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Generalizations of Grillet's theorem on maximal stable sets and maximal cliques in graphs

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Abstract

Grillet established conditions on a partially ordered set under which each maximal antichain meets each maximal chain. Berge pointed out that Grillet's theorem can be stated in terms of graphs, made a conjecture that strengthens it, and asked a related question. We exhibit a counterexample to the conjecture and answer the question; then we prove four theorems that generalize Grillet's theorem in the spirit of Berge's proposals.

1. Results

Grillet [2] proved that in every partially ordered set containing no quadruple (a, b, c, d) such that

$a < b, c < d, b$ covers c,

and the remaining three pairs of elements are incomparable,

each maximal antichain meets each maximal chain. (Throughout this paper, the adjective *maximal* is always meant with respect to set-inclusion rather than size.) Berge [1] pointed out that Grillet's theorem can be stated in terms of graphs rather than partially ordered sets: if a comparability graph has the property that every induced P_4 is contained in an induced A (see Fig. 1), then every maximal stable set meets each maximal clique. (The vertices of a *comparability graph* are the elements of a partially ordered set, with two vertices adjacent if and only if they are comparable.)

Then he went on to suggest possible generalizations of this statement. First, call a graph *beautifully ordered* if it has an acyclic orientation with no induced H_1 and no

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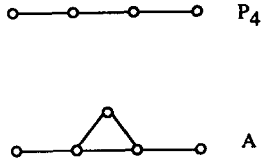


Fig. 1.

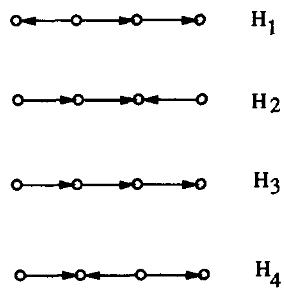


Fig. 2.

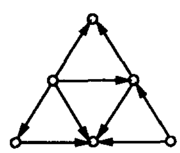


Fig. 3.

induced H_2 (see Fig. 2). Clearly every comparability graph is beautifully ordered. Berge asked:

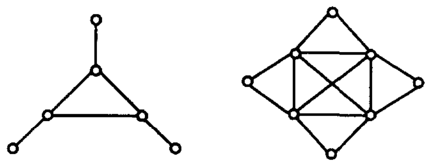
If a beautifully ordered graph has the property that every induced P_4 is contained in an induced A then does every maximal stable set meet each maximal clique?

The graph in Fig. 3 shows that the answer to the question is negative. Next, Berge made the following conjecture:

If G does not contain H_1, H_2, or H_3 as induced subdigraphs and if every induced H_4 can be embedded in an induced $\vec{A}$ (see Fig. 4) then every maximal stable set meets each maximal clique.

A counterexample to this conjecture is an orientation of the undirected graph with vertices $c_1, c_2, \ldots, c_7$ and $s_1, s_2, \ldots, s_7$ such that every two c_i's are adjacent, no two s_i's

Fig. 4.



F_1 F_2

Fig. 5.

are adjacent, and a c_i is adjacent to an s_j if and only if $i \neq j$. We direct each edge between c_i and c_j from c_i to c_j if and only if, with arithmetic modulo 7, j is one of $i + 1$, $i + 2$, $i + 4$; we direct each edge between s_i and c_j from s_i to c_j if and only if the edge between c_i and c_j is directed from c_i to c_j.

Note that no beautifully ordered graph contains a subgraph isomorphic to either of the graphs F_1 and F_2 shown in Fig. 5. Chvátal (personal communication) proposed the following conjecture as a variation on Berge's problem concerning beautifully ordered graphs:

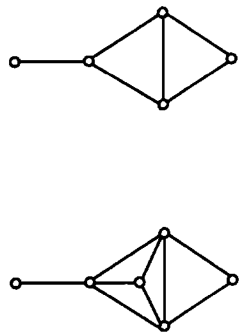
Let G be a graph with no induced subgraph isomorphic to F_1 or $\bar{F}_1$. Then each maximal stable set in G meets each maximal clique in G if and only if each P_4 in G extends into an A.

We shall prove two theorems that are weaker than Chvátal's conjecture but stronger than Grillet's theorem:

Theorem 1. *Let G be a graph with no induced subgraph isomorphic to F_1, $\bar{F}_1$, or F_2. Then each maximal stable set in G meets each maximal clique in G if and only if each P_4 in G extends into an A.*

Theorem 2. *Let G be a graph with no induced subgraph isomorphic to F_1 or $\bar{F}_1$. Then each maximal stable set in G meets each maximal clique in G if and only if each P_4 in G extends into an A and each plump P_4 in G extends into a plump A (see Fig. 6).*

In addition, we shall prove two theorems that generalize Grillet's theorem in the spirit of Berge's conjecture. The first of these theorems features the counterexample



Plump P_4

Plump A

Fig. 6.

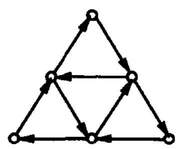


Fig. 7.

from Fig. 3; we shall refer to this directed graph as the *acyclic pyramid*; the *cyclic pyramid* featured in Theorem 4 is shown in Fig. 7.

Theorem 3. *Let G be an oriented graph with no induced acyclic pyramid. If each P_4 in G extends into an $\vec{A}$, then each maximal stable set in G meets each maximal clique in G.*

Theorem 4. *Let G be an oriented graph with no induced cyclic pyramid. If each P_4 in G extends into an $\vec{A}$, then each maximal stable set in G meets each maximal clique in G.*

Note that the hypotheses of Theorems 3 and 4 imply that each P_4 in G is oriented as H_4 in Fig. 2. *Acyclic* oriented graphs in which each P_4 is oriented as H_4 were introduced and studied by Hoàng and Reed [3] under the name of *P_4-comparability graphs.*

2. Proofs

Lemma 1. *Let H be an F_1-free graph whose set of vertices is partitioned into a stable set S and a clique C. If each vertex in C has some neighbor in S, then there must exist two vertices in S such that each vertex in C is adjacent to at least one of them.*

Proof. We proceed by induction on the number of vertices in C. Let v be a vertex in C. The induction hypothesis guarantees the existence of two vertices u_1 and u_2 in S such that each vertex in $C - v$ is adjacent to u_1 or u_2. If v is adjacent to u_1 or u_2, then we are done; otherwise, let u in S be a neighbor of v. If each vertex in C is adjacent to u_1 or u then we are done; if each vertex in C is adjacent to u_2 or u then we are done; hence we may assume that some v_2 in C is adjacent neither to u_1 nor to u and that some v_1 in C is adjacent neither to u_2 nor to u. But then u_1, u_2, u, v_1, v_2, v induce an F_1, a contradiction. □

As usual, we shall let $N(v)$ denote the set of all the neighbors of v and we shall set $N[v] = \{v\} \cup N(v)$.

Proof of Theorem 1. The "only if" part is trivial. To prove the "if" part, suppose to the contrary that a maximal stable set S shares no vertex with a maximal clique C. Let v_1, v_2 be two nonadjacent vertices outside C such that each vertex in C is adjacent to at least one of v_1, v_2 and such that, subject to this constraint, the size of $N(v_1) \cap N(v_2) \cap C$ is large as possible. (Such vertices exist by Lemma 1.)

Let I_1 (resp. I_2) denote the set of all the vertices in C which are adjacent to v_1 (resp. v_2) but nonadjacent to v_2 (resp. v_1), and let I_0 denote the set of all the vertices in C which are outside I_1 and I_2. (Both I_1 and I_2 are nonempty, for otherwise C is not a maximal clique.)

Let w be a vertex outside $N[v_1] \cup N[v_2]$ such that w has neighbors in each of I_1 and I_2 and such that, subject to this constraint, the size of $N(w) \cap (I_1 \cup I_2)$ is as large as possible. (Such a vertex exists, since for each choice of u_1 in I_1 and u_2 in I_2, there is an induced A with vertices v_1, u_1, u_2, v_2, w.) We shall distinguish between two cases.

Case 1: $I_1 \subseteq N(w)$ or $I_2 \subseteq N(w)$. Switching I_1 and I_2 if necessary, we may assume that $I_1 \subseteq N(w)$. Observe that each vertex in I_0 is adjacent to w (otherwise this vertex, v_1, v_2, w, and u_1 in I_1, and any u_2 in $N(w) \cap I_2$ would induce an $\bar{F}_1$, a contradiction). Hence each vertex in C is adjacent to at least one of w, v_2 and $|N(w) \cap N(v_2) \cap C| > |N(v_1) \cap N(v_2) \cap C|$, contradicting our choice of v_1, v_2.

Case 2: $I_1 \nsubseteq N(w)$ and $I_2 \nsubseteq N(w)$. Let a_1 be a vertex in $I_1 - N(w)$ and let a_2 be a vertex in $I_2 - N(w)$. By assumption, the path $v_1 a_1 a_2 v_2$ extends into an induced A; let v denote the fifth vertex of this A. Then vertex v is outside $C \cup \{v_1, v_2, w\}$. Now, v and w are nonadjacent (otherwise v_1, v_2, a_1, a_2, v, w would induce an F_1, a contradiction). Now we shall distinguish between two subcases.

Subcase 2.1: $N(w) \cap I_1 \subseteq N(v)$ or $N(w) \cap I_2 \subseteq N(v)$. Switching I_1 and I_2 if necessary, we may assume that $N(w) \cap I_1 \subseteq N(v)$. Now we must have $N(w) \cap I_2 \subseteq N(v)$ (otherwise v_2, a_2, v, w along with any u_1 in $N(w) \cap I_1$ and any u_2 in $(N(w) \cap I_2) - N(v)$ would induce an $\bar{F}_1$ in G, a contradiction). Since $a_1 \in (N(v) \cap I_1) - N(w)$ (and $a_2 \in (N(v) \cap I_2) - N(w)$), vertex v contradicts our choice of w.

Subcase 2.2: $N(w) \cap I_1 \nsubseteq N(v)$ and $N(w) \cap I_2 \nsubseteq N(v)$. In this subcase, v_1, v_2, a_1, a_2, v, w along any nonneighbor of v in $N(w) \cap I_1$ and any nonneighbor of v in $N(w) \cap I_2$ induce a F_2 in G, a contradiction. □

Proof of Theorem 2. The "only if" part is trivial. To prove the "if" part, we proceed as in the proof of Theorem 1 until we arrive at Case 2. Let u_2 be a vertex in $N(w) \cap I_2$. By virtue of the hypothesis on the plump P_4 induced by v_1, v_2, a_1, u_2, a_2, we get a vertex v such that $v_1, v_2, a_1, u_2, a_2, v$ induce a plump A in G. As in Case 2 of Theorem 1, v and w are nonadjacent; furthermore, each vertex in $N(w) \cap I_1$ must be adjacent to v (otherwise this vertex along with v_1, a_1, u_2, v, w would induce an F_1). The rest is the same as Subcase 2.1 of Theorem 1. □

Theorems 3 and 4 concern oriented graphs. In their proofs, we still let $N(v)$ stand for the set of all the neighbors of v (i.e. the set of all the in-neighbors and all the out-neighbors). Again, $N[v] = \{v\} \cup N(v)$.

Proof of Theorem 3. The argument follows the lines of the proof of Theorem 1. For each choice of u_i in I_i, the path $v_1u_1u_2v_2$ alternates; hence (switching subscripts if necessary) we may assume that each arc between v_1 and I_1 is directed towards I_1 and that each arc between v_2 and I_2 is directed towards v_2. The choice of w is a little different: w is a vertex outside $N[v_1] \cup N[v_2]$ such that there is at least one arc directed from w to I_1, there is at least one arc directed from I_2 to w, and such that, subject to these constraints, the size of $N(w) \cap (I_1 \cup I_2)$ is as large as possible. (Such a vertex exists, since for each choice of u_1 in I_1 and u_2 in I_2, there is an induced $\vec{A}$ with vertices v_1, u_1, u_2, v_2, w.) The subsequent case analysis goes as in the proof of Theorem 1 with minor modifications: (i) the fact that G is F_1-free and F_2-free follows from the assumption that every P_4 in G alternates, (ii) if I_1 and I_2 are switched in Subcase 2.1 then the directions of all arcs are reversed, (iii) the pyramid in Case 1 can be found by choosing u_1 (resp. u_2) in I_1 (resp. $N(w) \cap I_2$) so that the arc between u_1 (resp. u_2) and w is directed towards u_1 (resp. w); the pyramid found in Subcase 2.1 can be forced to be acyclic by the fact that both the arc between v_2 and u_2 and the arc between v_2 and a_2 are directed towards v_2. □

Lemma 2. *Let G be an oriented graph with no induced cyclic pyramid; assume that each P_4 in G extends into an $\vec{A}$. Let C be a clique in G and let s_1, s_2 be two nonadjacent vertices outside C such that, with $C_1 = N(s_1) \cap C$ and $C_2 = N(s_2) \cap C$:*

(a) *C_1 and C_2 are nonempty, disjoint, and their union is C;*
(b) *all arcs between s_1 and C_1 are directed from s_1 to C_1;*
(c) *all arcs between s_2 and C_2 are directed from C_2 to s_2.*

Then there exists a vertex w outside $N[s_1] \cup N[s_2]$ such that

(a) *w is adjacent to all the vertices in C;*
(b) *at least one arc is directed from w to C_1;*
(c) *at least one arc is directed from C_2 to w.*

Proof. We apply induction on the number of vertices in C. If $C_1 = \{u_1\}$ and $C_2 = \{u_2\}$ then the fifth vertex of any $\vec{A}$ that contains the path $s_1u_1u_2s_2$ can play the role of w. So we proceed to the induction step and assume, without loss of generality, that C_1 includes at least two vertices.

Let $u_1, u_2, \ldots, u_m$ be the vertices of C_1. Then, for each u_i, the induction hypothesis guarantees the existence of vertex w_i outside $N[s_1] \cup N[s_2]$ such that the conclusion of Lemma 2 holds with $C - u_i$ in place of C and with $C_1 - u_i$ in place of C_1. We may assume that w_i and u_i are nonadjacent for all i (else we are done with $w = w_i$). Now the mapping that assigns w_i to u_i is one-to-one. Note that no w_i is adjacent to another w_j (else at least one of the paths $w_j w_i u_j s_1$ and $w_i w_j u_i s_1$ would not alternate, a contradiction). It follows that C_1 contains no cyclic triangle (else the cyclic triangle, say $u_i u_j u_k u_i$, would extend by w_i, w_j, w_k to an induced cyclic pyramid, a contradiction), and so C_1 is a transitive tournament. Without loss of generality, suppose that $u_1, u_2, \ldots, u_m$ are enumerated in such an order that each edge $u_i u_j$ with $i < j$ is directed from u_i to u_j. Since each path $w_m u_i u_m w_i$ with $i < m$ alternates, each arc between $C_1 - u_m$ and w_m is directed towards w_m, a contradiction. □

Lemma 3. *Let G be an oriented graph with no induced cyclic pyramid; assumed that each P_4 in G extends into an $\vec{A}$. Let C be a clique in G and let s_1, s_2 be two nonadjacent vertices outside C such that, with $C_1 = N(s_1) \cap C$, $C_2 = N(s_2) \cap C$, and $I_1 = C_1 - C_2$, $I_2 = C_2 - C_1$:*

(a) *$I_1 \neq \emptyset$, $I_2 \neq \emptyset$, and $C_1 \cup C_2 = C$.*
(b) *all arcs between s_1 and I_1 are directed from s_1 to I_1;*
(c) *all arcs between s_2 and I_2 are directed from I_2 to s_2.*

Then some vertex w outside C satisfies at least one of the following two conditions:

(1) *w is adjacent to all the vertices of C and nonadjacent to s_1; furthermore, at least one arc is directed from I_2 to w.*
(2) *w is adjacent to all the vertices of C and nonadjacent to s_2; furthermore, at least one arc is directed from I_1 to w.*

Proof. We shall use induction on the number of vertices in I_0, the set of all the vertices in C which are outside I_1 and I_2. Since (1) follows directly from Lemma 2 when I_0 is empty, we proceed to the induction step.

Let $c_1, c_2, \ldots, c_s$ be the vertices in I_0. For each c_i, the induction hypothesis guarantees the existence of a w_i which satisfies (1) or (2) in place of w (with $C - c_i$ in place of C). Note that $w_i \notin I_0$, and so $w_i \neq c_i$ (and so $w_i \notin C$). We may assume that w_i and c_i are nonadjacent for all i (else we are done with $w = w_i$).

Fact 1. *Each arc between s_1 and I_0 is directed from s_1 to I_0.*

Proof. Suppose to the contrary that there is an arc directed from some c_i to s_1. Then s_1 and w_i must be nonadjacent (otherwise s_2 and w_i would be nonadjacent and, for any vertex v in I_2, at least one of paths $s_1 w_i v s_2$ and $w_i s_1 c_i s_2$ would fail to alternate, a contradiction).

If w_i satisfies (1) in place of w (and with $C - c_i$ in place of C), then there is an arc directed from some v in I_2 to w_i and path $s_1c_ivw_i$ does not alternate, a contradiction. Thus w_i must satisfy (2) in place of w (and with $C - c_i$ in place of C); in particular, there is an arc directed from some u in I_1 to w_i. But then s_1, s_2, u, c_i, w_i along with an arbitrary vertex in I_2 induce a cyclic pyramid, a contradiction. □

Proof of Lemma (*continued*). For each i, write $w_i \in W_1$ if w_i is nonadjacent to s_1 and $w_i \in W_2$ otherwise; set $D_1 = \{c_j \in I_0 \mid w_j \in W_1\}$ and $D_2 = \{c_j \in I_0 \mid w_j \in W_2\}$; note that each vertex in W_2 is nonadjacent to s_2.

Fact 2. *Each arc between s_2 and D_2 is directed from s_2 to D_2.*

Proof. Directly from Fact 1 and the fact that all paths $s_2c_is_1w_i$ with $c_i \in D_2$ alternate. □

Fact 3. *If $W_k \neq \emptyset$ (and $k = 1$ or $k = 2$) then there is some w_i in W_k such that each arc between w_i and D_k is directed towards D_k.*

Proof. Note that no w_i in W_k is adjacent to another w_j in W_k (else at least one of the paths $w_jw_ic_js_1$ and $w_iw_jc_is_1$ would not alternate, a contradiction). It follows that D_k contains no cyclic triangle (else the cyclic triangle, say $c_ic_jc_lc_i$, would extend by w_i, w_j, w_l to an induced cyclic pyramid, a contradiction), and so D_k is a transitive tournament. Without loss of generality, suppose that the elements of D_k are $c_1, c_2, \ldots, c_r$ and that each edge c_ic_j with $i < j$ is directed from c_i to c_j. Since each path $w_1c_ic_1w_i$ with $i > 1$ alternates, each arc between D_k and w_1 is directed towards D_k. □

Fact 4. *There is some w_i in $W_1 \cup W_2$ such that each arc between w_i and I_0 is directed towards I_0.*

Proof. Fact 3 allows us to assume that $W_1 \neq \emptyset$ and $W_2 \neq \emptyset$. By Fact 3 (with $s = 1$), there is a vertex w_i in W_1 such that each arc between w_i and D_1 is directed from w_i to D_1. Note that w_i is not adjacent to s_2 (else, for any u in I_1, at least one of the paths $s_1uw_is_2$ and $s_1c_is_2w_i$ would not alternate, a contradiction). For each vertex c_j in D_2, the arc between w_i and c_j is directed towards c_j: consider path $w_ic_js_1w_j$ if w_i and w_j are nonadjacent and path $s_2c_jw_iw_j$ if w_i and w_j are adjacent. □

Proof of Lemma 3 (*conclusion*). With w_i as in Fact 4, we shall distinguish between the following two cases:

Case 1: w_i satisfies (1) in place of w (and with $C - c_i$ in place of C). Note that w_i is not adjacent to s_2 (else, for any u in I_1, at least one of the paths $s_1uw_is_2$ and $s_1c_is_2w_i$ would not alternate, a contradiction). Furthermore, for each v in I_2, the arc between w_i and v is directed towards w_i: consider the path $s_1c_ivw_i$.

Subcase 1.1.: The arc between s_2 and c_i is directed towards c_i. Note that, for each u in I_1, the arc between u and w_i is directed towards w_i (consider the path $s_2c_iuw_i$). By Lemma 2 with s_2 in place of s_1, w_i in place of s_2, c_i in place of C_1, and I_1 in place of C_2, we find a vertex w outside $N[s_2] \cup N[w_i]$ such that w is adjacent to all the vertices in $\{c_i\} \cup I_1$; the arc between w and c_i is directed towards c_i; for some u in I_1, the arc between u and w is directed towards w. For each c_j in $I_0 - c_i$, the arc between c_j and w_i is directed towards c_j (by our choice of w_i), and so c_j must be adjacent to w (else $wc_ic_jw_i$ would not alterante, a contradiction); each v in I_2 must also be adjacent to w (else s_2vuw would not alternate, a contradiction). Hence w satisfies (2).

Subcase 1.2: The arc between s_2 and c_i is directed towards s_2. Note that, for each u in I_1, the arc between u and w_i is directed towards u (consider the path $s_2c_iuw_i$). By Lemma 2 with w_i in place of s_2, c_i in place of C_1, and I_2 in place of C_2, we find a vertex w outside $N[s_1] \cup N[w_i]$ such that w is adjacent to all the vertices in $\{c_i\} \cup I_2$; p the arc between w and c_i is directed towards c_i; for some v in I_2, the arc between v and w is directed towards w. For each x in $I_1 \cup (I_0 - \{c_i\})$, the arc between x and w_i is directed towards x, and so x must be adjacent to w (else wc_ixw_i would not alternate, a contradiction). Hence w satisfies (1).

Case 2: w_i satisfies (2) in place of w (and with $C - c_i$ in place of C). If w_i is not adjacent to s_1 then the condition of Case 1 is satisfied (since each path $s_1c_ivw_i$ with $v \in I_2$ alternates); hence may assume that w_i is adjacent to s_1. Since $w_is_1c_is_2$ alternates, the arc between s_2 and c_i is directed towards c_i. The remainder of the argument follows the lines of Subcase 1.1. □

Proof of Theorem 4. Suppose to the contrary that a maximal stable set S shares no vertices with a maximal clique C. Since each P_4 in G alternates, G is F_1-free; hence Lemma 1 guarantees the existence of two vertices, say s_1 and s_2, in S such that each vertex in C is adjacent to at least one of s_1 and s_2. Let I_1 (resp. I_2) denote the set of all the vertices in C which are adjacent to s_1 (resp. s_2) but nonadjacent to s_2 (resp. s_1), and let I_0 denote the set of all the vertices in C which are outside I_1 and I_2. (Both I_1 and I_2 are nonempty, for otherwise C is not a maximal clique.) Since each P_4 in G alternates, we may assume (switching s_1 and s_2 if necessary) that each arc between s_1 and I_1 is directed towards I_1, and each arc between s_2 and I_2 is directed towards s_2. By Lemma 3, C is not a maximal clique, a contradiction. □

3. Complexity

Let us call a graph *grillet* if it has the property that each of its maximal stable sets meets each of its maximal cliques. A natural question is this: how difficult is it to recognize graphs which are *not* grillet? Obviously, this problem is in NP; we are inclined to believe that it is NP-complete. Our Theorem 2 implies that this problem can be solved in polynomial time for graphs which contain no subgraph isomorphic to F_1 or $\bar{F}_1$.

If G happens to be not grillet then this fact cannot be certified by exhibiting a "forbidden" induced subgraph of G: every G is an induced subgraph of a grillet graph. To see this, let $C_1, C_2, \ldots, C_k$ be all the maximal cliques of G, add to G pairwise nonadjacent vertices $v_1, v_2, \ldots, v_k$, and connect v_i to all the vertices in C_i for each $1 \leqslant i \leqslant k$.

The related problem of recognizing pairs (G, S) such that G is a graph and S is a maximal stable set in G disjoint at least one maximal clique of G is NP-complete: we shall reduce the satisfiability problem into this problem. Given a boolean formula as a conjunction of clauses $C_1, C_2, \ldots, C_k$, consider the graph G whose vertex-set consists of pairwise disjoint stable sets $S_1, S_2, \ldots, S_k$ and S. Vertices in each S_i are labeled by the literals that occur in C_i; two vertices in distinct S_i's are nonadjacent if and only if they are labeled by x and $\bar{x}$ for some x; vertices of S are $v_1, v_2, \ldots, v_k$ and each v_i is adjacent to all the vertices in all S_j such that $j \neq i$. It is easy to see that S is disjoint from at least one maximal clique of G if and only if the formula is satisfiable.

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