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A nice labelling for tree-like event structures of degree 3 ☆

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ABSTRACT

We address the problem of finding nice labellings for event structures of degree 3. We develop a minimum theory by which we prove that the index of an event structure of degree 3 is bounded by a linear function of the height. The main theorem of the paper states that event structures of degree 3 whose causality order is a tree have a nice labelling with 3 colors. We exemplify how to use this theorem to construct upper bounds for the index of other event structures of degree 3.

1. Introduction

Event structures, introduced in [17], are nowadays a widely recognized model of true concurrent computation and have found many uses since then. They are an intermediate abstract model that makes it possible to relate more concrete models such as Petri Nets or higher dimensional automata [26]. They provide formal semantics of process calculi [23,25]. More recently, logicians became interested in event structures with the aim of constructing models of proof systems that are invariant under the equalities induced by the cut elimination procedure [8,13].

Our interest for event structures stems from the fact that they combine distinct approaches to the modeling of concurrent computation. On one side, language theorists have developed the theory of partially commutative monoids [3] as the basic language to approach concurrency. Classes of automata that properly model concurrent processes – such as asynchronous automata [27] or concurrent automata [5] – have been studied as part of this theory. On the other hand, the framework of domain theory and, ultimately, order theoretic ideas have often been proposed as the proper tools to handle concurrency, see for example [18]. In this paper we pursue a combinatorial problem that lies at the intersection of these two approaches. It is the problem of finding nice labellings for event structures of fixed degree. To our knowledge, this problem has not been investigated any longer since it was posed in [19,20] and partially solved in [1].

Let us recall that an event structure is made up of a set of local events E partially ordered by the causality relation $\leqslant$. Causally independent events may also be in the conflict relation $\smile$. A global state of the computation, comprehensive of its history, is modeled as a subset of events, lower closed with respect to the causality relation, which also is an independent set with respect to the conflict relation. These global states may be organized into a poset, the domain of an event structure, representing all the concurrent non-deterministic executions. The Hasse diagram of this poset codes the state-transition graph of the event structure as an abstract process. By labeling the transitions of this graph with letters from some alphabet, we can enrich the graph with the structure of a deterministic concurrent automaton. The nice labelling problem asks to find a labelling that uses an alphabet of minimum size. The size of this alphabet is called the *index* of the event structure.

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The problem is actually equivalent to a graph coloring problem in that we can associate to an event structure a graph, of which we are asked to compute the chromatic number. The *degree* of an event structure is the maximum out-degree of a node in the Hasse diagram of the associated domain, that is, the maximum number of upper covers of some element. Under the graph theoretic translation of the problem, the degree coincides with the clique number, and therefore it is a lower bound for the cardinality of a solution. A main contribution in [1] was to prove that event structures of degree 2 have index 2, i.e. they posses a nice labelling with 2 letters. On the other hand, it was proved there that event structures of higher degrees may require strictly more letters than the degree.

The labelling problem can be seen as a generalization of the problem of covering a poset by disjoint chains. Dilworth's Theorem [4] states that the minimal cardinality of such a cover equals the maximal cardinality of an antichain. This theorem and the results of [1] constitute the only knowledge on the problem presently available to us. For example, we cannot state that there is some fixed $k > n$ for which every event structure of degree n has a nice labelling with at most k letters. In light of standard graph theoretic results [14], the above statement should not be taken for granted.

We present here some first results on the nice labelling problem for event structures of degree 3. We develop a basic theory that shows that the graph of a degree 3 event structure, when restricted to an antichain, is almost acyclic and can be colored with 3 letters. This observation allows to construct an upper bound to the labelling number of such event structure as a linear function of its height. We focus then on event structures whose causality order is a tree or a forest. Intuitively, these tree-like event structures represent concurrent systems where processes are only allowed to fork or to take local non-deterministic choices. Our main theorem states that tree-like event structures of degree 3 have a nice labelling with 3 letters. Finally, we suggest how to use this and other theorems to construct upper bounds for the index of other event structures of degree 3. These general upper bounds depend on some parameter. To exemplify the scope of theory, we prove a constant upper bound on a simple class of degree 3 event structures.

The general question we address, whether there exists a finite common upper bound to the indexes of event structures of degree 3, remains open. Conscious that this question might be difficult to answer in its full generality – as usual for graph coloring problems – we felt worth to present these partial results and to encourage other researchers to pursue this and other combinatorial problems that arise from concurrency. Let us mention why these problems deserve an in-depth investigation. The theory of event structures is presently being applied to automated verification of systems. Some model checkers – see for example [22] and [16] – make explicit use of trace theory and of the theory of partially ordered sets to represent the state space of a concurrent system. The combinatorics of posets is then exploited to achieve an efficient exploration of the global states of concurrent systems [7,12,15]. Thus, having a solid theoretical understanding of such combinatorics is a prerequisite and a complement for designing efficient algorithms for these kind of tools.

The paper is structured as follows. After recalling the order theoretic concepts we shall use, we introduce event structures and the nice labelling problem in Section 2. In Section 3 we develop the first properties of event structures of degree 3. As a result, we devise an upper bound for the labelling number of such event structures as a linear function of the height. In Section 4 we present our main result stating that event structures whose underlying order is a tree may be labeled with 3 colors. In Section 5 we develop a general approach to construct upper bounds to the labelling number of event structures of degree 3. Using this approach and the results of the previous section, we compute a constant upper bound for a class of degree 3 event structures that have some simplifying properties, that consequently we call simple.

1.1. Order theoretic preliminaries

Let us anticipate that part of an event structure is a set E of events which is partially ordered by the causality relation $\leqslant$. As in this paper we shall heavily rely on order theoretic concepts, we introduce them here together with the notation we shall use. All these concepts will apply to the poset $\langle E, \leqslant\rangle$ of an event structure.

A finite poset is a pair $\langle P, \leqslant\rangle$ where P is a finite set and $\leqslant$ is a reflexive, transitive and antisymmetric relation on P. A subset $X \subseteq P$ is a *lower set* if $y \leqslant x \in X$ implies $y \in X$. If $Y \subseteq P$, then we denote by $\downarrow Y$ the least lower set containing Y. Explicitly, $\downarrow Y = \{\, x \in P \mid \exists y \in Y \text{ s.t. } x \leqslant y \,\}$. Two elements $x, y \in P$ are *comparable* if and only if either $x \leqslant y$ or $y \leqslant x$. We write $x \simeq y$ to mean that x, y are comparable. Also, we write $x < y$ if $x \leqslant y$ but $x \neq y$. A *chain* is sequence $x_0, \ldots, x_n$ of elements of P such that $x_0 < x_1 < \cdots < x_n$. The integer n is the length of the chain. The *height* of an element $x \in P$, noted $\mathrm{h}(x)$, is the length of the longest chain in $\downarrow\{\, x \,\}$. The height of P is $\max\{\, \mathrm{h}(x) \mid x \in P \,\}$. An *antichain* is a subset $X \subseteq P$ such that $x \not\simeq y$ for each pair of distinct $x, y \in X$. The *width* of $\langle P, \leqslant\rangle$, noted $\mathrm{w}(P, \leqslant)$, is the integer $\max\{\, \mathrm{card}(A) \mid A \text{ is an antichain} \,\}$. If the interval $\{\, z \in P \mid x \leqslant z \leqslant y \,\}$ is the two elements set $\{\, x, y \,\}$, then we say that x is a *lower cover* of y or that y is *an upper cover* of x. We denote this relation by $x \prec y$. The Hasse diagram of $\langle P, \leqslant\rangle$ is the directed graph $\langle P, \prec\rangle$. For $x \in P$, the *outdegree* of x, noted $\delta^+(x)$, is the number of upper covers of x. That is, $\delta^+(x)$ is the outdegree of x in the Hasse diagram. The outdegree of $\langle P, \leqslant\rangle$, noted $\delta^+(P, \leqslant)$, is the integer $\max\{\, \delta^+(x) \mid x \in P \,\}$. We shall denote by $\delta^-(x)$ the number of lower covers of x (i.e. the indegree of x in the Hasse diagram). The poset $\langle P, \leqslant\rangle$ is *graded* if $x \prec y$ implies $\mathrm{h}(y) = \mathrm{h}(x) + 1$.

2. Event structures and the nice labelling problem

Event structures are a basic model of concurrency introduced in [17]. The definition we present here is from [26].

Definition 2.1. An *event structure* is a triple $\mathcal{E} = \langle E, \leqslant, \mathcal{C}\rangle$ such that

(1) $\langle E, \leqslant\rangle$ is a poset, such that for each $x \in E$ the lower set $\downarrow\{x\}$ is finite,
(2) $\mathcal{C}$ is a collection of subsets of E such that:
- $\{x\} \in \mathcal{C}$ for each $x \in E$,
- $X \subseteq Y \in \mathcal{C}$ implies $X \in \mathcal{C}$,
- $X \in \mathcal{C}$ implies $\downarrow X \in \mathcal{C}$.

In this paper we shall consider finite event structures only, so that that $\downarrow\{x\}$ is always finite. The order $\leqslant$ of an event structure $\mathcal{E}$ is known as the *causality* relation between events. The collection $\mathcal{C}$ is known as the set of configurations of $\mathcal{E}$. A configuration $X \in \mathcal{C}$ of causally unrelated events – that is, an antichain with respect to $\leqslant$ – is a sort of snapshot of the global state of some distributed computation. A snapshot X may be transformed into a description of the computation that takes into account its history. This is done by adding to X the events that causally have determined events in X. That is, the history-aware description is the lower set $\downarrow X$ generated by X. We shall be particularly interested in the collection of history-aware configurations, defined as

$$\mathcal{H} = \{ Y \in \mathcal{C} \mid \downarrow Y = Y \}.$$

Observe that $X \in \mathcal{C}$ if and only if $\downarrow X \in \mathcal{H}$ so that $\mathcal{C}$ is determined from $\mathcal{H}$. Consequently, we do not lose information if we focus on the collection of history-aware configurations.

Two events $x, y \in E$ are said to be *concurrent* if $x \not\simeq y$ and there exists $X \in \mathcal{C}$ such that $x, y \in X$. We shall write $x \frown y$ to mean that x, y are concurrent. It is useful to introduce a weakened version of the concurrency relation where we allow events to be comparable: $x \mathrel{\overset{\frown}{\simeq}} y$ if and only if $x \frown y$ or $x \simeq y$. Equivalently, $x \mathrel{\overset{\frown}{\simeq}} y$ if and only if there exists $X \in \mathcal{C}$ such that $x, y \in X$. The set of configurations that arise from many concrete models is completely determined by the concurrency relation. For example, this is the case for event structures that code the behavior of 1-safe Petri Nets.

Definition 2.2. An event structure $\mathcal{E}$ is *coherent* if $\mathcal{C}$ is the set of cliques of the weak concurrency relation: $X \in \mathcal{C}$ if and only if $x \mathrel{\overset{\frown}{\simeq}} y$ for every pair of events $x, y \in X$.

Coherent event structures are also known as *event structures with binary conflict*. To understand this name, let us explicitely introduce the conflict relation and two other derived relations:

(1) *Conflict*: $x \smile y$ if and only if $x \not\simeq y$ and $x \not\frown y$.
(2) *Minimal conflict*: $x \mathrel{\underline{\smile}} y$ if and only (i) $x \smile y$, (ii) $x' < x$ implies $x' \mathrel{\overset{\frown}{\simeq}} y$ and (iii) $y' < y$ implies $x \mathrel{\overset{\frown}{\simeq}} y'$.
(3) *Orthogonality*: $x \mathrel{\overset{\frown}{\smile}} y$ if and only if $x \mathrel{\underline{\smile}} y$ or $x \frown y$.

A coherent event structure is completely described by the triple $\langle E, \leqslant, \smile\rangle$ where the conflict relation is symmetric and irreflexive, and moreover is such that $x \smile z$ whenever $x \smile y$ and $y \leqslant z$.

The concurrency relation, being the restriction to uncomparable elements of the complement of the conflict relation, satisfies the following conditions:

$$x \frown y \text{ implies } x \not\simeq y, \tag{C1}$$
$$x \frown y \text{ and } z \leqslant x \text{ implies } z \frown y \text{ or } z \leqslant y. \tag{C2}$$

We deal in this paper mainly with coherent event structures and, unless explicitly stated, event structure will be a synonym for coherent event structure. Yet, the combinatorial problems we analyze make sense for the more general notion of event structure and are worth investigating in this setting. In our opinion, the general notion is more appropriate for modelling concurrency, as the constraint on coherency is not often realized.

Coloring the graph of an event structure

The orthogonality relation clearly is symmetric. Thus, by identifying an ordered pair (x, y) such that $x \mathrel{\overset{\frown}{\smile}} y$ with the unordered pair $\{x, y\}$, we shall focus on the undirected graph induced by the orthogonality [1] relation. This graph, formally defined by

$$\mathcal{G}(\mathcal{E}) = \langle E, \mathrel{\overset{\frown}{\smile}}\rangle,$$

1 Let us observe that two orthogonal events are called independent in [1]. An independent set in the complement undirected graph $\langle V, E^c\rangle$ is a clique of the graph $\langle V, E\rangle$, thus explaining terminology used in [1]. In this paper we shall focus on the structural properties of the graph $\mathcal{G}(\mathcal{E}) = \langle E, \mathrel{\overset{\frown}{\smile}}\rangle$ and not of its complement, and therefore we prefer to deviate from the existing terminology.

will be called the graph of $\mathcal{E}$. Let us list some properties of the orthogonality relation:

$$x \mathrel{\overline{\frown}} y \text{ if and only if} \tag{O1}$$
$$\text{(i) } x \not\succeq y, \text{ (ii) } x' < x \text{ implies } x' \mathrel{\overset{\frown}{\underline{\smile}}} y, \text{ (iii) } y' < y \text{ implies } x \mathrel{\overset{\frown}{\underline{\smile}}} y',$$
$$\text{if } x \mathrel{\overline{\frown}} y \text{ and } z \leqslant x \text{ then } z \mathrel{\overline{\frown}} y \text{ or } z \leqslant y. \tag{O2}$$

Together with the following property:

$$\text{if } x \smile y \text{ then there exists } x' \leqslant x \text{ and } y' \leqslant y \text{ such that } x' \mathrel{\underline{\smile}} y', \tag{O3}$$

which ties up the conflict relation with the orthogonality through the minimal conflict, these properties shall be our main working tool. We leave the proof of them as an exercise for the reader.

Definition 2.3. A *nice labelling* of an event structure $\mathcal{E}$ is a coloring of the graph $\mathcal{G}(\mathcal{E})$. That is, it is a pair (λ, Σ) with Σ a finite alphabet and $\lambda : E \longrightarrow \Sigma$ such that $\lambda(x) \neq \lambda(y)$ whenever $x \mathrel{\overline{\frown}} y$.

For a graph G, let $\chi(G)$ denote its chromatic number and let $\omega(G)$ be its clique number, i.e. the size of the largest clique of G.

Definition 2.4. The *degree* of $\mathcal{E}$, $\omega(\mathcal{E})$, is the clique number of $\mathcal{G}(\mathcal{E})$, i.e. the number $\omega(\mathcal{G}(\mathcal{E}))$. The *index* of $\mathcal{E}$, $\chi(\mathcal{E})$, is the chromatic number of $\mathcal{G}(\mathcal{E})$, i.e. the number $\chi(\mathcal{G}(\mathcal{E}))$.

The ***nice labelling problem*** asks to compute $\chi(\mathcal{E})$ for a given event structure $\mathcal{E}$. It was shown to be an NP-complete problem in [1]. The graph theoretic definition of the index and of the degree makes it clear that $\omega(\mathcal{E}) \leqslant \chi(\mathcal{E})$. More generally, given a class $\mathcal{K}$ of event structures, the ***nice labelling problem for the class*** $\mathcal{K}$ asks to compute the index of $\mathcal{K}$, defined as

$$\chi(\mathcal{K}) = \max\{\, \chi(\mathcal{E}) \mid \mathcal{E} \in \mathcal{K} \,\}.$$

Of course, $\chi(\mathcal{K})$ might not be a finite number. A necessary condition for the relation $\chi(\mathcal{K}) < \infty$ to hold is the existence of a finite upper bound on the size of cliques of the graphs $\mathcal{G}(\mathcal{E})$ with $\mathcal{E} \in \mathcal{K}$. Thus, of particular interest are the classes of event structures $\mathcal{K}_n$ defined by

$$\mathcal{K}_n = \{\, \mathcal{E} \mid \omega(\mathcal{E}) \leqslant n \,\}.$$

It is time to recall the known results on the nice labelling problem for classes of event structures. The first one is the celebrated Dilworth's theorem.

Theorem 2.5 (Dilworth [4])**.** *If the conflict relation of* $\mathcal{E}$ *is empty, then* $\chi(\mathcal{E}) = \omega(\mathcal{E})$.

As a matter of fact, if the conflict relation is empty, then $x \mathrel{\overline{\frown}} y$ if and only if x, y are not comparable, so that nice labellings of $\mathcal{E}$ are in bijection with coverings of the poset $\langle E, \leqslant \rangle$ by disjoint chains. Notice next that the conflict relation of $\mathcal{E}$ is empty if and only if there is no pair of events $x, y \in E$ such that $x \mathrel{\underline{\smile}} y$, i.e. that are in minimal conflict. Dilworth's Theorem, as a statement about event structures with a limited number of minimal conflicts, has the following generalization:

Theorem 2.6 (Assous et al. [1])**.** *If*

$$\mathcal{K}_{n,m} = \{\, \mathcal{E} \in \mathcal{K}_n \mid \mathrm{card}\{\, (x, y) \mid x \mathrel{\underline{\smile}} y \,\} \leqslant m \,\},$$

then $\chi(\mathcal{K}_{n,m}) < \infty$.

Dilworth's theorem, as a particular case of the previous theorem, states that $\mathcal{K}_{n,0} = n$. The next result, dealing with event structures of degree 2, has been our motivating starting point.

Theorem 2.7 (Assous et al. [1])**.** $\chi(\mathcal{K}_2) = 2$ *and* $\omega(\mathcal{K}_n) > n$ *for* $n > 2$.

2.1. Computational interpretation of the nice labelling problem

The rest of this section is meant to clarify the role of the orthogonality relation and of the graph $\mathcal{G}(\mathcal{E})$. The computational interpretation we shall give is part of the folklore in concurrency theory, see for example [2], but it is worth recalling. Let us first review the definition of the domain of an event structure.

Definition 2.8. The domain $\mathcal{D}(\mathcal{E})$ of an event structure $\mathcal{E} = \langle E, \leqslant, \mathcal{C}\rangle$ is the poset $\langle \mathcal{H}, \subseteq \rangle$, where $\mathcal{H}$ is the collection of history-aware configurations of $\mathcal{E}$.

Following a standard axiomatization in theoretical computer science $\mathcal{D}(\mathcal{E})$ is a *stable L-domain*, see [6,26]. This property roughly means that $\mathcal{D}(\mathcal{E})$ almost is a *distributive* lattice. Let us stress this point, as most of the following considerations are elementary observations of the theory of distributive lattices.

The collection $\mathcal{H}$ being closed under binary intersections, the poset $\mathcal{D}(\mathcal{E})$ is a finite meet semilattice – or a chopped lattice as defined in [10, Chapter 4]. It is distributive in the following sense[2]: the equation $z \wedge (x \vee y) = (z \wedge x) \vee (z \vee y)$ is satisfied whenever $x \vee y$, the least upper bound of $\{ x, y \}$, exists. The following Lemma asserts that finite distributive meet semilattices are essentially the same structures as the domains of (possibly not coherent) event structures.

Lemma 2.9. *Every finite distributive meet semilattice is isomorphic to the domain of an event structure.*

Proof. Since the ideas on which the proof relies are well known, we only sketch it. Let L be a finite distributive meet semilattice, say that $x \in L$ is prime if it has a unique lower cover and denote by $J(L)$ the set of prime elements of L. As usual from lattice theory, argue that $x \leqslant z \vee y$ implies $x \leqslant z$ or $x \leqslant y$ whenever $x \in J(L)$ and the least upper bound $z \vee y$ exists. For $X \subseteq J(L)$ say that $X \in \mathcal{C}$ if the least upper bound of X exists in L. Let then $\mathcal{E} = \langle J(L), \leqslant, \mathcal{C}\rangle$, it is a standard exercise to prove that $\mathcal{D}(\mathcal{E})$ is order isomorphic to L. $\square$

A lower set in $\mathcal{H}$ represents a state of the global computation, comprehensive of its history. For $I, J \in \mathcal{H}$, $I \subseteq J$ intuitively means that the global state J may take place after the global state I. The Hasse diagram of $\mathcal{D}(\mathcal{E})$ therefore represents the state-transition graph of $\mathcal{E}$ as a process. We obtain a representation of the process $\mathcal{E}$ as an automaton if we color the edges of the Hasse diagram by letters (or colors) from some alphabet. It is quite natural, however, to ask this coloring to satisfy the following conditions.

Determinism: Transitions outgoing from the same state have different colors.

Concurrency: Every square of the diagram has to be colored according to the following pattern, suggesting that actions σ, τ may take place in parallel:

$$\begin{tikzcd} & J_0 \cup J_1 & \\ J_0 \arrow[ur, dash, "\sigma"] & & J_1 \arrow[ul, dash, "\tau"'] \\ & I \arrow[ul, dash, "\tau"] \arrow[ur, dash, "\sigma"'] & \end{tikzcd}$$

$$\tilde{\lambda}(I \prec J_1) = \sigma = \tilde{\lambda}(J_0 \prec J_0 \cup J_1). \tag{1}$$

Let us analyze what it means for an edge-coloring to be concurrent. Consider that if $I \prec J$ is an edge of the Hasse diagram of $\mathcal{D}(\mathcal{E})$, then $J = I \cup \{ x \}$ for some $x \in E \backslash I$ such that $y \in E$ whenever $y < x$. Thus, if we start from a labelling $\lambda : E \longrightarrow \Sigma$ and define $\tilde{\lambda}(I \prec I \cup \{ x \}) = \lambda(x)$, then condition (1) is fulfilled:

$$\begin{aligned} \tilde{\lambda}(I \prec J_0) &= \tilde{\lambda}(I \prec I \cup \{ x_0 \}) = \lambda(x_0) \\ &= \tilde{\lambda}(I \cup \{ x_1 \} \prec I \cup \{ x_0, x_1 \}) = \tilde{\lambda}(J_1 \prec J_0 \cup J_1). \end{aligned}$$

In order to see that every concurrent edge-coloring arise in this way, observe that by translating down colors along opposite side of concurrent squares as in (1), a concurrent edge-coloring is determined by the ideals in $\mathcal{D}(\mathcal{E})$ with a unique lower covers; these are of the form $\downarrow\{ x \}$. Thus we have observed.

Lemma 2.10. *There is a bijection between concurrent edge-colorings of the Hasse diagram of $\mathcal{D}(\mathcal{E})$ and functions $\lambda : E \longrightarrow \Sigma$.*

We analyze next how the condition on determism of a concurrent edge-coloring transfers to a function $\lambda : E \longrightarrow \Sigma$. The following is the key Lemma to understand the role of the orthogonality relation.

Lemma 2.11. *A set $\{ x_1, \ldots, x_n \}$ is a clique of $\mathcal{G}(\mathcal{E})$ if and only if there exists an history-aware configuration I such that $I \prec I \cup \{ x_i \}$, $i = 1, \ldots, n$, are distinct edges of the Hasse diagram of $\mathcal{D}(\mathcal{E})$.*

Proof. Suppose that $I \cup \{ x_i \}$ and $I \cup \{ x_j \}$ are distinct upper covers of some I in $\mathcal{D}(\mathcal{E})$. Then $\{x_i, x_j\}$ is an antichain since $x_i \leqslant x_j$ implies that $I \cup \{ x_i \} \subseteq I \cup \{ x_j \}$. If $x' < x_i$ then $x' \in I \subseteq I \cup \{ x_j \}$. Since $I \cup \{ x_j \}$ is a clique for the weak concurrency

[2] Usually, a meet semilattice is said to be distributive if its filter completion is a distributive lattice, see for example [24].

relation, then $x' \frown x_j$. Similarly $y' < x_j$ implies $x_i \frown y'$ and therefore $x_i \asymp x_j$, by (O1). In particular, distinct upper covers of some I give rise to a clique in $\mathcal{G}(\mathcal{E})$.

Conversely, let us suppose that $x_i \asymp x_j$ whenever $i \neq j$ and observe that, by (O1) and (C2), $x_i \asymp x_j$ implies $x' \frown y'$ for $x' < x$ and $y' < y$. Thus, if we let $I = \bigcup_{i=1}^{n} \{ x' \mid x' < x_i \}$, then $I \in \mathcal{D}(\mathcal{E})$ and $I \cup \{ x_i \} \in \mathcal{D}(\mathcal{E})$ as well, for $i = 1, \ldots, n$. If $i \neq j$, then x_i, x_j are not comparable and therefore $I \cup \{ x_i \}$ and $I \cup \{ x_j \}$ are distinct upper covers of I. □

Let us remark that the Lemma strongly depends on $\mathcal{E}$ being a coherent event structure. The Lemma also implies that $\omega(\mathcal{E})$, the degree of $\mathcal{E}$ or the maximum size of a clique in $\mathcal{G}(\mathcal{E})$, coincides with $\delta^{+}(\mathcal{D}(\mathcal{E}))$, the outdegree of $\mathcal{D}(\mathcal{E})$ or the maximum outdegree of some configuration in the Hasse diagram of $\mathcal{D}(\mathcal{E})$, introduced on p. 653. Considering the case $n = 2$ in the statement of Lemma 2.11, we deduce the following Proposition:

Proposition 2.12. *There is a bijection between concurrent deterministic edge-colorings of the Hasse diagram of $\mathcal{D}(\mathcal{E})$ and colorings the graph of $\mathcal{G}(\mathcal{E})$.*

Consequently, the size of a minimal alphabet by which we can transform the Hasse diagram of $\mathcal{D}(\mathcal{E})$ into a deterministic concurrent automaton coincides with the chromatic number of $\mathcal{G}(\mathcal{E})$, what we called the index of $\mathcal{E}$.

3. Cycles and antichains

From now on, in this and the following sections, $\mathcal{E} = \langle E, \leqslant, \mathcal{C} \rangle$ will be a fixed coherent event structure of degree at most 3. We begin our investigation of the nice labelling problem for $\mathcal{E}$ by studying the restriction to an antichain of the graph $\mathcal{G}(\mathcal{E})$. The main tool we shall use is the following Lemma, a straightforward generalization of [1, Lemma 2.2] to degree 3. In [21] we proposed generalizations of this Lemma to higher degrees and pointed out the geometrical flavor of the resulting statements.

Lemma 3.1. *Let $\{ x_0, x_1, x_2 \}, \{ x_1, x_2, x_3 \}$ be two size 3 cliques in the graph $\mathcal{G}(\mathcal{E})$ sharing the same face $\{ x_1, x_2 \}$. Then x_0, x_3 are comparable.*

Proof. Let us suppose that x_0, x_3 are not comparable. It is not possible that $x_0 \asymp x_3$, since then we have a size 4 clique in the graph $\mathcal{G}(\mathcal{E})$. Thus $x_0 \smile x_3$ and, by (O3), we can find $x_0' \leqslant x_0$ and $x_3' \leqslant x_3$ such that $x_0' \simeq x_3'$. We claim that $\{ x_0', x_1, x_2, x_3' \}$ is a size 4 clique in $\mathcal{G}(\mathcal{E})$, thus reaching a contradiction. If $x_0' \not\asymp x_1$, then $x_0' \leqslant x_1$, by (O2). However, $x_0' \leqslant x_1 \asymp x_3$ implies $x_0' \frown x_3$, and hence $x_0' \frown x_3'$. The latter relation contradicts $x_0' \simeq x_3'$. Similarly, $x_0' \asymp x_2, x_3' \asymp x_1, x_3' \asymp x_2$. □

We are going to improve the previous lemma. To this goal, let us say that a sequence $x_0 x_1 \ldots x_{n-1} x_n$ is a *straight cycle* if $x_n = x_0$, $x_i \asymp x_{i+1}$ for $i = 0, \ldots, n-1$, $x_i \not\asymp x_j$ whenever $i, j \in \{ 0, \ldots, n-1 \}$ and $i \neq j$. As usual, the integer n is the length of the cycle. Observe that a straight cycle is simple, i.e., a part from the endpoints of the cycle, it does not visit twice the same vertex. The height of a straight cycle $C = x_0 x_1 \ldots x_n$ is the integer

$$\mathrm{h}^{+}(C) = \sum_{i=0,\ldots,n-1} \mathrm{h}^{+}(x_i),$$

where $\mathrm{h}^{+}(x) = \mathrm{h}(x) + 1$ is the augmented height of an event. By assigning to each element of the cycle a non-zero weight, we can ensure that if C' is another straight cycle visiting a proper subset of the vertexes visited by C, then $\mathrm{h}^{+}(C') < \mathrm{h}^{+}(C)$. This will apply for example when C' is obtained from C as a shortcut through a chord.

Proposition 3.2. *The graph $\mathcal{G}(\mathcal{E})$ does not contain a straight cycle of length strictly greater than* 3.

Proof. Let SC_4 be the collection of straight cycles in $\mathcal{G}(\mathcal{E})$ whose length is at least 4. We shall show that if $C \in \mathrm{SC}_4$, then there exists $C' \in \mathrm{SC}_4$ such that $\mathrm{h}^{+}(C') < \mathrm{h}^{+}(C)$. If $\mathrm{SC}_4 \neq \emptyset$, then we construct an infinite descending chain of positive integers.

Let C be the straight cycle $x_0 \asymp x_1 \asymp x_2 \ldots x_{n-1} \asymp x_n = x_0$ where $n \geqslant 4$. Let us suppose that this cycle has a chord. Such a chord cut the cycle into two cycles of length $m_0 + 1$ and $m_1 + 1$, with $m_0 + m_1 = n$. By Lemma 3.1, we cannot have $m_0 = m_1 = 2$, and therefore $m_i \geqslant 3$ for some $i \in \{ 0, 1 \}$. That is, the chord divides the cycle into two straight cycles, one of which still has length at least 4. Moreover its height is strictly less than the height of C, since it contains a smaller number of vertices.

Otherwise C has no chord and $x_0 \not\asymp x_2$. By (O1), this means that either there exists $x_0' < x_0$ such that $x_0' \not\frown x_2$, or there exists $x_2' < x_2$ such that $x_0 \not\frown x_2'$. By symmetry, we can assume the first case holds. As in the proof of Lemma 3.1 $\{ x_0', x_1, x_2, x_3 \}$ form an antichain, and $x_0' x_1 x_2 x_3$ is a path. Let C' be the set $\{ x_0' x_1, \ldots x_{n-1} x_0' \}$. If C' is an antichain, then C' is a straight cycle such that $\mathrm{h}^{+}(C') < \mathrm{h}^{+}(C)$. Otherwise the set $\{ j \in \{ 4, \ldots, n-1 \} \mid x_j \geqslant x_0' \}$ is not empty; let i be the minimum in this set. Observe that $x_{i-1} \asymp x_i$ and $x_0' \leqslant x_i$ but $x_0' \not\leqslant x_{i-1}$ implies $x_{i-1} \asymp x_0'$, by (O2). Thus $\tilde{C} = x_0' x_1 x_2 x_3 \ldots x_{i-1} x_0'$ is a straight cycle of length at least 4 such that $\mathrm{h}^{+}(\tilde{C}) < \mathrm{h}^{+}(C)$. □

Corollary 3.3. *Any subgraph of $\mathcal{G}(\mathcal{E})$ induced by an antichain can be colored with 3 colors.*

Proof. Since the only cycles have length at most 3, such an induced graph is chordal and its clique number is 3. It is well known that the chromatic number of chordal graphs equals their clique number [9]. $\square$

In the rest of this section we exploit the previous observations to construct upper bounds for the index of $\mathcal{E}$. We remark that these upper bounds might appear either too abstract or trivial. On the other hand, they well illustrate the obstacles that might arise when trying to build complex event structures of index greater than 4.

A *stratifying function* for $\mathcal{E}$ is a function $h : E \longrightarrow \mathbb{N}$ such that, for each $n \geqslant 0$, the set $\{ x \in E \mid h(x) = n \}$ is an antichain. The height function is a stratifying function. Also $\varsigma(x) = \mathrm{card}\{ y \in E \mid y < x \}$ is a stratifying function. With respect to a stratifying function h the h-skewness of $\mathcal{E}$ is defined by

$$\mathrm{skew}_h(\mathcal{E}) = \max\{ |h(x) - h(y)| \mid x \frown y \}.$$

More generally, the skewness of $\mathcal{E}$ is defined by

$$\mathrm{skew}(\mathcal{E}) = \min\{ \mathrm{skew}_h(\mathcal{E}) \mid h \text{ is a stratifying function} \}.$$

Proposition 3.4. *If* $\mathrm{skew}(\mathcal{E}) < n$ *then* $\chi(\mathcal{G}(\mathcal{E})) \leqslant 3n$.

Proof. Let h be a stratifying function such that $|h(x) - h(y)| < n$ whenever $x \frown y$. For each $k \geqslant 0$, let $\lambda_k : \{ x \in E \mid h(x) = k \} \longrightarrow \{ a, b, c \}$ be a coloring of the graph induced by $\{ x \in E \mid h(x) = k \}$ – the existence of such a coloring being a consequence of Corollary 3.3. Define $\lambda : E \longrightarrow \{ a, b, c \} \times \{ 0, \ldots, n-1 \}$ as follows:

$$\lambda(x) = (\lambda_{h(x)}(x), h(x) \bmod n).$$

Let us suppose that $x \frown y$ and $h(x) \geqslant h(y)$, so that $0 \leqslant h(x) - h(y) < n$. If $h(x) = h(y)$, then by construction $\lambda_{h(x)}(x) = \lambda_{h(y)}(x) \neq \lambda_{h(y)}(y)$. Otherwise $h(x) > h(y)$ and $0 \leqslant h(x) - h(y) < n$ implies $h(x) \bmod n \neq h(y) \bmod n$. In both cases we obtain $\lambda(x) \neq \lambda(y)$. $\square$

An immediate consequence of Proposition 3.4 is the following upper bound for the index of $\mathcal{E}$:

$$\chi(\mathcal{E}) \leqslant 3(\mathrm{h}(\mathcal{E}) + 1).$$

To appreciate the upper bound, consider that another approximation to the index of $\mathcal{E}$ is provided by Dilworth's theorem [4], stating that $\gamma(\mathcal{G}(\mathcal{E})) \leqslant \mathrm{w}(\mathcal{E})$. To compare the two bounds, consider that there exist event structures of degree 3 whose width is an exponential function of the height.

Finally, we observe that in order to obtain a constant upper bound on some class of event structures, we can simply define the class

$$\mathcal{K}_n^k = \left\{ \mathcal{E} \in \mathcal{K}_n \,\middle|\, \max_{x,y \in E} |\mathrm{h}(x) - \mathrm{h}(y)| < k \right\},$$

so that Proposition 3.4 ensures that $\chi(\mathcal{K}_3^k) \leqslant 3k$. For $n > 3$, it can still be shown that $\chi(\mathcal{K}_n^k) < \infty$, even if the upper bounds available are not so tight as for $n = 3$. Let us observe that the condition

$$\max_{x,y \in E} |\mathrm{h}(x) - \mathrm{h}(y)| < k$$

appears to be quite natural for an event structure $\mathcal{E}$. An interpretation of this condition in terms of concurrent processes appears in the work [11] whose main purpose is to study the logical theories of infinite regular event structures.

4. An optimal nice labelling for trees and forests

We prove in this section the main contributions of this paper, Theorems 4.12 and 4.14. Assuming $\langle E, \leqslant \rangle$ is a tree or a forest, we shall define a labelling with 3 colors and prove it is a nice labelling. Since clearly we can construct a tree which needs at least three colors, such a labelling is optimal.

Before defining the labelling, we shall develop some observations about events having the same lower covers. These observations hold under the assumption that the degree of $\mathcal{E}$ is at most 3.

Definition 4.1. We say that two distinct events are *brothers* if they have the same set of lower covers.

Clearly if x, y are brothers, then $z < x$ if and only if $z < y$. More important, if x, y are brothers, then the relation $x \frown y$ holds. As a matter of fact, if $x' < x$ then $x' < y$, hence $x' \simeq y$. Similarly, if $y' < y$ then $y' \simeq x$. Thus (O1) implies $x \frown y$. It follows that a set of events having the same lower covers form a clique in $\mathcal{G}(\mathcal{E})$, hence it has at most the degree of an event structure, 3 in the present case. To introduce the next Lemmas, if $x \in E$, define

$$\mathtt{F}_x = \{\, z \in E \mid z \mathrel{\underline{\frown}} x \text{ and } y \leqslant z, \text{ for some brother } y \text{ of } x \,\},$$
$$\mathtt{S}_x = \{\, z \in E \mid z \mathrel{\underline{\frown}} x \text{ and } y \not\leqslant z, \text{ for every brother } y \text{ of } x \,\}.$$

That is, we are splitting the neighborhood of x into its $\mathtt{F}$*amily*, those events that are descendant of a brother of x, and its $\mathtt{S}$*ociety*, those events that are related to x but have no immediate connection with x. Intuitively, events in the family of x are at least as old as x, and this will limit our interest in the family. We shall instead engage in studying properties of the societies.

Lemma 4.2. *If x has two brothers, then $\mathtt{S}_x = \emptyset$.*

Proof. Let y, z be the two brothers of x. Let us suppose that $w \in \mathtt{S}_x$. If $w \mathrel{\underline{\frown}} y$, then $w \simeq z$ by Lemma 3.1. Since $z \not\leqslant w$, then $w < z$. However this implies $w < x$, contradicting $w \mathrel{\underline{\frown}} x$. Hence $w \not\mathrel{\underline{\frown}} y$ and we use (O3) to find $w' \leqslant w, y' \leqslant y$ such that $w' \mathrel{\underline{\smile}} y'$. It cannot be the case that $y' < y$: otherwise $y' < x$ and $x \mathrel{\underline{\frown}} w$ implies $y' \mathrel{\overset{\frown}{\simeq}} w$, by (O1); then $y' \mathrel{\overset{\frown}{\simeq}} w$ and $w \geqslant w'$ implies $y' \mathrel{\overset{\frown}{\simeq}} w'$, see (C2), contradicting $w' \mathrel{\underline{\smile}} y'$. We have therefore $w' < w, y' = y$ and $w' \mathrel{\underline{\smile}} y$. We claim that $w' \in \mathtt{S}_x$. As a matter of fact, w' cannot be above any of x, y, z, otherwise w would have the same property. Noticing that $w \mathrel{\underline{\frown}} x$ and $w' < w$, we use (O2) to deduce that $w' \mathrel{\underline{\frown}} x$ or $w' \leqslant x$. If $w' \leqslant x$, then $w' < x$, so that $w' < y$, contradicting $w' \mathrel{\underline{\smile}} y$: therefore $w' \mathrel{\underline{\frown}} x$ and $w' \in \mathtt{S}_x$. Observe now that $\{\, w', x, y \,\}, \{\, x, y, z \,\}$ are two 3-cliques sharing the same face $\{\, x, y \,\}$. As before, $w' \simeq z$, leading to a contradiction. □

Lemma 4.3. *If y is the only brother of x, then $\mathtt{S}_x, \mathtt{S}_y$ are comparable with respect to subset inclusion and the least of them, $\mathtt{S}_x \cap \mathtt{S}_y$, is linearly ordered by the causality relation.*

Proof. We observe first that if $z \in \mathtt{S}_x$ and $w \in \mathtt{S}_y$ then $z \simeq w$. An immediate consequence of this observation is that $\mathtt{S}_x \cap \mathtt{S}_y$ is linearly ordered.

Let us suppose that there exists $z \in \mathtt{S}_x$ and $w \in \mathtt{S}_y$ such that $z \not\simeq w$. Note that $\{\, z, x, y, w \,\}$ is an antichain: $y \not\leqslant z$, and $z < y$ implies $z < x$, which is not the case due to $z \mathrel{\underline{\frown}} x$. Thus $z \not\simeq y$ and, similarly, $w \not\simeq x$.

Since $z \mathrel{\underline{\frown}} x \mathrel{\underline{\frown}} y \mathrel{\underline{\frown}} w$ and there cannot be a length 4 straight cycle, we deduce $z \not\mathrel{\underline{\frown}} w$. We have therefore $z \not\frown w$ and, by the hypothesis, $z \not\simeq w$, and these two facts imply $z \smile w$. By (O3), there exists z' and w' such that $z' \leqslant z, w' \leqslant w$, and $z' \mathrel{\underline{\smile}} w'$. We show next that $z' \in \mathtt{S}_x$. We prove first that $z' \mathrel{\underline{\frown}} x$. If this is not the case, then (O2) together with $z' \leqslant z$ and $z \mathrel{\underline{\frown}} x$, implies $z' \leqslant x$; since $z' = x$ implies $x \leqslant z$ contradicting $z \mathrel{\underline{\frown}} x$, we deduce $z' < x$. As x, y are brothers, the relation $z' < x$ implies $z' < y$. Then, $z' < y, y \mathrel{\underline{\frown}} w$, and $w \geqslant w'$ implies $z' \frown w'$, which contradicts $z' \mathrel{\underline{\smile}} w'$. We have therefore $z' \mathrel{\underline{\frown}} x$. A second remark is that it is not the case that $y \leqslant z'$, since then $y \leqslant z$ and $z \not\in \mathtt{S}_x$. Thus, we have argued that $z' \in \mathtt{S}_x$ and, similarly, we argue that $w' \in \mathtt{S}_y$. As before $\{\, z', x, y, w' \,\}$ is an antichain, hence z', x, y, w' also form a length 4 straight cycle, a contradiction.

Our next observation is that $w \leqslant z \in \mathtt{S}_x$ and $w \not\leqslant x$ implies $w \in \mathtt{S}_x$. As usual, we use (O2) to deduce $w \mathrel{\underline{\frown}} x$ or $w \leqslant x$ from $w \leqslant z \mathrel{\underline{\frown}} x$; since $w \not\leqslant x$, then we have $w \mathrel{\underline{\frown}} x$. Also, if $y \leqslant w$ then $y \leqslant z$, which is not the case.

Our final observation is that $\mathtt{S}_x \supseteq \mathtt{S}_y$ whenever $\mathtt{S}_x \setminus \mathtt{S}_y \neq \emptyset$. Let $z \in \mathtt{S}_x \setminus \mathtt{S}_y$, pick any $w \in \mathtt{S}_y$ and recall that z, w are comparable. We cannot have $z \leqslant w$: considering that $z \not\leqslant x$, we deduce that $z \not\leqslant y$ as well; then $z \leqslant w \in \mathtt{S}_y$ and $z \not\leqslant y$ imply $z \in \mathtt{S}_y$, a contradiction. Hence $w < z \in \mathtt{S}_x$ and $w \not\leqslant x$ imply $w \in \mathtt{S}_x$, by our previous observation. □

The previous lemmas have the following interpretation. If x, y are two brothers, say that x *is more experienced than* y if $\mathtt{S}_x \supseteq \mathtt{S}_y$. Then the Lemmas state that we can always pick one of the brother who's more experienced than the other. Remark that the property is trivial if x, y, z are pairwise brothers, since in this case $\mathtt{S}_w = \emptyset$, $w \in \{\, x, y, z \,\}$. The property becomes interesting whenever y is the only brother of x, for which we formally introduce this relation.

Definition 4.4. We say that $\{x, y\} \subseteq E$ is a *proper pair of brothers* if y is the only brother of x.

The next lemma is an easy consequence of the previous Lemmas. While its significance might appear obscure right now, the Lemma will prove to be the key observation when later defining a nice labelling.

Lemma 4.5. *Let $x, y, z, w \in E$ be four events such that:*

(1) $\{x, y\}$ $\{z, w\}$ *are two proper pairs of brothers,*
(2) $w \not\leqslant x$,
(3) $z \in \mathtt{S}_x \cap \mathtt{S}_y$.

Then z is strictly more experienced than w, that is $\mathtt{S}_z \supset \mathtt{S}_w$.

Proof. If $\mathtt{S}_z \not\supset \mathtt{S}_w$, then $\mathtt{S}_z \subseteq \mathtt{S}_w$ by Lemma 4.3. If $w \leqslant y$, then either $w = y$ or $w < y$. We cannot have $w = y$, since we are assuming that w, y are distinct. We cannot either have $w < y$, since otherwise $w < x$, contradicting $w \not\leqslant x$. Hence we have $w \not\leqslant y$ and $x, y \in \mathtt{S}_z \subseteq \mathtt{S}_w$. It follows that $\{\, x, y, z, w \,\}$ is a size 4 clique, a contradiction. □

We come now to introduce trees, that are the particular subsets of E for which we shall define a nice labelling with 3 colors. The presence in trees of many brothers possibly is the intuitive reason for a nice labelling with 3 colors to exist.

Definition 4.6. A subset $T \subseteq E$ is a *tree* if and only if

(1) each $x \in T$ has exactly one lower cover $\pi(x) \in E$,
(2) T is convex: $x, z \in T$ and $x < y < z$ implies $y \in T$,
(3) if x, y are minimal in T, then $\pi(x) = \pi(y)$.

The *height* of x in T, noted $\mathrm{h}_T(x)$, is the cardinality of the set $\{ y \in T \mid y < x \}$. Observe that two events x, y of a tree are brothers if and only if $\pi(x) = \pi(y)$.

In this context, for a linear ordering we shall mean a transitive *irreflexive* relation $\lhd$ which, moreover, is total: $x \lhd y$ or $y \lhd x$. A linear ordering $\lhd$ on a tree T is said to be compatible with the height if it satisfies

$$\mathrm{h}_T(x) < \mathrm{h}_T(y) \text{ implies } x \lhd y. \tag{HEIGHT}$$

It is a standard result that such a linear ordering always exists. Once fixed such a linear ordering, we shall think of it as imposing a precise age on events of T; that is, the relation $x \lhd y$ shall be read as asserting that x is older than y. Observe that the condition (HEIGHT) implies that an ancestor x of y is older than y.

With the idea of defining a labelling of T greedily by means of a fixed linear ordering $\lhd$, let us define

$$\mathtt{N}_x^{\lhd} = \{ y \in T \mid y \mathrel{\overset{\frown}{-}} x \text{ and } y \lhd x \}, \qquad x \in T.$$

That is, $\mathtt{N}_x^{\lhd}$ is the neighborhood of x within T, restricted to older events. We represent $\mathtt{N}_x^{\lhd}$ as the disjoint union of $\mathtt{B}_x^{\lhd}$ and $\mathtt{S}_x^{\lhd}$ where

$$\mathtt{B}_x^{\lhd} = \{ y \in \mathtt{N}_x^{\lhd} \mid \pi(x) < y \},$$
$$\mathtt{S}_x^{\lhd} = \mathtt{N}_x^{\lhd} \setminus \mathtt{B}_x^{\lhd}.$$

With respect to these sets $\mathtt{B}_x^{\lhd}$, $\mathtt{S}_x^{\lhd}$, $x \in T$, we develop a series of observations.

Lemma 4.7. *If $y \in \mathtt{B}_x^{\lhd}$ then y is an older brother of x. Consequently there can be at most two elements in $\mathtt{B}_x^{\lhd}$.*

Proof. If $y \in \mathtt{B}_x^{\lhd}$, then $y \lhd x$ and $\mathrm{h}_T(y) \leqslant \mathrm{h}_T(x)$. Since $\pi(x) < y$ then $\mathrm{h}_T(\pi(x)) < \mathrm{h}_T(y)$ and $\mathrm{h}_T(x) = \mathrm{h}_T(\pi(x)) + 1 \leqslant \mathrm{h}_T(y)$. We deduce therefore that $\mathrm{h}_T(x) = \mathrm{h}_T(y)$, showing that $\pi(x)$ is a lower cover of y, so that y is a brother of x. $\square$

Lemma 4.8. *$\mathtt{S}_x^{\lhd}$ is a lower set of $\mathtt{S}_x$. That is, $\mathtt{S}_x^{\lhd} \subseteq \mathtt{S}_x$ and $z' \leqslant z \in \mathtt{S}_x^{\lhd}$ implies $z' \in \mathtt{S}_x^{\lhd}$, provided that $z' \in \mathtt{S}_x$.*

Proof. If $z \in \mathtt{S}_x^{\lhd}$ then $z \mathrel{\overset{\frown}{-}} x$ and $\pi(x) \not< z$, hence $\pi(x) \not\leqslant z$. If y is a brother of x, then relation $y \leqslant z$ implies $\pi(x) = \pi(y) \leqslant y \leqslant z$ and contradicts $z \mathrel{\overset{\frown}{-}} \pi(x)$. Hence $y \not\leqslant z$ and $z \in \mathtt{S}_x$. Let us suppose that $z' < z$ and $z' \mathrel{\overset{\frown}{-}} x$. Then $\mathrm{h}_T(z') < \mathrm{h}_T(z)$, $z' \lhd z \lhd x$, and $z' \lhd x$. Since $z' \mathrel{\overset{\frown}{-}} x \geqslant \pi(x)$ then, by (O2), either $z' \mathrel{\overset{\frown}{-}} \pi(x)$, or $\pi(x) \leqslant z'$. However, the latter property implies $\pi(x) \leqslant z$, which is not the case. Therefore $z' \mathrel{\overset{\frown}{-}} \pi(x)$ and $z' \in \mathtt{S}_x^{\lhd}$. $\square$

Lemma 4.9. *If both $\mathtt{B}_x^{\lhd}$ and $\mathtt{S}_x^{\lhd}$ are not empty, then $\mathtt{B}_x^{\lhd}$ is a singleton $\{ y \}$ and $\{x, y\}$ is a proper pair of brothers.*

Proof. By the previous Lemma, $\mathtt{S}_x^{\lhd} \subseteq \mathtt{S}_x$. Hence, if $\mathtt{S}_x^{\lhd}$ is not empty, then $\mathtt{S}_x$ is not empty as well, so that by Lemma 4.2 x can have at most one brother. Since $\mathtt{B}_x^{\lhd}$ is not empty, and every element in $\mathtt{B}_x^{\lhd}$ is a brother of x, then $\mathtt{B}_x^{\lhd}$ has a unique element y, and $\{ x, y \}$ form a proper pair of brothers. $\square$

Let us remark that $x, y \in T$ are a *proper pair of brothers* if they are brothers and $\{ z \mid \pi(z) = \pi(x) \} = \{ x, y \}$. The previous observations suggest to look for a linear order $\lhd$ that enforces a strictly more experience brother to be an eldest brother.

Definition 4.10. We say that a linear order $\lhd$ on T is compatible with proper pair of brothers if it satisfies (HEIGHT) and moreover

$$\mathtt{S}_x \supset \mathtt{S}_y \text{ implies } x \lhd y, \tag{BROTHERS}$$

for each proper pair of brothers x, y.

Again, it is not difficult to see that such a linear order always exists. In the following we shall assume that $\lhd$ satisfies both (HEIGHT) and (BROTHERS).

We are ready to define a partial labelling λ of the event structure $\mathcal{E}$. The function λ will have T as its domain. Let us fix a three elements totally ordered alphabet $\Sigma = \{ a_0, a_1, a_2 \}$. The labelling $\lambda : T \longrightarrow \Sigma$ is formally defined by the clauses (1)–(4) to follow.

Before introducing the formal definition, let us set up some ideas – as well as some terminology – that might help understanding the definition of λ and the proof of Theorem 4.12. With respect to the linear order $\lhd$, we shall say that $x \in T$ is an *eldest brother* if $\mathtt{B}_x^{\lhd} = \emptyset$; otherwise, we shall say that x is a *younger brother*. The clauses (1) and (2) may be understood as stating that an eldest brother x inherits the property $\lambda(\pi(x))$ of his father $\pi(x)$. This stipulation will never create conflicts. The main concern when defining the labelling is to understand how younger brothers can enrich themselves – that is, get a property from the set Σ – without entering in conflict with members of their neighborhood. Clause (3) observes that if at the date of his birth a younger brother is related to his older brothers only, then these brothers can be at most two and it would not be a problem getting an unused color from the alphabet Σ. Clause (4) is the subtlest. If at his birth a younger brother x has some relation outside his family, then he has just one brother y who, by condition (BROTHERS), is more experienced than x. In particular, we shall see that the society of x has a main ancestor z_0 and is a main lineage of z_0, meaning that all of its members are eldest descendants of z_0. Thus, assuming that such a lineage has inherited the same color from its ancestor, we shall see that the colors used in the neighborhood of x are just 2; an unused color from Σ is therefore still available.

Definition 4.11. The labelling $\lambda : T \longrightarrow \Sigma$ is formally defined by induction on $\lhd$ by following clauses:

(1) If $x \in T$ is an eldest brother and $\mathrm{h}_T(x) = 0$, then we let $\lambda(x) = a_0$.
(2) If $x \in T$ is an eldest brother and $\mathrm{h}_T(x) \geqslant 1$, let $\pi(x)$ be its unique lower cover. Since $\pi(x) \in T$ and $\pi(x) \lhd x$, $\lambda(\pi(x))$ is defined and we let $\lambda(x) = \lambda(\pi(x))$.
(3) If x is a younger brother and $\mathtt{S}_x^{\lhd} = \emptyset$, then, by Lemma 4.7, we let $\lambda(x)$ be the least symbol not in $\lambda(\mathtt{B}_x^{\lhd})$.
(4) If x is a younger brother and $\mathtt{S}_x^{\lhd} \neq \emptyset$ then:
   - by Lemma 4.9 $\mathtt{B}_x^{\lhd} = \{ y \}$ is a singleton and $\{x, y\}$ is a proper pair of brothers,
   - by Lemma 4.8 $\mathtt{S}_x^{\lhd}$ is a lower set of $\mathtt{S}_x$. By the condition (BROTHERS), $\mathtt{S}_x \subseteq \mathtt{S}_y$, so that $\mathtt{S}_x$ is a linear order. Let therefore z_0 be the common least element of $\mathtt{S}_x^{\lhd}$ and $\mathtt{S}_x$.

   We let $\lambda(x)$ be the unique symbol not in $\lambda(\{ y, z_0 \})$.

Theorem 4.12. *For each $x, y \in T$, if $x \frown y$ then $\lambda(x) \neq \lambda(y)$.*

Proof. It suffices to prove that $\lambda(y) \neq \lambda(x)$ if $y \in \mathtt{N}_x^{\lhd}$. The statement is proved by induction on $\lhd$. Let us suppose the statement is true for all $z \lhd x$.

(i) If $\mathrm{h}_T(x) = 0$ then x is minimal in T, so that $\mathtt{N}_x^{\lhd} = \mathtt{B}_x^{\lhd}$. If moreover x is an eldest brother then $\mathtt{N}_x^{\lhd} = \mathtt{B}_x^{\lhd} = \emptyset$, so that the statement holds trivially.

(ii) If x is an eldest brother and $\mathrm{h}_T(x) \geqslant 1$, then its unique lower cover $\pi(x)$ belongs to T. We observe next that $y \frown \pi(x)$ whenever $y \in \mathtt{N}_x^{\lhd}$: we use (O2) to deduce $\pi(x) \frown y$ or $\pi(x) \leqslant y$; however, if $\pi(x) \leqslant y$, then $\mathrm{h}_T(y) \geqslant \mathrm{h}_T(x)$; considering that $y \lhd x$ and condition (HEIGHT), we deduce $\mathrm{h}_T(y) = \mathrm{h}_T(x)$, so that y is an older brother of x, a contradiction. Since $y \lhd x$ and $\pi(x) \lhd x$, and either $y \in \mathtt{N}^{\lhd}(\pi(x))$ or $\pi(x) \in \mathtt{N}_y^{\lhd}$, it follows that $\lambda(x) = \lambda(\pi(x)) \neq \lambda(y)$ from the inductive hypothesis.

(iii) If x is a younger brother and $\mathtt{S}_x^{\lhd} = \emptyset$, then $\mathtt{N}_x^{\lhd} = \mathtt{B}_x^{\lhd}$ and, by construction, $\lambda(y) \neq \lambda(x)$ whenever $y \in \mathtt{N}_x^{\lhd}$.

(iv) If x is a younger brother and $\mathtt{S}_x^{\lhd} \neq \emptyset$, then let $\mathtt{B}_x^{\lhd} = \{ y \}$ and let z_0 be the common least element of $\mathtt{S}_x^{\lhd}$ and $\mathtt{S}_x$. Since by construction $\lambda(x) \neq \lambda(y)$, to prove that the statement holds for x, it is enough to pick $z \in \mathtt{S}_x^{\lhd}$ and argue that $\lambda(z) \neq \lambda(x)$. We claim that each element $z \in \mathtt{S}_x^{\lhd} \setminus \{ z_0 \}$ is an eldest brother. If the claim holds, then $\lambda(z) = \lambda(\pi(z))$, so that $\lambda(z) = \lambda(z_0)$ is inductively deduced.

Suppose therefore that there exists $z \in \mathtt{S}_x^{\lhd} \setminus \{ z_0 \}$ which is not an eldest brother and let $w \in \mathtt{B}_z^{\lhd}$. Recall first from Lemma 4.9 that $\{x, y\}$ form a proper pair of brothers. Similarly, $\{w, z\}$ form a proper pair of brothers. Otherwise, if z, w, u are pairwise distinct brothers, then either $w \leqslant x$ or $u \leqslant x$ by Lemma 4.2. In both cases, however, we obtain $z_0 < x$ – since $z_0 < z, u, w$ – which contradicts $z_0 \frown x$. Clearly, x, y, z, w are pairwise distinct as well.

Since $y \lhd x$, condition (BROTHERS) implies $\mathtt{S}_x \subseteq \mathtt{S}_y$, and hence $z \in \mathtt{S}_x \cap \mathtt{S}_y$. If $w \in \mathtt{B}_z^{\lhd}$, then we cannot have $w \leqslant x$, since again we would deduce $z_0 \leqslant x$. Thus we deduce that $w \not\leqslant x$ and we can apply Lemma 4.5 to deduce $\mathtt{S}_z \supset \mathtt{S}_w$. On the other hand, $w \lhd z$ and condition (BROTHERS) imply $\mathtt{S}_z \subseteq \mathtt{S}_w$. Thus, we have reached a contradiction by assuming $\mathtt{B}_z^{\lhd} \neq \emptyset$. It follows that z is an eldest brother. $\square$

The obvious corollary of Proposition 4.12 is that if $\mathcal{E}$ is already sort of a tree, then it has a nice labelling with 3 colors. We state this fact as the following theorem, after we have made precise the meaning of the phrase "$\mathcal{E}$ is sort of a tree."

Definition 4.13. Let us say that $\mathcal{E}$ is a *forest* if every element has at most one lower cover. Let $\mathcal{F}_3$ be the class of event structures of degree 3 that are forests.

Theorem 4.14. *The index of the class $\mathcal{F}_3$ is 3.*

As a matter of fact, let $\mathcal{E}$ be a forest, and consider the event structure $\mathcal{E}_\perp$ obtained from $\mathcal{E}$ by adding a new bottom element $\perp$. Remark that the graph $\mathcal{G}(\mathcal{E}_\perp)$ is the same graph as $\mathcal{G}(\mathcal{E})$ apart from the fact that an isolated vertex $\perp$ has been added. The set of events E is a tree within $\mathcal{E}_\perp$, hence the graph induced by E in $\mathcal{G}(\mathcal{E}_\perp)$ can be colored with three colors. But this graph is exactly $\mathcal{G}(\mathcal{E})$.

To end this section, we mention that Theorem 4.14, stating the equality between the index and the degree for forests of degree 3, does not generalize to forests in higher degrees [1].

5. More upper bounds

We present in this section some concluding remarks that are meant to suggest some promising path toward a general solution of the nice labelling problem for event structures of degree 3.

The results presented in the previous sections point out a remarkable property of event structures of degree 3: many types of subsets of events induce a subgraph of $\mathcal{G}(\mathcal{E})$ that can be colored with 3 colors. These include:

(1) *antichains*, by Corollary 3.3,
(2) *trees* by Theorem 4.12,
(3) *history-aware configurations*, since if $X \in \mathcal{H}$, then $\mathrm{w}(X) \leqslant 3$, so that such a subset can be labeled with 3 colors by Dilworth's Theorem,
(4) the *stars* of events.

The star of an event $x \in E$ is the subgraph of $\mathcal{G}(\mathcal{E})$ induced by the subset $\{ x \} \cup \{ y \in E \mid y \frown\!\!\!\!\smile x \}$. To see that a star can also be labeled with 3 colors, let

$$N_x = \{ y \in E \mid y \frown\!\!\!\!\smile x \}$$

be the neighborhood of x in $\mathcal{G}(\mathcal{E})$, and consider the structure

$$\mathcal{E}_x = \langle N_x, \leqslant_{|N_x}, \mathcal{C}_{|N_x} \rangle,$$

where $\leqslant_{|N_x}$ is the restriction of $\leqslant$ to N_x and $\mathcal{C}_{|N_x} = \{ X \cap N_x \mid X \in \mathcal{C} \}$.

Lemma 5.1. *$\mathcal{E}_x$ is a coherent event structure with the property that $\mathcal{G}(\mathcal{E}_x)$ is the subgraph of $\mathcal{G}(\mathcal{E})$ induced by N_x. Consequently $\omega(\mathcal{E}_x) < \omega(\mathcal{E})$.*

Proof. We leave the reader to verify that $\mathcal{E}_x$ is an event structure whose concurrency relation $\frown_x$ is the restriction of $\frown$ to the set N_x. Consequently $\mathcal{C}_{|N_x}$ is the set of cliques for $\frown_x$ and $\mathcal{E}_x$ is coherent. Let $y \frown\!\!\!\!\smile_x z$ be the orthogonality relation of $\mathcal{E}_x$, let us verify that, for $y, z \in N_x$, $y \frown\!\!\!\!\smile_x z$ if and only if $y \frown\!\!\!\!\smile z$.

If $y \frown\!\!\!\!\smile z$ then y, z are not comparable. If $y' \in N_x$ and $y' < y$, then either $y' \leqslant z$ or $y' \frown z$, that is $y' \frown_x z$. By symmetry, $z' < z$ with $z' \in N_x$ implies $z' \leqslant y$ or $y \frown_x z'$, thus $y \frown\!\!\!\!\smile_x z$.

Let us suppose in the other direction that $y \frown\!\!\!\!\smile_x z$. Then y, z are not comparable. If $y' < y$ and $y' \in N_x$, then $y' \frown_x z$ or $y' \leqslant z$, which implies $y' \stackrel{\frown}{\leqslant} z$. If $y' < y$ but $y' \notin N_x$, then $y' < x$. From $z \frown\!\!\!\!\smile x$ and $y' < x$ it follows that $y' \stackrel{\frown}{\leqslant} x$. Similarly, if $z' < z$ then $z' \stackrel{\frown}{\leqslant} x$ and therefore we can deduce $y \frown\!\!\!\!\smile z$.

Finally, observe that, by adding the event x, a size n clique in $\mathcal{G}(\mathcal{E}_x)$ gives rise to a size $n + 1$ clique in $\mathcal{G}(\mathcal{E})$. □

We finalize our discussion by observing that if $\omega(\mathcal{E}) = 3$, then $\omega(\mathcal{E}_x) \leqslant 2$ so that $\mathcal{E}_x$ has a labelling with 2 colors, by [1]. It follows that star of x, the $\{ x \} \cup N_x$, can be labeled with 3 colors.

We might ask whether this property can be exploited to construct nice labellings. A tentative answer comes from a standard technique in graph theory [28]. Consider a partition of the set of events $\mathcal{P} = \{ [z] \mid z \in E \}$ and define the quotient graph $\mathcal{G}(\mathcal{P}, \mathcal{E})$ as follows: its vertexes are the equivalence classes of $\mathcal{P}$ and $[x] \frown\!\!\!\!\smile [y]$ if and only if there exists $x' \in [x]$, $y' \in [y]$ such that $x' \frown\!\!\!\!\smile y'$.

Proposition 5.2. *If each equivalence class $[z] \in \mathcal{P}$ has a labelling with 3 colors and the graph $\mathcal{G}(\mathcal{P}, \mathcal{E})$ is n-colorable, then $\mathcal{E}$ has a labelling with $3n$ colors.*

Proof. For each equivalence class $[x]$ choose a coloring $\lambda_{[x]}$ of $[x]$ with an alphabet with 3 colors. Let λ_0 a coloring of the graph $\mathcal{G}(\mathcal{P}, \mathcal{E})$ and define $\lambda(x) = (\lambda_{[x]}(x), \lambda_0([x]))$. Then λ is a coloring of $\mathcal{E}$: if $x \frown\!\!\!\!\smile y$ and $[x] = [y]$, then $\lambda_{[x]}(x) = \lambda_{[y]}(x) \neq \lambda_{[y]}(y)$ and otherwise, if $[x] \neq [y]$, then $[x] \frown\!\!\!\!\smile [y]$ so that $\lambda_0([x]) \neq \lambda_0([y])$. □

The reader should remark that the technique suggested by Proposition 5.2 has already been used within Proposition 3.4.

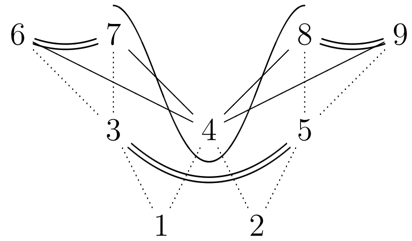

Fig. 1. The event structure $\mathcal{S}$.

We conclude our discussion by exemplifying how to use the Labelling Theorem for trees 4.12 in connection with Proposition 5.2 to construct a finite upper bound for the index of a particular class of event structures. This class shall be called simple due to the additional simplifying properties of its structures.

Consider the event structure depicted in Fig. 1, named $\mathcal{S}$. In this picture we use dotted lines for the edges of the Hasse diagram of $\langle E, \leqslant\rangle$, with lower events lying towards the bottom. We use simple lines for maximal concurrent pairs and double lines for minimal conflicts. Concurrent pairs $x \frown y$ that are not maximal, i.e. for which there exists x', y' such that $x' \asymp y'$ and either $x < x'$ or $y < y'$, are not drawn. We leave the reader to verify that a nice labelling of $\mathcal{S}$ needs at least 4 colors. On the other hand, it is not difficult to find a nice labelling with 4 colors. To obtain it, take apart events with at most 1 lower cover from the others, as suggested in the picture. Use then the results of the previous section to label with three colors the elements with at most one lower cover, and label the only element with two lower covers with a forth color.

A formalization of this intuitive method leads to the following definition and proposition.

Definition 5.3. We say that an event structure is *simple* if

(1) it is graded, i.e. $\mathrm{h}(x) = \mathrm{h}(y) - 1$ whenever $x \prec y$,
(2) every size 3 clique of $\mathcal{G}(\mathcal{E})$ contains a minimal conflict.

The event structure $\mathcal{S}$ is simple and proves that even simple event structures cannot be labeled with just 3 colors.

Proposition 5.4. *Every simple event structure $\mathcal{E}$ of degree 3 has a nice labelling with 12 colors.*

Proof. Recalling that $\delta^-(x)$ is the number of lower covers of an event x, let $E_n = \{ x \in E \mid \delta^-(x) = n \}$. Observe that a simple $\mathcal{E}$ is such that $E_3 = \emptyset$: if $x \in E_3$, then its three lower covers form a clique of concurrent events. Also, by considering the lifted event structure $\mathcal{E}_\perp$, introduced at the end of section 4, we can assume that $\mathrm{card}(E_0) = 1$, i.e. $\mathcal{E}$ has just one minimal element which necessarily is isolated in the graph $\mathcal{G}(\mathcal{E})$.

Let $\lhd$ be a linear ordering of E compatible with the height: $\mathrm{h}(x) < \mathrm{h}(y)$ implies $x \lhd y$. With respect to this linear ordering we shall use a notation analogous to the one used in the previous section. We let

$$\begin{aligned} \mathtt{N}_x^\lhd &= \{ y \in E,\ y \asymp x \mid y \lhd x \}, \\ \mathtt{B}_x &= \{ y \in E,\ y \asymp x \mid y' \prec y \ \text{ implies } \ y' \leqslant x \}. \end{aligned}$$

Next, we organize the proof around a sequence of technical Claims.

Claim 5.5. *For each $x \in E$, $\mathtt{B}_x$ has cardinality at most 2.*

Let us show that $y, z \in \mathtt{B}_x$ implies $y \asymp z$. Let $y, z \in \mathtt{B}_x$ and $y' < y$: we have then $y' < x$, so that $x \asymp z$ and (O1) implies $y' \gtrsim z$. Symmetrically $z' < z$ implies $y \gtrsim z'$, and we can deduce $y \asymp z$ using again (O1) and the fact that y, z cannot be comparable. It follows that $\{ x \} \cup \mathtt{B}_x$ is a clique of the relation $\asymp$ and, consequently, $\mathtt{B}_x$ may have at most 2 elements. □ *Claim*

While the previous claim holds only on the base that the degree of $\mathcal{E}$ is 3, the next Claim crucially uses the fact that $\mathcal{E}$ is simple.

Claim 5.6. *If $x \in E_0$ or $x \in E_2$ then $\mathtt{N}_x^\lhd \subseteq \mathtt{B}_x$.*

The Claim obviously holds if $x \in E_0$ since then, by our assumptions, $x = \perp$ is the least element of $(E, \leqslant)$, hence $\mathtt{N}_x^\lhd = \emptyset$.

Let us assume, therefore, that $x \in E_2$ and let x_1, x_2 be the two lower covers of x. Consider then $y \in \mathtt{N}_x^\lhd$. From $x_i < x \asymp y$ it follows $x_i < y$ or $x_i \frown y$, by (O1). If $x_i \frown y$ for $i = 1, 2$, then y, x_1, x_2 is a clique of concurrent events, which contradicts the fact that $\mathcal{E}$ is simple. Therefore, at least one lower cover of x is below y, let us say $x_1 < y$. It follows that $\mathrm{h}(x) = \mathrm{h}(x_1) + 1 \leqslant \mathrm{h}(y)$, and since $y \lhd x$ implies $\mathrm{h}(y) \leqslant \mathrm{h}(x)$, then x, y have the same height. We deduce that $x_1 \prec y$. Next, if y' is another lower cover of y, distinct from x_1, then y' and x_2 must be comparable, otherwise y', x_1, x_2 is a clique of concurrent events.

As y' and x_2 have the same height, it follows that $y' = x_2$. We have argued that every lower cover of y is below x, that is, $y \in \mathtt{B}_x$. $\square$ *Claim*

Claim 5.7. *The subgraph of $\mathcal{G}(\mathcal{E})$ induced by E_2 can be colored with* 3 *colors.*

If $x \in E_2$, then $\mathtt{N}_x^{\lhd} \subseteq \mathtt{B}_x$ has at most 2 elements which form a clique. This implies that the restriction of $\lhd$ to E_2 is a perfect elimination ordering, that is, a linear order with the property that, for each $x \in E_2$,

$$\{ y \in E_2, y \frown x \mid y \lhd x \} = \mathtt{N}_x^{\lhd} \cap E_2$$

is a clique, of size at most 2. As usual, existence of such ordering implies that we can color E_2 with 3 colors. $\square$ *Claim*

The remarks developed until now allow us to construct a partition the graph $\mathcal{G}(\mathcal{E})$ with some of properties needed to apply Proposition 5.2. Namely, for $x \in E_1$, let

$$\rho(x) = \max\{ z \in E \mid z \leqslant x, z \not\in E_1 \}, \qquad [x] = \{ y \in E_1 \mid \rho(y) = \rho(x) \},$$

and define then

$$\mathcal{P} = \{ E_0 \} \cup \{ [x] \mid x \in E_1 \} \cup \{ E_2 \}.$$

Observe that E_0 is a singleton, each $[x]$, $x \in E_1$, is a tree, and E_2 has the property stated in Claim 5.7. Thus, the partition $\mathcal{P}$ is such that each equivalence class induces a 3-colorable subgraph of $\mathcal{G}(\mathcal{E})$.

We prove next that the graph $\mathcal{G}(\mathcal{P}, \mathcal{E})$ is colorable with 4 colors. Since E_0 is isolated in $\mathcal{G}(\mathcal{P}, \mathcal{E})$, is it enough to prove that the subgraph of $\mathcal{G}(\mathcal{P}, \mathcal{E})$ induced by the trees $\{ [x] \mid x \in E_1 \}$ is 3-colorable. To this goal, we define a linear ordering $\lhd$ on the set of trees – by stating that $[y] \lhd [x]$ if and only if $\rho(y) \lhd \rho(x)$ – and then let

$$\mathtt{N}_{[x]}^{\lhd} = \{ [y] \frown [x] \mid [y] \lhd [x] \}.$$

We shall show that $\mathtt{N}_{[x]}^{\lhd}$ may contain at most two equivalence classes – and consequently that trees can be colored with 3 colors – by defining a function $f : \mathtt{N}_{[x]}^{\lhd} \longrightarrow \mathtt{B}_{\rho(x)}$ and proving that it is injective.

Let therefore $[x]$ be fixed. For each $[y] \in \mathtt{N}_{[x]}^{\lhd}$, let us pick an event $y' \in [y]$ such that $y' \frown x'$ for some $x' \in [x]$ – the existence of such a y' is ensured by the definition of the graph $\mathcal{G}(\mathcal{P}, \mathcal{E})$. Define then

$$f([y]) = \min\{ z \in E \mid \rho(y) \leqslant z \leqslant y' \text{ and } z \not\leqslant \rho(x) \}.$$

Claim 5.8. *For each* $[y] \in \mathtt{N}_{[x]}^{\lhd}$, $f([y]) \in \mathtt{B}_{\rho(x)}$.

We prove first that $f([y]) \frown \rho(x)$. Notice that $y' \frown \rho(x)$: from $\rho(x) \leqslant x' \frown y'$ we deduce, by (O2), $\rho(x) \frown y'$ or $\rho(x) \leqslant y'$. The latter, however, implies $\rho(x) < \rho(y)$ by the definition of ρ, and this relation contradicts $\rho(y) \lhd \rho(x)$. Next, $f([y]) \leqslant y' \frown \rho(x)$ and $f([y]) \not\leqslant \rho(x)$ implies $f([y]) \frown \rho(x)$, again by (O2).

To show that $f([y]) \in \mathtt{B}_{\rho(x)}$, we split our reasoning into two cases. If $f([y]) = \rho(y)$, then $f([y]) = \rho(y) \lhd \rho(x)$, so that $f([y]) \in \mathtt{N}_{\rho(x)}^{\lhd}$. Since $\rho(x) \in E_0 \cup E_2$, Claim 5.6 ensures that $f([y]) \in \mathtt{B}_{\rho(x)}$. If $f([y]) \neq \rho(y)$, then we directly prove that $z \leqslant \rho(x)$ whenever $z \prec f([y])$. This is an immediate consequence of the definition of f: $f([y]) \in E_1$ and its unique lower cover z is such that $z \leqslant \rho(x)$. $\square$ *Claim*

Thus the set $f(\mathtt{N}_{[x]}^{\lhd})$ has cardinality at most 2 and, to argue that $\mathtt{N}_{[x]}^{\lhd}$ has at most 2 elements, we are left to prove:

Claim 5.9. *The function f is injective.*

Let us suppose that $f([y]) = f([z])$. If $f([y]) = f([z]) \in E_0 \cup E_2$, then $\rho(y) = f([y]) = f([z]) = \rho(z)$, so that $[y] = [z]$. If $f([y]) = f([z]) \in E_1$, then $w \leqslant y'$ if and only if $w \leqslant z'$, for each $w \in E_0 \cup E_2$. We have therefore $\rho(y) = \rho(z)$, i.e. $[y] = [z]$. $\square$ *Claim*

We have terminated arguing that $\mathcal{G}(\mathcal{P}, \mathcal{E})$ can be colored with 4 colors. Therefore we can use Proposition 5.2 and deduce that $\mathcal{G}(\mathcal{E})$ has a labelling with 12 colors. $\square$

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