moor and the Chernov tensors, in the 4-dimensional case. Using [29, Theorem 1, p. 1312], after tedious computations, we infer:

**Theorem 3.1** [4]. Consider the \(H_4\) Berwald-Moor symmetric tensor \(A\) given in (1.2). Then

(a) the \(Z\)-eigenvalues of \(A\) are \(\lambda \in \sigma_Z(A) = \{0, \pm \frac{1}{16}\}\), with the associated \(Z\)-eigenvectors
\[
\begin{align*}
S_{\lambda=0} &= \{(x_1, x_2, x_3, x_4) | \exists i < j; i, j \in \{1, 4\}, \exists \theta \in [0, 2\pi), \\
&\quad x_i = \cos \theta, x_j = \sin \theta, x_k = 0, \forall k \in \{1, 4\} \setminus \{i, j\}\}, \\
S_{\lambda=1/16} &= \{(x_1, x_2, x_3, x_4) | x_1, x_2, x_3, x_4 \in \{\pm 1/2\}, x_1x_2x_3x_4 > 0\}, \\
S_{\lambda=-1/16} &= \{(x_1, x_2, x_3, x_4) | x_1, x_2, x_3, x_4 \in \{\pm 1/2\}, x_1x_2x_3x_4 < 0\};
\end{align*}
\]

(b) the \(H\)-eigenvalues of \(A\) are \(\lambda \in \sigma_H(A) = \{0, \pm \frac{1}{4}\}\), with the \(H\)-eigenspaces
\[
\begin{align*}
S_{\lambda=0} &= \bigcup_{1 \leq i < j \leq 4} \text{Span}\{e_i, e_j\}, \\
S_{\lambda=1/4} &= \text{Span}\{(1, 1, 1, 1)\} \cup \text{Span}\{(1, 1, -1, -1)\&\}, \\
S_{\lambda=-1/4} &= \text{Span}\{(1, 1, 1, -1)\&\};
\end{align*}
\]

where \(\{e_k\}_{k=1}^{4}\) is the canonical basis of \(\mathbb{R}^4\);

(c) the \(E\)-eigenvalues of \(A\) are \(\lambda \in \sigma_E(A) = \sigma_H(A) \cup \{\pm \frac{1}{4}\}\), where the \(E\)-eigenspaces are formed of the previous associated real \(H\)-eigenspaces and the supplementary complex \(E\)-eigenspaces
\[
\begin{align*}
S_{\lambda=1/4} &= \text{Span}\{(\epsilon, \epsilon, 1, -1)\&\} \cup \text{Span}\{(\epsilon, -\epsilon, 1, 1)\&\}, \\
S_{\lambda=-1/4} &= \text{Span}\{(\epsilon, \epsilon, 1, 1)\&\} \cup \text{Span}\{(\epsilon, -\epsilon, 1, -1)\&\}, \\
S_{\lambda=\epsilon} &= \text{Span}\{(1, 1, 1, \epsilon)\&\} \cup \text{Span}\{(-1, 1, 1, -\epsilon)\&\} \cup \text{Span}\{(1, 1, 1, -\epsilon)\&\},
\end{align*}
\]

The \(Z\)-spectra are tightly related to the geometric features of the Finslerian scaled indicatrix of the associated \(m\)-root structure, as follows:

**Corollary 3.2.** The hyperquartic surface \(\Sigma: Ay^4 = c (c > 0)\) does not surround a bounded region in \(\mathbb{R}^4\), is nonempty and the distance \(d = \text{dist}(O, \Sigma) = (c/\lambda_{\text{max}})^{1/4}\) is determined by the maximal eigenvalue \(\lambda_{\text{max}} = 1/16\), \(d = 2\sqrt{c}\) which is attended at \(8 = 2 + (\frac{1}{2})\) points of \(\Sigma\), whose position vectors \(y_* \cdot d\) are

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7 By \(\&\) we denote the taking into consideration of all the possible symmetric permutations of the four components of the vector.

8 Here \(\epsilon \in \{\pm i\}\) and the sign remains constant along the description of each subspace.
provided by the eigenvectors associated to $\lambda_{\max}$, namely
\[ y_\ast \in \left\{ 2^{-1}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \mid \varepsilon_{1,2,3,4} \in \{\pm 1\}, \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1 \right\}. \] (3.1)

As well, for the Chernov tensor (1.6) in $\mathbb{R}^4$, the three types of spectra are given by the following result:

**Theorem 3.3.** Consider the Chernov symmetric tensor $B$ given in (1.6). Then:

(a) The Z-eigenvalues of $B$ are $\lambda \in \sigma_Z(B) = \{0, \pm 1/2, \pm 4\theta/3, (-6\varepsilon^3 + 2\varepsilon)/3\}$, where $\theta = \{\pm 1/\sqrt{21}\}$ and $\varepsilon = \{\sqrt{5} \pm 1\}/\sqrt{2}, (-\sqrt{5} \pm 1)/\sqrt{2}\}$, with the associated Z-eigenvectors
\[
\begin{align*}
S_{\lambda=0} &= \{(\pm 1, 0, 0, 0)^8\}, \\
S_{\lambda=1/2} &= \{2^{-1}(1, 1, 1, 1)\}, \\
S_{\lambda=-1/2} &= \{-2^{-1}(1, 1, 1, 1)\}, \\
S_{\lambda=\pm 4\theta/3} &= \{\theta(-2, -2, -2, 3)^8\}, \\
S_{\lambda=-\varepsilon^3+2\varepsilon} &= \{\varepsilon(1, 1, 3(2\varepsilon^2 - 1), 3(2\varepsilon^2 - 1))^8\}.
\end{align*}
\]

(b) The H-eigenvalues of $B$ are $\lambda \in \sigma_H(B) = \{0, 1, -1/3\}$, with the associated H-eigenvectors
\[
\begin{align*}
S_{\lambda=0} &= \text{Span}\{(1, 0, 0, 0)^8\}, \\
S_{\lambda=1} &= \text{Span}\{(1, 1, 1, 1)\}, \\
S_{\lambda=-1/3} &= \text{Span}\{(1, -1, -1)^8\}.
\end{align*}
\]

(c) The E-eigenvalues of $B$ are $\lambda \in \sigma_E(B) = \sigma_H(B) \cup \{-2\varepsilon, 2\theta+1, 1\}$, where $\varepsilon = \{(-3 \pm i\sqrt{15})/6\}$ and $\theta \in \{(-3 \pm i\sqrt{15})/4\}$, with the E-eigenvectors given by the corresponding H-eigenvectors and the associated supplementary complex eigenvectors
\[
\begin{align*}
S_{\lambda=-2\varepsilon/3} &= \text{Span}\{(1, \varepsilon, \varepsilon, \varepsilon)^8\}, \\
S_{\lambda=2\theta+1/3} &= \text{Span}\{(1, 1, 1, \theta)^8\}, \\
S_{\lambda=1} &= \text{Span}\{(1, \eta, \eta, -\eta^2\eta^2/2\eta^2+1)^8\} \cup \text{Span}\{(1, \eta, \eta, -\eta^2\eta^2/2\eta^2+1)^8\} \\
&\quad \cup \{(\alpha, \beta, \rho, -\rho - \alpha - \beta)^8 | \alpha, \beta \in \mathbb{R}\},
\end{align*}
\]

where
\[
\eta \in \{(-1 \pm i\sqrt{2})/3\}, \quad \nu \in \{(-1 \pm i\sqrt{2})/2\}, \quad \varepsilon \in \{(-3 \pm i\sqrt{15})/6\}, \\
\rho = [-2, -2, -2, 3(2\varepsilon^2 - 1), 3(2\varepsilon^2 - 1))^8, \quad \theta \in \{(-3 \pm i\sqrt{5})/4\}.
\]

A series of properties regarding recession vectors and rank, which were generically described in [29, Theorem 3, p. 1315], point out that in our cases the mixed action of the multilinear forms on vectors provide alternate descriptions of angles in the Berwald-Moor and Chernov 4-dimensional relativistic

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9 Here the signs in $\theta$ and $\varepsilon$ remain constant along the description of each subset.

10 Here, the choice of sign within the description of the parameters is preserved along the description of the eigensets.
Namely, we have the geometric correspondence between the pairwise actions of $T$ and the corresponding polyangle for unit vectors:

$$Tx^2y^2 = \langle x, y \rangle_T^2, \quad Tx^3y = \langle x, y \rangle_T^3.$$  

Another geometric consequence is that the associated scaled indicatrix $\Sigma$ is not a cylinder. E.g., in $H_4$, $\Sigma$ is the Tzitzeica surface given by $\Sigma : y_1y_2y_3y_4 = c$, which for $c > 0$ has 8 connected sheets. As well, the Chernov scaled indicatrix $\Sigma : \sum_{1 \leq i < j < k \leq 4} y_iy_jy_k = c (c > 0)$ is non-void and unbounded.

4. Berwald-Moor, Chernov and Bogoslovsky cases in $\mathbb{R}^3$

The 3-dimensional models determined by the Finsler metrics (1.3)–(1.9) prove to be useful to understanding the higher-dimensional related models provided by $m$-root symmetric Grobner polynomials [8, 17, 25, 26]. The 3-dimensional polyangles have been recently studied by Pavlov and Kokarev [27]. We shall further describe the spectra and corresponding eigenvectors of the Berwald-Moor (1.4), Chernov (1.8) and Bogoslovsky (1.10) structures, in the 3-dimensional case. To this aim, we have the following results:

**Theorem 4.1.** Consider the $H_3$ Berwald-Moor symmetric tensor (1.4). Then:

(a) The $Z$-eigenvalues and the eigenspace generators are $\sigma_Z = \{0, \pm 1/3\sqrt{3}\}$ and:

$$S_{\lambda=0} = \{\pm(1, 0, 0)^\mathbb{R}\},$$

$$S_{\lambda=\pm 1/3\sqrt{3}} = \{(1, -1, -1)^\mathbb{R}/\sqrt{3}\} \cup \{(1, 1, 1)/\sqrt{3}\}.$$

(b) The $H$-spectral data of the tensor are $\lambda \in \sigma_H = \{0, 1/3\}$ and

$$S_{\lambda=0} = \text{Span}\{(1, 0, 0)^\mathbb{R}\},$$

$$S_{\lambda=1/3} = \text{Span}\{(1, 1, 1)\}.$$  

(c) The $E$-eigenvalues of the 3d-BM form are $\sigma_E = \sigma_H \cup \{\omega^k/3| k = 1, 2\}$, with the $E$-eigenvectors given by the corresponding $H$-eigenvectors and the associated supplementary complex eigenvectors

$$S_{\lambda=1/3} = \text{Span}\{(1, \omega, \omega^2)^\mathbb{R}\},$$

$$S_{\lambda=\omega/3} = \text{Span}\{(1, 1, \omega)^\mathbb{R}\} \cup \text{Span}\{(1, \omega^2, \omega)^\mathbb{R}\},$$

$$S_{\lambda=\omega^2/3} = \text{Span}\{(1, 1, \omega^2)^\mathbb{R}\} \cup \text{Span}\{(1, \omega, \omega^2)^\mathbb{R}\},$$

where $\omega = \exp(i\pi/3)$.

**Theorem 4.2.** Consider the Chernov symmetric tensor in $\mathbb{R}^3$. Then

(a) The $Z$-eigenvalues are $\sigma_Z = \{-1, 2\}$ and the eigenspace generators are:

$$S_{\lambda=-1} = \left\{\left(-\frac{t \pm \nu}{2}, -\frac{t \mp \nu}{2}, t\right) \bigg| t \in D = \left[-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right], \nu = \sqrt{2 - 3t^2}\right\},$$

$$S_{\lambda=2} = \{\pm(1, 1, 1)/\sqrt{2}\}. $$
(b) The $H$-eigenvalues are $\lambda \in \sigma_H = \sigma_Z$, with the same associated $Z$- and $H$-eigenvectors. We have $\sigma_E = \sigma_H$, and the $E$-eigenvectors are the ones from item (a), supplemented by the complex eigenvectors $\lambda \in \sigma_E = \sigma_H = \sigma_Z$. We provide a brief account on the existing theory and on the Berwald-Moor and Chernov cases, for the case $\mathbb{R}$ the base index associated to the symmetric tensor. We further provide a brief account on the existing theory and on the Berwald-Moor and Chernov cases, for the case $\mathbb{R}^4$.

5. Asymptotic behavior of the Finsler indicatrix

The spectral data provide information on the asymptotic behavior of the Finsler indicatrix of the associated $m$-root Finsler structure. The main tools are the asymptotic rays, the degeneracy vectors and the base index associated to the symmetric tensor. We further provide a brief account on the existing theory and on the Berwald-Moor and Chernov cases, for the case $\mathbb{R}^4$.

5.1. Asymptotic rays and recession vectors

Consider a symmetric tensor field $T \in \mathcal{T}_m^0(\mathbb{R}^n)$ on $V = \mathbb{R}^n$.

**Definition 5.1.**

(a) We say that the semi-line $L = \{\alpha y | \alpha \geq 0 \} \subset \mathbb{R}^n$ with $||y|| = 1$ is an asymptotic ray for $T \in \mathcal{T}_m^0(\mathbb{R}^n)$ and $\Sigma : Ty = c \ (c > 0)$ if $\alpha y \notin \Sigma$, $\forall \alpha \geq 0$ and there exists a sequence of points $u^{(k)} \in \Sigma$, $k \geq 1$, such that $||u^{(k)}|| \to \infty$ and $d(u^{(k)}, L) \to 0$, for $k \to \infty$, where the Euclidean norm and distance are used.

(b) An asymptotic ray $L$ of $T$ is of degree $d$ if there exists a sequence of points as above, such that $||u^{(k)} - \text{pr}_y u^{(k)}|| = O\left(||u^{(k)}||^{-1} \cdot u^{(k)} - y\right)^d$.

It is known that any $y$-generated an asymptotic ray $L$ of $T$ satisfies $Ty = 0$; as well there exists the following description of asymptotic rays:

**Theorem 5.1 [29].** Let $||y|| = 1$ and $Ty = 0$. Then:

(a) $L$ is an asymptotic ray for $T$ if and only if $y$ does not belong to the set of recession vectors $R = \{y \in \mathbb{R}^n | \forall x \in \mathbb{R}^n, \forall \alpha \in \mathbb{R}, T(x + \alpha y)^m = Tx^m\}$. 
5.2. Degeneracy vectors and base index

We have the decomposition $S_0 = R \oplus S_{OB}$, where $S_{OB} = S_0 \cap R^\perp$ and $S_0 = \{ y \in \mathbb{R}^4 | Ty^m = 0 \}$.

**Definition 5.2.**

(a) A vector $y \in R^\perp$ is called degeneracy vector of degree $k$ of $T$ if $Ty^{m-k} = 0$, the set of such vectors being denoted with $D_k$, $k \in \mathbb{N}$, $m = 1$.

(b) If $D_{k-1} \neq \emptyset$ but $D_k = \{0\}$, then we call $d = k$ the base index. If $D_0 = \{0\}$, then we put $d = 0$.

The following result provides a general characterization of degeneracy sets:

**Theorem 5.2.** Let $T \in T_m^0(\mathbb{R}^n)$. Then

(a) The following chain holds: $\mathbb{R}^4 \supset D_0 \supset D_1 \supset D_2 \supset \cdots \supset D_{m-2} \supset \{0\}$.

(b) $\forall k \in \mathbb{N}$, $m = 2$, $(x \in D_k \Rightarrow \text{Span}(x) \subset D_k)$.

(c) If $T$ is positive or negative semi-definite, then $D_1 \neq \{0\}$.

(d) If $k + j \geq m$, then $D_k + D_j \subset D_{k+j-m}$.

Then, for the $H_4$ Berwald-Moor tensor, we have

**Corollary 5.3.** The degeneracy sets of the $H_4$ form (1.2) are

$$
D_0 = \bigcup_{i=1}^{4} \{ y | y_i = 0 \}, \quad D_1 = \{ y | Ay^3 = 0 \} = \bigcup_{i<j} \{ y | y_i = y_j = 0 \},
$$

$$
D_2 = \bigcup_{i=1 \atop j \neq i}^{4} \{ y | y_j = t \delta_{ij}, t \in \mathbb{R} \}, \quad D_3 = \{ y | Ay = 0 \} = \{0\} = R
$$

and item (a) provides the chain $\mathbb{R}^4 \supset D_0 \supset D_1 \supset D_2 \supset D_3 \supset \{0\}$.

We note that in $H_4$ we have $D_2 \neq \{0\}$, and hence the base index is $d = 3$. Moreover, in this case item (c) does not apply for $A$, and $D_2 + D_3 \subset D_4$. From geometric point of view, the vectors of a degeneracy set $D_k$ provide null Pavlov polyangles with complimentary $k$-copies of vectors, for $k \in \mathbb{N}$.

Moreover, it was shown [29] that in the $H_4$ case, the characterization points [29, Theorem 9, p. 1325,30] coincide with the points previously described in (3.1).

As well, for the Chernov form (1.6) in $R^4$, the only nontrivial degeneracy set is the ruled hypersurface $D_0 = \{ y \in \mathbb{R}^4 | \sum_{1 \leq i \leq 4, \ k \leq 4} y_i y_k = 0 \}$. The chain of degeneracy sets reduces to: $\mathbb{R}^4 \supset D_0 \supset D_1 = D_2 = D_3 = \{0\}$. Since $D_0 \neq \{0\}$, the base index is 1. Item (c) does not apply and $m = 2, D_1 + D_1 = \{0\} \subset D_0$. 


\[ \text{(b)} \text{ } L \text{ is an asymptotic ray of degree } 1 - \frac{1}{m} \text{ iff } y \text{ is not a } Z\text{-eigenvector associated to the } Z\text{-eigenvalue } \lambda = 0. \]

\[ \text{(c)} \text{ } L \text{ is an asymptotic ray of degree } 1 - \frac{k}{m} \text{ iff } y \text{ is a } Z\text{-eigenvector associated to the } Z\text{-eigenvalue } \lambda = 0 \text{ and } Ty^{m-k} \neq 0, Ty^{m-k+1} = 0. \]

Then, for the $H_4$ Berwald-Moor tensor, we have $R = \{0\}$, the asymptotic rays are characterized by $Ay^4 = y_1 y_2 y_3 y_4 = 0$, and belong to the union of the four coordinate hyperplanes of $\mathbb{R}^4$. Moreover, we have the degrees of asymptotic rays $3/4; 1/2; 1/4$ for $y \in D_0 \setminus D_1 \setminus y \in D_1 \setminus D_2$ or $y \in D_2 \setminus D_3$, respectively, according to the definitions of degeneracy sets from below. As well, for the Chernov case in $\mathbb{R}^3$, one gets $R = \{0\}$ and the asymptotic rays are the directions of the $E^4_1$ Minkowski light-cone $\sum_{i<j} y_i y_j = 0$. 


6. The best rank-one approximation

We define the best rank-one super-symmetric approximation of \( T \in T_m^0(\mathbb{R}^n) \), as the homogeneous polynomial \( y \)-dependent tensor \( \lambda y^m \equiv \lambda y \otimes \cdots \otimes y \), which is global minimizer for the distance \( ||T - \lambda y^m||_F \) for \( \lambda \in \mathbb{R} \), \( ||y||_2 = 1 \), where \( || \cdot ||_F \) is the Frobenius norm, and where \( y^m \) can be regarded as an \( m \)th order \( n \)-dimensional rank-1 tensor with components \( y_1, \ldots, y_m \). A useful result in this respect is:

**Theorem 6.1** [29, Theorem 9, p. 1235,30]. Consider \( T \in T_m^0(\mathbb{R}^n) \). Then:

(a) For \( \lambda \in \sigma_Z(T) \) and \( y \) its associated Z-eigenvector, we have \( \lambda = Ty \) and \( ||T - \lambda y^m||_F^2 = ||T||_F^2 - \lambda^2 \geq 0 \).

(b) The best rank one approximation of \( T \) is provided by considering the eigenvalue \( \lambda = \max \sigma_Z(T) \), and is given by \( \lambda y^m \).

The best rank-one approximation has several notable applications in signal processing (e.g. [22,23]). One can show that the minimization problem defined above is equivalent to the dual problem of maximizing

\[
f(y) = \sum_{i_1,\ldots,i_m=1}^{\infty} T_{i_1,\ldots,i_m} y_{i_1} \cdots y_{i_m} = \langle T, y^m \rangle, \quad \text{for} \ ||y||_2 = 1,
\]
equivalent to the maximization of the Rayleigh quotient

\[
q(y) = \langle T, y^m \rangle^2 \cdot (y, y)^{-m} = (\sum_{i_1,\ldots,i_m=1}^{\infty} T_{i_1,\ldots,i_m} y_{i_1} \cdots y_{i_m})^2 \cdot (y, y)^{-m}.
\]

In the case of the 4-dimensional Gröbner associated symmetric tensors, we infer

**Theorem 6.2.** The best rank one-approximations for the \( m \)-root Berwald-Moor and Chernov tensors in \( \mathbb{R}^4 \), are provided by the spectral data of the tensors, as follows:

(a) For the \( H_4 \) Berwald-Moor tensor (1.2), the minimizer \( \lambda = 1/16 \), is attended at the eigenvectors of \( \lambda \), by the symmetric tensors

\[
\hat{A} = (16^2 \cdot 4!)^{-1} \cdot \sum_{i_1,2,3,4 \in T_4} \varepsilon_{i_1} \varepsilon_{i_2} \varepsilon_{i_3} \varepsilon_{i_4} \ dx^{i_1} \otimes dx^{i_2} \otimes dx^{i_3} \otimes dx^{i_4},
\]

where \( \varepsilon_{1,2,3,4} \in \{ \pm 1 \}, \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 = 1 \), which provide the quartic forms

\[
\hat{A} y^4 = 16^{-2} \cdot (\varepsilon_1 y_1 + \varepsilon_2 y_2 + \varepsilon_3 y_3 + \varepsilon_4 y_4)^4.
\]

(b) For the Chernov tensor (1.6) in \( \mathbb{R}^4 \), the minimizer \( \lambda = 1/2 \), is attended at the eigenvectors of \( \lambda \), by the super-symmetric tensor

\[
\hat{A} = (32 \cdot 4!)^{-1} \cdot \sum_{i_1,2,3,4 \in T_4} dx^{i_1} \otimes dx^{i_2} \otimes dx^{i_3} \otimes dx^{i_4},
\]

which provide the quartic form \( \hat{A} y^4 = 32^{-1} \cdot (y_1 + y_2 + y_3 + y_4)^4 \).

7. Conclusions

The spectral properties of five symmetric tensors related to notable \( m \)-root geometric Finsler structures (Berwald-Moor, Chernov and Bogoslovsky) are studied. Their \( Z, H \) and \( E \) spectra and eigenspaces are determined. For the first two structures, in the 4-dimensional case, the degeneracy vectors, characterization points, rank, asymptotic rays, base index and the best rank-one approximation are investigated. The geometric relevance of the spectral properties of the multilinear forms is discussed, emphasizing the relation to the geometric properties of the Finsler associated indicatrix and to the poly-scalar product of the Berwald-Moor framework [5–7,10,12,25–27].
Acknowledgements

The present work is part of GAR 4/3.06.2010, developed under the auspices of between Romania and Republic of Belarus. The author is grateful for the useful discussions on the present subject to Liqun Qi, Yiming Long and Dmitry Pavlov.

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