Strain energy approach to compute stress intensity factors for isotropic homogeneous and bi-material V-notches

Muhammad Treifi, S. Olutunde Oyadiji *

School of Mechanical, Aerospace and Civil Engineering, University of Manchester, Manchester M13 9PL, UK

A R T I C L E   I N F O

Article history:
Received 18 October 2012
Received in revised form 8 March 2013
Available online 30 March 2013

Keywords:
Strain energy
Stress intensity factor
Notch
Finite element method
Bi-material
Fracture mechanics

A B S T R A C T

A strain energy approach (SEA) is developed to compute the general stress intensity factors (SIFs) for isotropic homogeneous and bi-material plates containing cracks and notches subject to mode I, II and III loading conditions. The approach is based on the strain energy of a control volume around the notch tip, which may be computed by using commercial finite element packages. The formulae are simple and easy to implement. Various numerical examples are presented and compared to corresponding published results or results that are computed using different numerical methods to demonstrate the accuracy of the SEA. Many of those results are new, especially for the cases of bi-material notches where the problem is quite complicated.

1. Introduction

The importance of studying stress intensities caused by the presence of sharp corners/notches has led to much research devoted to the analysis of sharp notches. The presence of sharp notches causes stress intensities around the notch tip. This area is vulnerable to a crack initiation that may lead to structural failure or shortening of the service life of a structure. Most of the research in literature is for isotropic homogeneous cases. Very little is done for bi-material notch problems, due to its complexity.

It is well known that, in linear elastic fracture mechanics, the stresses at a notch tip become infinite (singular) (Williams, 1952, 1957). Based on experimental findings by Seweryn (1994), it was demonstrated that simple failure criteria based on the notch SIFs exist. Therefore, some researchers tried to establish a failure criterion for notch problems, such as Kněsl (1991), Gómez and Elices (2003) and Carpinteri et al. (2008). Other researchers developed different methods and procedures to compute the notch SIFs such as the boundary co-locating method (Gross and Mendelson, 1972), the boundary element singularity subtraction technique (Portela et al., 1991), singular finite elements (Lin and Tong, 1980) and finite element post-processing approaches (Babuška and Miller, 1984). Semi-analytical methods are also developed to compute the SIFs of a notch such as the hybrid crack element (HCE) (Tong et al., 1973), the scaled boundary finite element subtraction technique (SBFEM) (Wolf, 2003), and the fractal-like finite element method (FFEM) (Leung and Su, 1994; Treifi et al., 2008, 2009a,b, 2007; Treifi and Oyadiji, 2009).

Those methods are capable of computing not only the SIFs but also the higher order terms of the notch tip asymptotic field.

Most of the work mentioned above dealt only with homogeneous crack/notch problems. For bi-material problems, published results are available mainly for interfacial crack problems. For bi-material notches, published SIFs are rare, because the problem is quite complicated. For interface crack cases, Williams (1959) investigated configurations of dissimilar materials containing interface cracks. Lin and Mar (1976) developed a hybrid crack element. Lee and Choi (1988) used a boundary element method which employed the multi-region technique and the double-point concept. Yau and Wang (1984) developed a procedure based on the evaluation of conservation integrals. Matsumoto et al. (2000) used an approach based on the interaction energy release rates to compute the SIFs of interface cracks. Researchers who dealt with bi-material notch problems are few, and their research work was more about studying the stress and displacement fields and the behaviour of the singular eigenvalues. Early work was carried out by Bogy (1968, 1970) and Bogy and Wang (1971). Carpenter and Byers (1987) used a reciprocal work contour integral method. Tan and Meguid (1997) developed a singular finite element formulation using expressions of the singular stress and displacement fields of a bi-material notch. Chen and Sze (2001) developed a hybrid finite element formulation using numerically obtained asymptotic stress and displacement fields. Carpinteri et al. (2006) presented an approximate analytical model based on the theory of multi-layered beams to compute mode I SIFs for a general notch perpendicular to a bi-material interface. Paggi and Carpinteri (2008) presented a comprehensive review of interface mechanical problems leading to stress singularities.
In this paper, we present a strain energy based approach to compute the SIFs for homogeneous and bi-material notch problems subject to mode I, II and III loading conditions. The approach is based on the work of Lazzarin and Zambardi (2001), Lazzarin and Berto (2005), Lazzarin and Filippi (2006), Lazzarin and Zappalorto (2008), Lazzarin et al. (2010), Berto and Lazzarin (2007), Radij et al. (2009), Zappalorto and Lazzarin (2011) and Zappalorto et al. (2008) who developed the idea of using averaged strain energy over a control volume around a notch tip to compute the SIFs for sharp and rounded notches. They dealt mainly with homogeneous pure mode I, II or III cases. For mixed mode I and II cases, they usually neglected the effect of mode II SIF (Lazzarin and Zambardi, 2001) where they used examples with non-singular mode II stress components, but in a recent publication (Lazzarin et al., 2010) they suggested using two concentric volumes to compute mode I and II notch SIFs. However, this approach does not always work as will be discussed later. In the current work, we simply partition the control volume and the integral accordingly to compute mode I and II SIFs for homogeneous mixed-mode and bi-material notch problems. This new strategy is operationally very simple to implement. It involves simple mathematical operations that can be carried out numerically. The strain energy for the control volume could be computed by using commercial finite element packages. In most of these packages, it is not possible to compute notch SIFs, but the SEA empowers analysts to compute notch SIFs. The accuracy of the approach is tested via many examples of isotropic homogeneous and bi-material notches under different loading conditions. The results are compared to available published results and results computed numerically using different numerical methods; the agreement is very good. Also, new results are presented.

2. Strain energy approach

The strain energy of a finite volume around a notch-tip can be written as (Bower, 2010)

\[ W^{(e)} = \int_V W^{(e)} dV \]

where \( W^{(e)} \) is the strain energy density and can be computed as follows

\[ W^{(e)} = \int \sigma : \varepsilon dV \]

where \( \sigma \) and \( \varepsilon \) are stress and strain tensors, respectively. For an isotropic material, the strain energy density \( W^{(e)} \) for a generalised state of stress can be written as

\[ W^{(e)} = \frac{1}{2} \left[ \sigma_{xx} \varepsilon_{xx} + \sigma_{yy} \varepsilon_{yy} + \sigma_{zz} \varepsilon_{zz} + 2\sigma_{xy} \varepsilon_{xy} + 2\sigma_{xz} \varepsilon_{xz} + 2\sigma_{yz} \varepsilon_{yz} \right] \]

The strains can be written in terms of the stresses by using Hooke’s law

\[ \varepsilon_{xx} = \frac{1}{E} \left[ \sigma_{xx} - v(\sigma_{yy} + \sigma_{zz}) \right] \]
\[ \varepsilon_{yy} = \frac{1}{E} \left[ \sigma_{yy} - v(\sigma_{xx} + \sigma_{zz}) \right] \]
\[ \varepsilon_{zz} = \frac{1}{E} \left[ \sigma_{zz} - v(\sigma_{xx} + \sigma_{yy}) \right] \]
\[ \gamma_{xy} = \frac{1}{G} \tau_{xy} \]
\[ \gamma_{yz} = \frac{1}{G} \tau_{yz} \]
\[ \gamma_{xz} = \frac{1}{G} \tau_{xz} \]
\[ G = \frac{E}{2(1+v)} \]

where \( E, G, v \) are Young’s modulus, shear modulus and Poisson’s ratio, respectively. For simplicity the stresses can be expressed as

\[ \sigma_{ij} = f(K_i, K_{ii}, K_{iii}, r, \theta) \]

where \( K_i, K_{ii} \) and \( K_{iii} \) are the mode I, II and III SIFs, respectively. By substituting the stress expressions into Eq. (1) and carrying out the integration over a finite volume around the notch tip, Eq. (1) becomes a representation of a direct relation between the strain energy for a finite volume and the SIFs. The strain energy could be easily computed using a commercial finite element package. Most FE packages are not, to our knowledge, capable of computing the SIFs for general notches. Therefore, this approach is quite useful to extract SIFs for general notches by using current commercial FE packages. Eq. (1) could be partitioned to deal with bi-material or mixed mode cases where two equations are needed to compute mode I and mode II SIFs. This will be discussed in detail in the next sections.

2.1. Isotropic homogeneous notch

2.1.1. Relationships between stress intensity factors and strain energy of a finite volume around a notch tip under in-plane loading conditions (mode I, II and mixed mode)

For the in-plane problem, the strain energy density is

\[ W^{(e)} = \frac{1}{2E} \left[ \sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 - 2v(\sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz}) \right] + 2(1 + v)\tau_{xy}^2 \]

where \( \sigma_{zz} = 0 \) under plane-stress and \( \sigma_{zz} = v(\sigma_{xx} + \sigma_{yy}) \) under plane-strain.

The stress expressions for a general homogeneous notch as seen in Fig. 1 are (Williams, 1952; Portela et al., 1991)

\[ \sigma_{xx} = x^l \tau^{l-1} A_1 \left[ (2 + x^l \cos 2\alpha + \cos 2x^l \alpha) \cos (x^l - 1) \theta - (x^l - 1) \cos (x^l - 3) \theta \right] + x^l \tau^{l-1} A_2 \left[ (2 + x^l \cos 2\alpha - \cos 2x^l \alpha) \sin (x^l - 1) \theta + (x^l - 1) \sin (x^l - 3) \theta \right] \]

\[ \sigma_{yy} = x^l \tau^{l-1} A_1 \left[ (2 - x^l \cos 2\alpha - \cos 2x^l \alpha) \cos (x^l - 1) \theta + (x^l - 1) \cos (x^l - 3) \theta \right] + x^l \tau^{l-1} A_2 \left[ (2 + x^l \cos 2\alpha - \cos 2x^l \alpha) \sin (x^l - 1) \theta - (x^l - 1) \sin (x^l - 3) \theta \right] \]

\[ \tau_{xy} = x^l \tau^{l-1} A_1 \left[ (2 \cos 2\alpha + \cos 2x^l \alpha) \sin (x^l - 1) \theta + (x^l - 1) \sin (x^l - 3) \theta \right] + x^l \tau^{l-1} A_2 \left[ (2 \cos 2\alpha - \cos 2x^l \alpha) \cos (x^l - 1) \theta + (x^l - 1) \cos (x^l - 3) \theta \right] \]

Fig. 1. Isotropic homogeneous notch geometry.
where \( \lambda' \) and \( \lambda'' \) are eigenvalues and are computed using the following characteristic equations

\[
\lambda' \sin 2\alpha + \sin 2\lambda' \alpha = 0
\]

\( \lambda'' \sin 2\alpha - \sin 2\lambda'' \alpha = 0 \) 

(10)

(11)

\( A_1 \) and \( A_2 \) are constants related to the mode I and mode II SIFs

\[
K_I = \sqrt{2\pi E}(1 + \lambda' - \lambda' \cos 2\alpha - \cos 2\lambda' \alpha)A_1
\]

(12)

\[
K_{II} = \sqrt{2\pi E}(1 - \lambda'' + \lambda'' \cos 2\alpha + \cos 2\lambda'' \alpha)A_2
\]

(13)

Eqs. (7)-(9) are eigenfunction series expansions (the \( \Sigma \) symbol is dropped for simplicity). In the SEA, only the singular terms are considered. The stress expressions in Eqs. (7)-(9) can be rewritten for simplicity as

\[
\sigma_{xx} = A_1 \tau^{r-1} f_x(\theta) + A_2 \tau^{r-1} g_x(\theta) = A_1 \tau^{r-1} f_x + A_2 \tau^{r-1} g_x
\]

(14)

\[
\sigma_{yy} = A_1 \tau^{r-1} f_y(\theta) + A_2 \tau^{r-1} g_y(\theta) = A_1 \tau^{r-1} f_y + A_2 \tau^{r-1} g_y
\]

(15)

\[
\tau_{xy} = A_1 \tau^{r-1} f_{xy}(\theta) + A_2 \tau^{r-1} g_{xy}(\theta) = A_1 \tau^{r-1} f_{xy} + A_2 \tau^{r-1} g_{xy}
\]

(16)

Under plane-stress, substituting the above equations into Eq. (6) gives

\[
W^{(c)} = \frac{1}{2E} \left[ A_1 \tau^{r,2-1} \left( f_x^2 + f_y^2 - 2\nu f_x f_y + 2(1 + \nu) f_{xy}^2 \right) + A_2 \tau^{r,2-1} \left( g_x^2 + g_y^2 - 2\nu g_x g_y + 2(1 + \nu) g_{xy}^2 \right) + A_1 A_2 \tau^{r,2-1} \left( 2f_x g_x + 2f_y g_y - 2\nu(f_x g_y + f_y g_x) + 4(1 + \nu)f_{xy} g_{xy} \right) \right]
\]

(17)

By substituting Eq. (17) into Eq. (1), the strain energy for a finite volume of a radius \( R_c \) around a notch tip is

\[
E^{(c)} = \int_0^{R_c} \int_0^{2\pi} W^{(c)} r dr d\theta
\]

\[
\begin{align*}
E^{(c)} &= \frac{1}{2E} \left[ A_1 \tau^{r,2-1} \int_0^{2\pi} \left( f_x^2 + f_y^2 - 2\nu f_x f_y + 2(1 + \nu) f_{xy}^2 \right) dr \right. \\
&\quad + A_2 \tau^{r,2-1} \int_0^{2\pi} \left( g_x^2 + g_y^2 - 2\nu g_x g_y + 2(1 + \nu) g_{xy}^2 \right) dr \\
&\quad + A_1 A_2 \tau^{r,2-1} \int_0^{2\pi} \left( 2f_x g_x + 2f_y g_y - 2\nu(f_x g_y + f_y g_x) + 4(1 + \nu)f_{xy} g_{xy} \right) dr \left. \\
&\quad + 4(1 + \nu)f_{xy} g_{xy} \int_0^{2\pi} dr \right]
\end{align*}
\]

(18)

Substituting Eqs. (12) and (13) into Eq. (18) gives a quadratic equation with two unknowns

\[
E^{(c)} = MK_I^2 + NK_{II}^2 + QK_I K_{II}
\]

(19)

It should be noted that the coefficients \( M, N \) and \( Q \) have different dimensional units. Sih (1974) reported a similar expression relating the strain energy density to the crack SIFs when he introduced a strain energy density factor as a fracture parameter for crack problems. The integration could be carried out numerically by using, for example, Composite Simpson’s rule. Eq. (19) illustrates a direct relation between the SIFs and the strain energy of a finite volume around the notch tip. For pure mode I or pure mode II, Eq. (19) can be used to compute mode I SIF, \( K_I \), or mode II SIF, \( K_{II} \) \((E^{(c)} = MK_I^2 \) for pure mode I and \( E^{(c)} = NK_{II}^2 \) for pure mode II). However, in the case of mixed mode problems, Eq. (19) represents an equation that contains two unknowns \( K_I \) and \( K_{II} \). To overcome this, Eq. (19) could be partitioned into two regions: one below the bisector \((-\alpha \) to 0) and the other above the bisector \((0 \) to \(+\alpha \) as shown in Fig. 2. This leads to the following two quadratic equations with two unknowns

\[
E_1^{(c)} = \int_0^{R_c} \int_0^{\pi} W^{(c)} r dr d\theta = M_1 K_I^2 + N_1 K_{II}^2 + Q_1 K_I K_{II}
\]

(20)

\[
E_2^{(c)} = \int_0^{R_c} \int_0^{\pi} W^{(c)} r dr d\theta = M_2 K_I^2 + N_2 K_{II}^2 + Q_2 K_I K_{II}
\]

(21)

Due to symmetry, the following relations hold

\[
M_1 = M_2
\]

\[
M_1 = N_2
\]

\[
Q_1 = -Q_2
\]

(22)

After some simple algebraic manipulations, the mode I and II SIFs can be computed using the following equations

\[
K_I = \frac{E_1^{(c)} - E_2^{(c)}}{2Q_1 K_I}
\]

(23)

\[
2M_1 K_I^2 - \left( E_1^{(c)} + E_2^{(c)} \right) K_I^2 + \frac{N_1 \left( E_1^{(c)} - E_2^{(c)} \right)^2}{2Q_1^2} = 0
\]

(24)

Obviously, Eqs. (23) and (24) would give more than one set of answers. Usually it is easy to determine the correct set. However, for less experienced analysts, the ratio of the relative displacements of the notch faces could be used to determine the right set of answers. For a crack problem, the following equation holds

\[
\frac{K_I}{K_{II}} = \frac{\Delta_1}{\Delta_2}
\]

(25)

where \( \Delta_1 = u_1(r_0, \alpha) - u_2(r_0, \alpha) \) and \( \Delta_2 = u_1(r_0, \alpha) - u_2(r_0, \alpha) \) are the relative displacements of the crack faces. Eq. (25) could be used as an approximation for notch cases too i.e. \( \frac{\Delta_1}{\Delta_2} \approx \frac{\Delta_2}{\Delta_1} \). Because \( \Delta_1 \) and \( \Delta_2 \) are computed numerically using finite element analysis, Eq. (25) is better considered as an approximation rather than an equality. The displacements of two nodes facing each other on the notch faces such as the nodes 1 and 2 as shown in Fig. 2 could be used. It is advisable to consider nodes that are reasonably far from the notch tip.

Lazzarin et al. (2010) suggested using two concentric circles with different radii to deal with mixed mode cases, but their suggestion cannot be used for the special case of a notch with an opening angle of zero, i.e. a crack.
2.1.2. Relationships between stress intensity factors and strain energy of a finite volume around a notch tip under out-of-plane loading conditions (mode III)

For the out-of-plane problem, the strain energy density is

$$W^{(y)} = \frac{1}{2G} \left[ \tau_{\alpha\beta}^2 + \tau_{\alpha\zeta}^2 \right]$$  \hspace{1cm} (26)

The stress expressions for a general homogeneous notch under mode III loading conditions are

$$\tau_{\alpha\beta} = G_{\alpha\beta} \epsilon^{(III)} \cdot \Phi_\beta$$

$$\tau_{\alpha\zeta} = G_{\alpha\zeta} \epsilon^{(III)} \cdot \Phi_\zeta$$  \hspace{1cm} (27)

where \( \epsilon^{(III)} = \frac{\pi}{2\pi} n = 1, 2, 3, \ldots \) and \( B \) is a constant. For detailed derivations, one may refer to Seweryn and Molski (1996). Eqs. (27) and (28) are eigenfunction series expansions (the \( \Sigma \) symbol is dropped for simplicity). In the SEA, only the singular term is considered, i.e. \( n = 1 \). The constant \( B \) associated with the singular eigenvalue is related to the mode III SIF

$$K_{\Pi III} = \sqrt{2\pi G} \epsilon^{(III)} B$$  \hspace{1cm} (29)

By substituting Eqs. (27)-(29) into Eqs. (26) and (1), and after some algebraic manipulations, the strain energy of a finite volume under mode III conditions can be written as

$$E^{(y)} = \frac{K_{\Pi III}^2}{4\pi G}$$  \hspace{1cm} (30)

This equation is in agreement with an expression presented by Lazzarin and Zappalorto (2008) relating the strain energy density to the mode III SIF. Eq. (30) represents a simple analytical formula that links the mode III SIF to the strain energy of a finite volume around a notch tip.

2.2. Bi-material notch

2.2.1. Relationships between stress intensity factors and strain energy of a finite volume around a notch tip under in-plane loading conditions (mode I, II and mixed mode)

For the in-plane bi-material problem, the expression for the strain energy density is

$$W^{(y)} = \frac{1}{2G} \left[ (\sigma_{\alpha\alpha})^2 + (\sigma_{\alpha\zeta})^2 + (\sigma_{\zeta\zeta})^2 + 2\nu \left( \sigma_{\alpha\zeta} \sigma_{\alpha\zeta} + \sigma_{\alpha\zeta} \sigma_{\zeta\zeta} + \sigma_{\alpha\zeta} \sigma_{\alpha\zeta} \right) \right] + \frac{1}{2G} \left( \tau_{\alpha\beta}^2 + \tau_{\alpha\zeta}^2 \right)$$  \hspace{1cm} (31)

where \( j \) refers to the material. Under plane-stress conditions \( \sigma_{\alpha\alpha} = 0 \), and under plane-strain conditions \( \sigma_{\alpha\alpha} = 1 \) \( (\sigma_{\alpha\alpha} + \sigma_{\zeta\zeta}) \).

The stress expressions for a general bi-material notch as seen in Fig. 3 are (Carpenter and Byers, 1987)

$$2\sigma^{(1)}_{\Pi x} = \frac{1}{2G_{\Pi x}} A_1 \left[ 3 \epsilon^{(III)}(\zeta - 1) - \epsilon^{(III)}(\zeta - 1) \epsilon^{(III)}(\zeta - 1) \right] + Y_1 \left[ 2 \epsilon^{(III)}(\zeta - 1) - \epsilon^{(III)}(\zeta - 1) + S_1 \epsilon^{(III)}(\zeta - 1) - S_2 \epsilon^{(III)}(\zeta - 1) \right]$$

$$2\sigma^{(2)}_{\Pi x} = \frac{1}{2G_{\Pi x}} \left[ 3 \epsilon^{(III)}(\zeta - 1) - \epsilon^{(III)}(\zeta - 1) \epsilon^{(III)}(\zeta - 1) \right] + Y_1 \left[ 2 \epsilon^{(III)}(\zeta - 1) - \epsilon^{(III)}(\zeta - 1) + S_1 \epsilon^{(III)}(\zeta - 1) - S_2 \epsilon^{(III)}(\zeta - 1) \right]$$

$$2\sigma^{(3)}_{\Pi x} = \frac{1}{2G_{\Pi x}} \left[ 3 \epsilon^{(III)}(\zeta - 1) - \epsilon^{(III)}(\zeta - 1) \epsilon^{(III)}(\zeta - 1) \right] + Y_1 \left[ 2 \epsilon^{(III)}(\zeta - 1) - \epsilon^{(III)}(\zeta - 1) + S_1 \epsilon^{(III)}(\zeta - 1) - S_2 \epsilon^{(III)}(\zeta - 1) \right]$$

Fig. 3. Bi-material notch geometry.
The integration over $\theta$ is from $-\pi$ to $0$ for $j=2$ (material 2) and from $0$ to $\pi$, for $j=1$ (material 1).

Equation (45) gives one equation per material in terms of two unknowns which are the real and imaginary parts of $A_1$ (or $K_1$).

For brevity and simplicity Eq. (45) can be rewritten as

$$E^{(e)} = A_1^2 M_j + A_1^2 N_j + A_1 A_i Q_j$$

Eq. (46) could be simplified further, because $N_j = \bar{M}_j$ and $Q_j$ is a real number, as

$$a^2 (2M_1 \text{Re}l + Q_1) + b^2 (Q_1 - 2M_1 \text{Re}l) - 4ab M_1 \text{Im}aginary - E^{(e)} = 0$$

for material 1, and

$$a^2 (2M_2 \text{Re}l + Q_2) + b^2 (Q_2 - 2M_2 \text{Re}l) - 4ab M_2 \text{Im}aginary - E^{(e)} = 0$$

for material 2, where $A_1 = a + ib$. The bi-material SIFs can be computed using Eqs. (47) and (48) after computing the strain energy in material 1 and material 2 within a finite region around the notch tip of radius $R$.

For real eigenvalues $\lambda$ and $\lambda_j$, the stress expressions are

$$2\sigma_{xx} = \sum_k \lambda_j r_{xj}^{-1} a_k \left\{ (p_{x1} + i p_{x2}) [e^{i(\theta - \pi)} (2 + \lambda_j (e^{2i\theta} - e^{-2i\theta})) + e^{2i\theta}] + (p_{x1} - i p_{x2}) [e^{i(\theta + \pi)} (2 + \lambda_j (e^{2i\theta} - e^{-2i\theta})) + e^{-2i\theta}] \right\}$$

for material 2,

$$2\sigma_{yy} = \sum_k \lambda_j r_{yj}^{-1} a_k \left\{ (p_{y1} + i p_{y2}) [e^{i(\theta - \pi)} (2 + \lambda_j (e^{2i\theta} - e^{-2i\theta})) + e^{-2i\theta}] - e^{2i\theta} \right\}$$

Under plane-stress, substituting the above equations into Eq. (31) gives

$$W^{(e)} = \sum_k \lambda_j r_{xj}^{-1} a_k \left\{ (p_{x1} + i p_{x2}) [e^{i(\theta - \pi)} (2 + \lambda_j (e^{2i\theta} - e^{-2i\theta})) + e^{2i\theta}] + (p_{x1} - i p_{x2}) [e^{i(\theta + \pi)} (2 + \lambda_j (e^{2i\theta} - e^{-2i\theta})) + e^{-2i\theta}] \right\}$$

The strain energy for a finite volume of a radius $R$, around a notch tip is obtained by substituting the above equation into Eq. (1),

$$E^{(e)} = \int_0^1 \int W^{(e)} r \, dr \, d\theta = \frac{1}{2 \pi R^2} \sum_k \left\{ (p_{x1} + i p_{x2}) [e^{i(\theta - \pi)} (2 + \lambda_j (e^{2i\theta} - e^{-2i\theta})) + e^{2i\theta}] + (p_{x1} - i p_{x2}) [e^{i(\theta + \pi)} (2 + \lambda_j (e^{2i\theta} - e^{-2i\theta})) + e^{-2i\theta}] \right\}$$

$$+ A_1 A_j (\alpha + \lambda_j) \int \left\{ (2f_j g_j + 2f_j g_j - 2v_j (f_j g_j + f_j g_j) + 4(1 + v_j) f_j g_j) \right\}$$

$$+ A_1 A_j (\alpha + \lambda_j) \int \left\{ (2f_j g_j + 2f_j g_j - 2v_j (f_j g_j + f_j g_j) + 4(1 + v_j) f_j g_j) \right\}$$

$$+ A_1 A_j (\alpha + \lambda_j) \int \left\{ (2f_j g_j + 2f_j g_j - 2v_j (f_j g_j + f_j g_j) + 4(1 + v_j) f_j g_j) \right\}$$

The integration over $\theta$ is from $-\pi$ to $0$ for $j=2$ (material 2) and from $0$ to $\pi$, for $j=1$ (material 1).
where \( k = I, II, s_{k1} = \frac{d_{k1} - d_{k2}}{d_{k2} - d_{k1}}, s_{k2} = \frac{d_{k5} - d_{k6}}{d_{k6} - d_{k5}} \), and to avoid division by zero, \( p_{k1} \) and \( p_{k2} \) can be computed using the expressions stated in Table 1 where

\[
q_{k11} = \text{Re}(t_{k1}) + \text{Re}(t_{k2}), \quad q_{k12} = \text{Im}(t_{k2}) - \text{Im}(t_{k1})
\]

\[
q_{k21} = \text{Im}(t_{k1}) + \text{Im}(t_{k2}), \quad q_{k22} = \text{Re}(t_{k1}) - \text{Re}(t_{k2})
\]

\[t_{k1} = s_{k2} - s_{k3}, \quad t_{k2} = s_{k1} - s_{k4}\]

To determine \( p_{k1} \) and \( p_{k2} \) from Table 1, it should be noted that the first step is to determine the largest absolute value of the \( q_{kij} \)'s, this is \( |q_{kl}| \). The corresponding expressions for \( p_{k1} \) and \( p_{k2} \) are obtained from the row containing this largest absolute value of \( q_{kl} \).

The SIFs expressions are

\[
K_I = \sqrt{2\pi} \left( \frac{\gamma_{k1}}{2} \left[ \frac{\gamma_{k1}}{2} \left( 1 + \gamma_{k1} \left( 1 - e^{-2\gamma_{k1}} \right) - e^{2\gamma_{k1}} \right) \right] \right)
\]

\[
+ \left( \frac{\gamma_{k1}}{2} - \gamma_{k2} \right) \left( 1 + \gamma_{k2} \left( 1 - e^{2\gamma_{k2}} \right) - e^{-2\gamma_{k2}} \right) \]

(55)

**Fig. 4.** (a) Notched plate subject to tension loading conditions (b) the plate FE mesh (c) control volume.

**Fig. 5.** SIFs for Notched plate \( \gamma = 60^\circ \) under tension loading conditions.
\[ K_{II} = \sqrt{2\pi} \frac{J_{II}}{2I} \left[ (p_{II} + ip_{II}) (\lambda_0 (1 - e^{-2i\xi_1}) + e^{2i\xi_1}) - (p_{II} - ip_{II}) (\lambda_0 (e^{2i\xi_1} - 1) - e^{-2i\xi_1}) \right] \]

(56)

The stress expressions in Eqs. (49)-(54) can be rewritten as

\[ \sigma_{xx} = a_{r^{-1}} f_{1x}(\theta) + a_{r^{-1}} g_{1x}^{(j)}(\theta) = a_{r^{-1}} f_{1x} + a_{r^{-1}} g_{1x}^{(j)} \]

(57)

\[ \sigma_{yy} = a_{r^{-1}} f_{1y}(\theta) + a_{r^{-1}} g_{1y}^{(j)}(\theta) = a_{r^{-1}} f_{1y} + a_{r^{-1}} g_{1y}^{(j)} \]

(58)

\[ \tau_{xy} = a_{r^{-1}} f_{1y}(\theta) + a_{r^{-1}} g_{1y}^{(j)}(\theta) = a_{r^{-1}} f_{1y} + a_{r^{-1}} g_{1y}^{(j)} \]

(59)

Assuming plane-stress state, substituting the above equations into Eq. (31) gives

\[ W^{(e)} = \frac{1}{2E} \left[ a_{r^{2i\xi-1}} (f_{1x}^2 + f_{1y}^2 + 2(1 + \nu)g_{1y}^{(j)} + 2) + a_{r^{2i\xi+1}} (g_{1x}^2 + g_{1y}^{(j)} - 2) + a_{r^{-1}(1+h-2)} (2f_{1x}g_{1x} + 2f_{1y}g_{1y} + 2 + (1 + \nu)g_{1y}^{(j)} + 4) \right] \]

(60)

by substituting the above equation into Eq. (1), the strain energy for a finite volume of a radius \( R_c \) around a bi-material notch tip with real singular eigenvalues is obtained

\[ E^{(e)} = \int_0^{R_c} \int_0^{\pi} W^{(e)} \rho d\theta d\rho \]

(61)

The integration over \( \theta \) is from \( -\xi_2 \) to 0 for \( j = 2 \) (material 2) and from 0 to \( \pi \) for \( j = 1 \) (material 1). Computing the strain energy for each region using a commercial FE package, and substituting it back into Eq. (60) gives

**Fig. 6.** (a) A slant centre cracked plate (b) the plate FE mesh.

**Fig. 7.** Mode I and II SIFs for the slant centre cracked plate.
Fig. 8. The FE mesh used in the FFEM of the slant centre cracked plate.

Fig. 9. (a) A slant centre notched plate $\gamma = 45^\circ$ (b) the plate FE mesh.

\[ E^{(e)} = MJK_e^2 + NJK_2^2 + QJK_{II} \]  

Eq. (62) represents two quadratic equations with two unknowns, \( K_I \) and \( K_{II} \), which can be solved easily using a programming software such as the MATLAB program.

2.2.2. Relationships between stress intensity factors and strain energy of a finite volume around a notch tip under out-of-plane loading conditions (mode III)

For the out-of-plane bi-material problem, the strain energy density in cylindrical-polar coordinates is

\[ W^{(e)} = \frac{1}{2\pi} \left[ \tau^{(e)}_x^2 + \tau^{(e)}_y^2 \right] \]  

The stress expressions for a general bi-material notch under mode III loading conditions are

\[ \tau^{(e)}_{xz} = G_1 \lambda^{III} r^{m-1} B \left( \frac{\cos \lambda^{III} x_1}{\sin \lambda^{III} x_1} \cos \lambda^{III} \theta + \sin \lambda^{III} \theta \right) \]  

\[ \tau^{(e)}_{yz} = G_1 \lambda^{III} r^{m-1} B \left( -\frac{\cos \lambda^{III} x_1}{\sin \lambda^{III} x_1} \sin \lambda^{III} \theta + \cos \lambda^{III} \theta \right) \]  

for material 1 and

\[ \tau^{(e)}_{xz} = G_2 \lambda^{III} r^{m-1} B \left( \frac{\cos \lambda^{III} x_1}{\sin \lambda^{III} x_1} \cos \lambda^{III} \theta + \frac{G_1}{G_2} \frac{\sin \lambda^{III} \theta}{\sin \lambda^{III} x_1} \right) \]  

\[ \tau^{(e)}_{yz} = G_2 \lambda^{III} r^{m-1} B \left( -\frac{\cos \lambda^{III} x_1}{\sin \lambda^{III} x_1} \sin \lambda^{III} \theta + \frac{G_1}{G_2} \cos \lambda^{III} \theta \right) \]  

for material 2, where \( \lambda^{III} \) is an eigenvalue that can be computed from

\[ \left( \frac{G_1}{G_2} + 1 \right) \sin \lambda^{III} (x_1 + x_2) + \left( \frac{G_1}{G_2} - 1 \right) \sin \lambda^{III} (x_1 - x_2) = 0 \]  

and \( B \) is a constant. Eqs. (64) and (67) are eigenfunction series expansions (the \( \Sigma \) symbol is dropped for simplicity). In the SEA, only the singular term is considered. The constant \( B \) associated with the singular eigenvalue is related to the mode III SIF

\[ K_{III} = \sqrt{2\pi G_1 \lambda^{III} B} \]  

For detailed derivations, one may refer to Qian and Hasebe (1997). By substituting Eqs. (64)–(67) into Eqs. (63) and (1), and after some algebraic manipulations, the strain energy of a finite volume under mode III conditions can be written as

\[ E^{(e)} = \int_0^R \int_{-\gamma}^{\gamma} W^{(e)} d\theta + \int_0^R W^{(e)} d\theta \right) r dr = \frac{I}{8\pi G_1 \lambda^{III} K_{III}^2} \]  

where \( I \) is an integral and its value is

\[ I = G_1 \lambda^{III} x_1 \left( \frac{\cos \lambda^{III} x_1}{\sin \lambda^{III} x_1} \right)^2 + \left( \frac{G_1}{G_2} \right)^2 \]  

\[ + G_1 \lambda^{III} x_1 \left( \frac{\cos \lambda^{III} x_1}{\sin \lambda^{III} x_1} \right)^2 + 1 \]  

Eq. (70) represents a simple analytical formula that can be used to determine mode III SIF values of bi-material notches after computing the strain energy of a finite volume of radius \( R \) around and a notch tip using available commercial FE packages.
3. Numerical examples and verification

The proposed approach is verified by means of comparison with available published results and numerical results computed using the ABAQUS FEA commercial software and/or the software developed by the authors using FFEM. The numerical examples are presented in sub-groups corresponding to the sub-sections in Section 2 starting with isotropic homogeneous and then bi-material examples. Some discussion on the choice of \( R_c \) is also presented for each case. In all the examples the strain energy values for a finite volume around the notch tip are predicted using ABAQUS version 6.8. In the FEA, it is well known that the mesh of a body has an effect on the results, and that good meshes should be used for the analysis. Here, too, a good mesh should be used to achieve accurate results. By ‘good mesh’ we mean that large and small elements should not be adjacent, but there should be a transitional change in size.

3.1. Isotropic homogeneous notch

3.1.1. Mode I, II and mixed mode

The effect of the radius \( R_c \) on the accuracy of the SEA to predict SIF values for isotropic notch cases subject to in-plane loading conditions is demonstrated through different examples. For all the examples the global mesh used is more or less the same, but the local mesh around the notch tip is different. For example, different numbers of finite elements are used to discretise the small region around the notch/crack tip. A notched plate with a notch opening angle of \( \gamma = 60^\circ \) under pure mode I loading conditions shown in Fig. 4(a) is considered first. The plate is of height \( H = 20 \) and width \( W = 10 \). The notch length is \( a \) where \( a/w = 0.4 \). Quadrilateral elements (which are designated as CPS8 in the ABAQUS FEA software) are used to model the plate as shown in Fig. 4(b). The small region around the notch tip is meshed layer by layer with a similarity ratio \( q = 0.6 \) as illustrated in Fig. 4(c). The radius of the first layer is

![Fig. 10. Mode I and II SIFs for the slant centre notched plate.](image)

![Fig. 11. The FE mesh used in the FFEM of the slant centre notched plate.](image)

<table>
<thead>
<tr>
<th>( \gamma (\alpha, k) )</th>
<th>( h_c/H )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>SEA</td>
<td>3.508</td>
<td>1.048</td>
<td>2.352</td>
<td>0.234</td>
<td>2.158</td>
</tr>
<tr>
<td>ABAQUS</td>
<td>3.504</td>
<td>1.045</td>
<td>2.349</td>
<td>0.232</td>
<td>2.151</td>
<td>0.054</td>
</tr>
<tr>
<td>(0.5,0.5)</td>
<td>SEA</td>
<td>3.557</td>
<td>1.439</td>
<td>2.369</td>
<td>0.308</td>
<td>2.174</td>
</tr>
<tr>
<td>ABAQUS</td>
<td>3.544</td>
<td>1.448</td>
<td>2.361</td>
<td>0.307</td>
<td>2.167</td>
<td>0.070</td>
</tr>
<tr>
<td>(0.5014,0.5982)</td>
<td>ABAQUS</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>60</td>
<td>SEA</td>
<td>–</td>
<td>–</td>
<td>2.464</td>
<td>0.423</td>
<td>2.265</td>
</tr>
<tr>
<td>FFE</td>
<td>–</td>
<td>–</td>
<td>2.472</td>
<td>0.404</td>
<td>2.263</td>
<td>0.090</td>
</tr>
<tr>
<td>(0.5122,0.7309)</td>
<td>ABAQUS</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>90</td>
<td>SEA</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>2.514</td>
</tr>
<tr>
<td>FFE</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>2.511</td>
<td>0.127</td>
</tr>
<tr>
<td>(0.5445,0.9085)</td>
<td>ABAQUS</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>
$R_0 = 0.6$. This is an arbitrary choice to provide transitional change in the size of the finite elements used to model the plate. $R_0$ has no significance other than to indicate the relative size of the mesh used around the notch tip with respect to the mesh size in the domain far from the notch tip. Initially, six layers are used within $R_0$. The mode I SIF values are computed based on the energy values for control volumes of sizes ranging from 1 layer to 6 layers (that is, the control radius is $R_c = 0.046656–0.6$). The same is repeated for 9 ($R_c = 0.0100777–0.6$), 16 ($R_c = 0.000282168–0.6$) and 20 ($R_c = 0.0000365376–0.6$) layers. The value of $R_0$ is not changed. Only the number of layers within $R_0$ and the radius of the control volume $R_c$ are changed. The results are plotted in Fig. 5. The graph shows clearly that convergence is achieved with increasing number of layers within $R_0$; that is, using smaller sizes of the control volume $R_c$ allowed by finer meshes within $R_0$. The converged SIF value (scaled by $\frac{\sigma \sqrt{\pi a^{1/2}}}{C_0}$) achieved is $K_I = 2.225$. Published results for this case are reported to be $K_I = 2.223$ by Gross and Mendelson (1972) and $K_I = 2.222$ by Portela et al. (1991). In this example, accurate results are achieved when computing the strain energy value for values of $R_c$ between 0.000101566 and 0.046656. In other words, the strain energy is computed for a control volume of size ranging from 3 to 15 layers out of the 20 layers that are used to model the region around the notch tip. Fig. 5 also shows that the size of the control volume $R_c$ has an important role and results are less dependent on the mesh within $R_c$. In this figure, the different curves mean that different meshes (number of layers within $R_c$) are used for each value of $R_c$. Considering a value of $R_c = 0.046656$ in Fig. 5, its projection on the green dashed curve shows that only

<table>
<thead>
<tr>
<th>$\gamma$ ($\psi_1, \psi_2$)</th>
<th>$b_c/H$</th>
<th>$0.1$</th>
<th>$0.2$</th>
<th>$0.3$</th>
<th>$0.4$</th>
<th>$0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$K_I$</td>
<td>$K_\Pi$</td>
<td>$K_I$</td>
<td>$K_\Pi$</td>
<td>$K_I$</td>
<td>$K_\Pi$</td>
</tr>
<tr>
<td>0</td>
<td>SEA</td>
<td>10.275</td>
<td>5.798</td>
<td>4.963</td>
<td>1.971</td>
<td>3.055</td>
</tr>
<tr>
<td></td>
<td>FFE</td>
<td>10.465</td>
<td>5.568</td>
<td>4.981</td>
<td>1.926</td>
<td>3.064</td>
</tr>
<tr>
<td>(0.5,0.5)</td>
<td>ABAQUS</td>
<td>10.505</td>
<td>5.591</td>
<td>4.993</td>
<td>1.927</td>
<td>3.069</td>
</tr>
<tr>
<td>30</td>
<td>SEA</td>
<td>10.754</td>
<td>7.681</td>
<td>5.035</td>
<td>2.601</td>
<td>3.085</td>
</tr>
<tr>
<td></td>
<td>FFE</td>
<td>10.732</td>
<td>7.716</td>
<td>5.020</td>
<td>2.582</td>
<td>3.077</td>
</tr>
<tr>
<td>(0.5014,0.5982)</td>
<td>ABAQUS</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>60</td>
<td>SEA</td>
<td>–</td>
<td>–</td>
<td>5.198</td>
<td>3.557</td>
<td>3.183</td>
</tr>
<tr>
<td></td>
<td>FFE</td>
<td>–</td>
<td>–</td>
<td>5.268</td>
<td>3.448</td>
<td>3.207</td>
</tr>
<tr>
<td>(0.5122,0.7309)</td>
<td>ABAQUS</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>90</td>
<td>SEA</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>3.476</td>
</tr>
<tr>
<td></td>
<td>FFE</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>3.462</td>
</tr>
<tr>
<td>(0.5445,0.9085)</td>
<td>ABAQUS</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Fig. 12. (a) Notched plate subject to tension loading conditions (b) notched plate subject to in-plane shear loading conditions (c) the plate FE mesh.
one layer of FE elements is used within \( R_c \). Its projection on the blue line shows that more layers, 4 layers of elements, are used within \( R_c \approx 0.046656 \), and so on for the red (11 layers) and black (15 layers) lines.

For mixed-mode problems, two examples, one of a slant crack and another of a slant notch, are considered. Fig. 6(a) illustrates a plate with a slant centre crack at angle of \( \beta = 45^\circ / C14 \). To compute the strain energy the plate is meshed using quadrilateral elements (CPS8) in ABAQUS as shown in Fig. 6(b). Like the previous example, the SIFs values are computed using the strain energy of different volumes around the crack tip. Coarse and fine meshes are used within \( R_0 = 0.6 \). The results are plotted in Fig. 7. It is clear again that better convergence is achieved by using finer meshes within \( R_0 \), i.e. smaller sizes of the control volume \( R_c < 0.1 \). This observation is in line with the general accepted size of the region governed by the singular terms around a crack tip, which is about \( a/10 \). The scaled SIFs predicted using the strain energy for a control volume of radius \( R_c = 0.00047019 \) (6 layers) when using 20 layers within \( R_0 = 0.6 \) (fine mesh) are \( K_I = 0.651 \) and \( K_{II} = 0.641 \). The SIFs values for this problem computed using the ABAQUS package are \( K_I = 0.655 \) and \( K_{II} = 0.640 \). In ABAQUS, quarter-point elements are used around the crack tip, and the plate mesh is the same as in Fig. 6(b). By using the fractal-like finite element method (FFEM), a method extended by the current authors to compute the notch SIFs, the SIFs values for this problem are \( K_I = 0.650 \) and \( K_{II} = 0.636 \). In FFEM, the plate is meshed using six-node triangular elements as shown in Fig. 8.

Now, a plate similar to the last example containing an inclined centre notch as shown in Fig. 9(a) is analysed. The notch opening angle is \( \gamma = 45^\circ / C14 \) and its length is \( 2a = 2 \). The plate dimensions are \( H = 2W = 10 \). The plate is meshed using CPS8 elements in ABAQUS as shown in Fig. 9(b). The SIFs values computed based on the strain energy values for different enclosed volumes around

![Figure 13](image13.png)

Fig. 13. (a) Notched plate subject to anti-plane shear loading conditions \( \gamma = 60^\circ / C14 \) (b) the plate FE mesh.

![Figure 14](image14.png)

Fig. 14. Mode III SIFs for the notched plate \( \gamma = 60^\circ / C14 \).

![Figure 15](image15.png)

Fig. 15. (a) Off-centre notched plate subject to anti-plane shear loading conditions (b) the plate FE mesh.
the notch tip (different \( R_c \)'s) are plotted in Fig. 10. Like the previous examples, using a finer mesh (and therefore smaller \( R_c \)) gives better convergence. The scaled SIFs predicted using the strain energy for a control volume of 4 layers (\( R_c = 0.000169262 \)) and 20 layers within \( R_0 = 0.6 \) are \( K_I = 0.657, K_II = 0.910 \). For this problem, Lazzarin et al. (2010)\(^1\) reported the following SIFs values \( K_I = 355, K_II = 325 \) not scaled \( (K_I = 0.641, K_II = 0.838 \) scaled \( ) \). Lazzarin et al. (2010) computed the mode I and II SIFs based on the strain energy density of two concentric circles. Using their approach, the authors computed the SIF values based on the strain energy of two concentric circles of radii \( R_c = 0.000169262 \) (3 layers) and \( R_c = 0.0002820407 \) (5 layers). The SIFs values obtained are \( K_I = 0.657, K_II = 0.910 \) which are in good agreement with our computed values using the SEA. The difference between the current results and those reported by Lazzarin et al. (2010) could be attributed to using different sizes of control volumes compared to the ones used in this paper.

Using the FFEM (Treifi et al., 2009a), the SIFs values for this example are \( K_I = 0.646, K_II = 0.912 \). In the FFEM, the plate is meshed as shown in Fig. 11 using six-node triangle elements. In the singular region, twenty layers of elements are used. The SIFs values predicted using the strain energy approach as described in Section 2.1.1 and the FFEM are in very good agreement. This proves that the current results are correct.

The approach based on two concentric circles is not always applicable, as the current authors did a test using two circles to compute the mixed mode SIFs for the previous crack example. The results obtained were totally unrealistic. This is due to the fact that the two concentric circles approach leads to an indeterminate system of equations in the case of a crack problem, i.e. a notch with a zero opening angle. However, the approach presented in this paper does not have this limitation. Therefore, the current procedure is a more general approach to compute mixed mode SIFs of a general notch including the special case of a crack. It should also be noted that Lazzarin et al. (2010) did not mention that their approach is applicable to mixed mode crack problems explicitly.

Different examples of notched plates with different notch opening angles and different locations under tension or shear loading conditions are presented in Tables 2 and 3. The notched plates and their FE meshes are similar to the ones shown in Fig. 12. The results are compared to those predicted by ABAQUS for the crack cases using the same mesh and the FFEM results reported by Treifi et al. (2008, 2009a) for crack and notch cases. The plate dimensions are \( H = 2W = 20 \), and the notch length is \( a \) where \( a/W = 0.4 \). Fine mesh of 20 layers of elements is used within \( R_0 = 0.6 \), and the strain energy used to predict the SIF values is computed for a volume of radius \( R_c = 0.000101566 \) (that is for 3 layers). The results are presented in Tables 2 and 3 and are in good agreement with the results predicted using the other numerical approaches. The accuracy of the SEA could be improved by computing the strain energy for different volumes and then looking at the converged regions as demonstrated in the previous examples.

### 3.1.2. Mode III

A convergence study of an edge notched plate subject to mode III loading conditions as shown in Fig. 13 is presented to demonstrate the effect of \( R_c \) on the accuracy of the SEA. The plate is modelled and analysed using different meshes within the region around the notch-tip. The plate dimensions are: \( H = 2W = 20 \), the plate thickness \( t = 1 \), the notch length \( a \) where \( a/W = 0.4 \), and the notch opening angle \( \gamma = 60^\circ \). Three dimensional FE elements (C3D20) are used to model the plate in ABAQUS in order to compute the strain energy, and anti-plane conditions are applied. The small region around the notch tip is meshed by layer with a similarity ratio \( \rho = 0.6 \). The radius of the first layer is taken as \( R_0 = 0.6 \). Initially, six layers are used within \( R_0 \). The mode I SIF values are computed based on the energy values for control volumes of sizes ranging from 1 layer to 6 layers (that is, the control radius is \( R_c = 0.046656 \)–0.6). The same is repeated for \( 9 \ (R_c = 0.0010777 \)–0.6), \( 16 \ (R_c = 0.000282168 \)–0.6) and \( 20 \ (R_c = 0.0000365376 \)–0.6). The value of \( R_0 \) is not changed. The mode III SIF values are plotted in Fig. 14. For the parameters considered \( (N_l = 6, 9, 16, 20) \), this figure shows clearly that convergence is achieved regardless of the mesh around the notch tip (within \( R_0 \)) and radius of the control volume \( (R_c) \). The converged SIF value (scaled by \( \tau \sqrt{a^3/\rho} \)) predicted by the SEA is \( K_III = 1.418 \). The mode III SIF for this case reported by Treifi et al. (2009b) using the FFEM is \( K_III = 1.417 \).

Different examples of notched plates with different notch opening angles and different locations under anti-plane loading conditions are analysed. The notched plates and their FE meshes are similar to the ones shown in Fig. 15. The results are compared in Table 4 to those predicted by ABAQUS for the crack cases using the same mesh and the FFEM results for the crack and notch cases reported by Treifi et al. (2009b) (the values are scaled by \( \tau \sqrt{a^3/\rho} \)). In the FFEM, the plate is meshed using 6-node triangular elements similar to the mesh shown in Fig. 16. The plate dimension

#### Table 4

Scaled SIFs \( (K_III/\tau \sqrt{a^3/\rho}) \) under anti-plane shear loading conditions.

<table>
<thead>
<tr>
<th>( (\rho/a) )</th>
<th>( h_o/H )</th>
<th>0.2</th>
<th>0.3</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>SEA</td>
<td>1.117</td>
<td>1.087</td>
<td>1.077</td>
</tr>
<tr>
<td>(0.5)</td>
<td>ABAQUS</td>
<td>1.117</td>
<td>1.087</td>
<td>1.077</td>
</tr>
<tr>
<td>30</td>
<td>SEA</td>
<td>1.282</td>
<td>1.245</td>
<td>1.234</td>
</tr>
<tr>
<td>(0.545455)</td>
<td>ABAQUS</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>60</td>
<td>SEA</td>
<td>1.478</td>
<td>1.432</td>
<td>1.417</td>
</tr>
<tr>
<td>(0.6)</td>
<td>ABAQUS</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>90</td>
<td>SEA</td>
<td>–</td>
<td>1.648</td>
<td>1.628</td>
</tr>
<tr>
<td>(0.666667)</td>
<td>ABAQUS</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Fig. 16. The FE mesh used in the FFEM of off-centre notched plate subject to anti-plane shear loading conditions.
A coarse mesh of 7 layers of elements is used within \( R_0 = 0.6 \), and the strain energy used to predict the SIF values is computed for a volume of radius \( R_c = 0.07776 \) (that is for 3 layers). The SEA results presented in Table 4 are in good agreement with the results predicted using ABAQUS for crack cases and the FFEM for crack and notch cases. The accuracy of the SEA is excellent when dealing with pure mode cases, so for those cases finer meshes are not necessary.

3.2. Bi-material notch

3.2.1. Mode I, II and mixed mode

To demonstrate the effect of \( R_c \) on the accuracy of SIF values predicted using the SEA relationships presented in Section 2.2.1 for a bi-material notch, an edge cracked plate consisting of two parts as shown in Fig. 17(a) is analysed for different material property ratios. The convergence study presented in Section 3.1.1 for single material notch cases demonstrated the need for a fine mesh around the notch-tip to obtain high accuracy results for mixed-mode I and II cases. Therefore, 20 layers will be used within the small region \( (R_0 = 0.6) \) containing the crack tip. The plate is meshed using CPS8 elements in ABAQUS as shown in Fig. 17(b). The cracked plate dimensions are \( H = 3W = 30 \), and the crack length \( a \) is given as \( a/W = 0.4 \). The Poisson’s ratios of both materials are taken as \( \nu_1 = \nu_2 = 0.3 \). The Young Modulus ratios considered are \( E_1/E_2 = 1, 2, 4, 10, 100 \). The real and imaginary parts of the complex SIF, representing Mode I and II SIFs, are computed.
using different values of \( R_c \) ranging from \( R_c = 0.0000365376 \) to 0.6 (that is, for 1 layer to 20 layers). The SIF values are plotted in Figs. 18–22. Because Eq. (62) gives two sets of valid roots, both sets are plotted. Corresponding SIFs values predicted by Matsumoto et al. (2000) are also plotted for comparison. Those figures show that convergence is achieved for only one set of the roots for \( E_1/E_2 = 1, 2, 4 \) (small differences in material properties). However, for large differences in material properties as is the case in \( E_1/E_2 = 10, 100 \), both sets of roots converge within different regions. When one of the sets converges, the other diverges. In addition, for small material property differences \( E_1/E_2 = 1, 2, 4 \), convergence is achieved using small \( R_c \), but for large material property differences \( E_1/E_2 = 100 \) convergence is achieved using large \( R_c \). In the cases studied here, good accuracy is achieved using \( R_c = 0.000282109 \) (that is, the control volume containing 5 layers) for the cases of small material property differences. For the cases of large material property differences, better convergence is achieved using \( R_c = 0.0167961 \) (that is, the control volume containing 13 layers).

Based on the above discussion, the SIFs for an edge-crack bi-material plate are computed for different crack lengths and different material property ratios. The cracked plate and its FE mesh are similar to those shown in Fig. 17. The plate dimensions are the same as of the previous example. 20 layers are used within the small region \((R_c = 0.6)\) containing the crack tip. The scaled SIF values \((K_i/\sigma \sqrt{\pi a^{1-R_c(2a)^{4(i)}}})\) computed using the SEA are tabulated in Table 5. Corresponding published results by Matsumoto et al. (2000) and computed results using ABAQUS are also tabulated for comparison. In ABAQUS, the mesh is used to compute the SIFs. The control volume radius is taken as \( R_c = 0.000282 \) (5 layers), \( R_c = 0.001306 \) (8 layers), \( R_c = 0.0006047 \) (11 layers) and \( R_c = 0.0016796 \) (13 layers) for \( E_1/E_2 = 1, 2, 4, 10, 100 \), respectively. The singular eigenvalues for a bi-material crack of \( E_1/E_2 = 1, 2, 4, 10, 100 \) are \( \lambda = 0.5, \lambda = 0.5 + 0.037306, \lambda = 0.5 + 0.0678545, \lambda = 0.5 + 0.0937743 \) and \( \lambda = 0.5 + 0.113817 \), respectively. Table 5 shows that the SEA results are in good agreement with the numerical results computed using ABAQUS and with those reported by Matsumoto et al. (2000).

**Table 5**
Scaled SIFs for a bio-material cracked plate.

<table>
<thead>
<tr>
<th>( a/W )</th>
<th>( E_1/E_2 )</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>10</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_i )</td>
<td>( K_a )</td>
<td>( K_i )</td>
<td>( K_a )</td>
<td>( K_i )</td>
<td>( K_a )</td>
<td>( K_i )</td>
</tr>
<tr>
<td>0.2</td>
<td>SEA</td>
<td>1.367</td>
<td>0.000</td>
<td>1.365</td>
<td>0.138</td>
<td>1.364</td>
</tr>
<tr>
<td></td>
<td>ABAQUS</td>
<td>1.368</td>
<td>0.000</td>
<td>1.368</td>
<td>0.137</td>
<td>1.368</td>
</tr>
<tr>
<td></td>
<td>Matsumto</td>
<td>–</td>
<td>–</td>
<td>1.367</td>
<td>0.137</td>
<td>1.368</td>
</tr>
<tr>
<td>0.3</td>
<td>SEA</td>
<td>1.660</td>
<td>0.000</td>
<td>1.657</td>
<td>0.160</td>
<td>1.652</td>
</tr>
<tr>
<td></td>
<td>ABAQUS</td>
<td>1.661</td>
<td>0.000</td>
<td>1.659</td>
<td>0.159</td>
<td>1.654</td>
</tr>
<tr>
<td></td>
<td>Matsumto</td>
<td>–</td>
<td>–</td>
<td>1.657</td>
<td>0.156</td>
<td>1.655</td>
</tr>
<tr>
<td>0.4</td>
<td>SEA</td>
<td>2.112</td>
<td>0.000</td>
<td>2.107</td>
<td>0.199</td>
<td>2.099</td>
</tr>
<tr>
<td></td>
<td>ABAQUS</td>
<td>2.112</td>
<td>0.000</td>
<td>2.109</td>
<td>0.198</td>
<td>2.101</td>
</tr>
<tr>
<td></td>
<td>Matsumto</td>
<td>–</td>
<td>–</td>
<td>2.109</td>
<td>0.195</td>
<td>2.102</td>
</tr>
<tr>
<td>0.5</td>
<td>SEA</td>
<td>2.827</td>
<td>0.000</td>
<td>2.819</td>
<td>0.268</td>
<td>2.805</td>
</tr>
<tr>
<td></td>
<td>ABAQUS</td>
<td>2.826</td>
<td>0.000</td>
<td>2.821</td>
<td>0.268</td>
<td>2.807</td>
</tr>
<tr>
<td></td>
<td>Matsumto</td>
<td>–</td>
<td>–</td>
<td>2.819</td>
<td>0.268</td>
<td>2.806</td>
</tr>
<tr>
<td>0.6</td>
<td>SEA</td>
<td>4.037</td>
<td>0.000</td>
<td>4.024</td>
<td>0.396</td>
<td>4.000</td>
</tr>
<tr>
<td></td>
<td>ABAQUS</td>
<td>4.035</td>
<td>0.000</td>
<td>4.025</td>
<td>0.398</td>
<td>4.002</td>
</tr>
<tr>
<td></td>
<td>Matsumto</td>
<td>–</td>
<td>–</td>
<td>4.024</td>
<td>0.398</td>
<td>4.001</td>
</tr>
<tr>
<td>0.7</td>
<td>SEA</td>
<td>6.363</td>
<td>0.000</td>
<td>6.338</td>
<td>0.665</td>
<td>6.288</td>
</tr>
<tr>
<td></td>
<td>ABAQUS</td>
<td>6.357</td>
<td>0.000</td>
<td>6.336</td>
<td>0.671</td>
<td>6.291</td>
</tr>
<tr>
<td></td>
<td>Matsumto</td>
<td>–</td>
<td>–</td>
<td>6.348</td>
<td>0.668</td>
<td>6.298</td>
</tr>
</tbody>
</table>

![Fig. 22](image-url) SIFs for the bi-material cracked plate \( E_1/E_2 = 100 \).

![Fig. 23](image-url) (a) A bi-material notched plate subject to tension (b) the plate FE mesh.
For a notch case, a bi-material notched plate with an opening angle of $\gamma = 60^\circ$ as shown in Fig. 23(a) is analysed. The plate dimensions are $H = 2W = 20$, and the crack length $a$ is taken as $a/W = 0.4$. The Poisson's ratios of both material are taken as $\nu_1 = \nu_2 = 0.3$. The SIFs are computed for different material property ratios $E_1/E_2 = 1.2, 4, 10, 100$. The plate is meshed using CPS8 elements in ABAQUS as shown in Fig. 23(b) to compute the strain energy. 20 layers are used within the small region ($R_0 = 0.6$) containing the crack tip. The results are tabulated in Table 6. The control volume radius for each case are either two real eigenvalues or one complex eigenvalue. The cases of two singular real eigenvalues give two real SIFs $K_I$ and $K_{II}$ for mode I and II, respectively. The cases of one singular complex eigenvalue give complex SIFs $K_c = K_I + iK_{II}$. The results in Table 6 are new and there are no available published results to compare with. However, the previous validation for crack cases, which are special cases of notch problems with opening angle of $\gamma = 0^\circ$, shows that the SEA gives accurate results. Therefore, the results in Table 6 are valid. It should be noted that the ABAQUS software only computes SIF values when the notch opening angle is zero, i.e. a crack, but for true notches it cannot compute the notch SIFs.

### 3.2.2. Mode III

The mode III SIFs for different cases of a bi-material notched plate subject to out-of-plane shear loading conditions as shown in Fig. 24 are computed using the SEA. The plate dimensions are: $H = 2W = 20$, the plate thickness $t = 1$, the notch length is $a$ where $a/W = 0.4$. Three dimensional FE elements (C3D20) are used to model the plate in ABAQUS in a similar way to that in Fig. 15(b) in order to compute the strain energy, and anti-plane conditions are applied. The results are compared to those predicted by ABAQUS for the crack cases and the FFEM (Treifi and Oyadiji, 2013) for the crack and notch cases. In the ABAQUS analysis for the crack cases, the same mesh as for the SEA is used. For the notch cases where ABAQUS cannot predict the notch SIFs, the results are compared to those predicted by the FFEM. In the FFEM, the plate is meshed using 6-node triangular elements similar to the mesh shown in Fig. 16. Based on the convergence study presented for

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$h_0/H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>FFE</td>
</tr>
<tr>
<td></td>
<td>ABAQUS</td>
</tr>
<tr>
<td>$30$</td>
<td>FFE</td>
</tr>
<tr>
<td></td>
<td>ABAQUS</td>
</tr>
<tr>
<td>$60$</td>
<td>FFE</td>
</tr>
<tr>
<td></td>
<td>ABAQUS</td>
</tr>
<tr>
<td>$90$</td>
<td>FFE</td>
</tr>
<tr>
<td></td>
<td>ABAQUS</td>
</tr>
</tbody>
</table>

### Table 7

Scaled SIFs ($K_{III}/t/\sqrt{a}^{1.5}$) under anti-plane shear loading conditions ($G_1/G_2 = 10$).

### Table 8

Scaled SIFs ($K_{III}/t/\sqrt{a}^{1.5}$) under anti-plane shear loading conditions ($G_1/G_2 = 4$).

### Table 9

Scaled SIFs ($K_{III}/t/\sqrt{a}^{1.5}$) under anti-plane shear loading conditions ($G_1/G_2 = 1/4$).

Fig. 24. A bi-material notched plate subject to anti-plane shear loading conditions.
the mode III isotropic cases, a coarse mesh of 7 layers of elements is used within \( R_0 = 0.6 \), and the strain energy used to predict the SIF values is computed for a volume of radius \( R = 0.07776 \) (3 layers). The SEA results compared to corresponding results predicted using ABAQUS and the FFEM are presented in Tables 7–10. The results of the three different approaches are in excellent agreement.

4. Conclusions

In this paper, a simple approach based on the strain energy of a control volume was developed to compute the mode I, II and III SIFs for isotropic homogeneous and bi-material crack/notch problems. The approach is simple to employ numerically. It relates the SIFs to the strain energy that may be computed using commercial FE packages; thus, enabling those packages to compute notch SIFs. The accuracy of the SEA was demonstrated via many different numerical examples of homogeneous and bi-material cracked and notched plates. For pure mode conditions, a coarse mesh (and therefore a larger size of the control volume) may be used to model the region around the notch tip, but it is recommended using finer meshes (and therefore a smaller size of the control volume) when dealing with mixed mode cases. The results generated using the SEA are in very good agreement with existing published results and numerical solutions.

References


Table 10

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( h_0/H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.083</td>
</tr>
<tr>
<td>SEA</td>
<td>1.078</td>
</tr>
<tr>
<td>FFE</td>
<td>1.077</td>
</tr>
<tr>
<td>ABAQUS</td>
<td>1.077</td>
</tr>
<tr>
<td>30</td>
<td>1.241</td>
</tr>
<tr>
<td>SEA</td>
<td>1.234</td>
</tr>
<tr>
<td>FFE</td>
<td>1.234</td>
</tr>
<tr>
<td>ABAQUS</td>
<td>–</td>
</tr>
<tr>
<td>60</td>
<td>1.426</td>
</tr>
<tr>
<td>SEA</td>
<td>1.418</td>
</tr>
<tr>
<td>FFE</td>
<td>1.417</td>
</tr>
<tr>
<td>ABAQUS</td>
<td>–</td>
</tr>
<tr>
<td>90</td>
<td>–</td>
</tr>
<tr>
<td>SEA</td>
<td>1.629</td>
</tr>
<tr>
<td>FFE</td>
<td>1.628</td>
</tr>
<tr>
<td>ABAQUS</td>
<td>–</td>
</tr>
</tbody>
</table>


