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A Nonlinear Volterra Equation Arising in the Theory of Superfluidity*

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1. In connection with his recently developed theory of superfluidity C. C. Lin [1] has formulated a boundary value problem which in certain situations becomes the following:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad x > 0, \qquad t > 0, \tag{1.1}$$

$$u(x,0) = 0 \tag{1.2}$$

$$\frac{\partial u}{\partial x}(0,t) = k\,[u(0,t) - c \sin t]^3, \qquad t > 0$$

where $k > 0$. This problem is somewhat more complicated than one arising in connection with the heat transfer between solids and gases under nonlinear boundary conditions by Mann and Wolf [2] but like that problem can be replaced by a nonlinear Volterra type integral equation.

Here the existence theory will be considered for (1.1) and (1.2) subject to

$$\frac{\partial u}{\partial x}(0,t) = \Phi[u(0,t) - f(t)], \qquad t > 0 \tag{1.3}$$

where Φ and $f(t)$ are assumed to be continuous. Moreover $\Phi(y)$ is monotone increasing and $\Phi(0) = 0$. The case where $f(t)$ is periodic will be discussed and it will be shown that in that case there exists a continuous periodic function $\phi(t)$ such that

$$\lim_{t \to \infty} [u(0,t) - \phi(t)] = 0.$$

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If the problem (1.1), (1.2) and (1.3) has a solution $u(x,t)$ which is continuous for $x \geqslant 0$, $t \geqslant 0$ and if $\partial u/\partial x(x,t)$ is continuous for $t > 0$ and bounded as $t \to +0$ then it can be readily verified that with

$$\psi(t) = \frac{\partial u}{\partial x}(0,t), \tag{1.4}$$

$$u(x,t) = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{\psi(\eta)}{\sqrt{t-\eta}} \exp\left(\frac{-x^2}{4(t-\eta)}\right) d\eta \tag{1.5}$$

is a solution of (1.1), (1.2) and (1.4).

From (1.5) follows

$$u(0,t) = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{\psi(\eta)}{\sqrt{t-\eta}} d\eta.$$

Combining this with (1.3) and (1.4)

$$u(0,t) = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{\Phi[u(0,\eta) - f(\eta)]}{\sqrt{t-\eta}} d\eta. \tag{1.6}$$

It is with (1.6) that we shall be concerned here. Once it is shown to have a solution $u(0,t)$, then $\partial u/\partial x(0,t)$ is determined by (1.3) which in turn determines $\psi(t)$ in (1.4) and $u(x,t)$ in (1.5).

The integral equation (1.6) is a special case of that treated by Padmavally in [3] and Theorem 1.1 which follows is more restricted than the results of [3]. However the lemmas used in proving it are needed for Theorems 1.2 and 1.3 in any case.

If in (1.6) we define

$$F(t) = u(0,t) - f(t) \tag{1.7}$$

the (1.6) becomes

$$F(t) + f(t) = -\pi^{-1/2} \int_0^t \Phi(F(\eta)) (t-\eta)^{-1/2} d\eta. \tag{1.8}$$

A function $f(t)$ is said to satisfy a Lipschitz condition of order $\alpha > 0$ at t if there exists a constant K such that

$$|f(t+h) - f(t)| \leqslant K|h|^\alpha \tag{1.9}$$

for small $|h|$.

THEOREM 1.1. *Let $f(t)$ be continuous for $0 \leqslant t < \infty$ and on any finite interval let f satisfy a uniform Lipschitz condition of order $\beta > 0$. Let $\Phi(y)$ be monotone increasing, $\Phi(0) = 0$, and for any $y_0 > 0$ let there exist a constant $K(y_0)$ such that*

$$|\Phi(y_2) - \Phi(y_1)| \leqslant K(y_0)|y_2 - y_1| \tag{1.10}$$

for $|y_1|$ and $|y_2| \leqslant y_0$. Then (1.8) possesses a unique continuous solution $F(t)$ an $(0, \infty)$.

THEOREM 1.2. *In addition to the hypothesis of Theorem 1.1 the further assumptions are made that $f(t)$ has period ω and that with* max $|f(t)| = M$, *there is a positive monotone increasing function $k(u)$ for $u > 0$ such that*

$$\Phi(y_2) - \Phi(y_1) \geqslant k(y_2 - y_1) \tag{1.11}$$

for $y_2 - y_1 > 0$ and $|y_1|$ and $|y_2| \leqslant 2M$. Then there is a continuous periodic function $\phi(t)$ of period ω and the solution $F(t)$ of (1.8) satisfies

$$\lim_{t\to\infty} [F(t) + f(t) - \phi(t)] = 0 \tag{1.12}$$

Moreover

$$|F(t) + f(t)| \leqslant \max |f(t)|. \tag{1.13}$$

Note that in the earlier formulation $u(0,t) = F(t) + f(t)$ and hence (1.12) and (1.13) imply

$$\lim_{t\to\infty} [u(0,t) - \phi(t)] = 0$$

and

$$|u(0,t)| \leqslant \max |f(t)|.$$

THEOREM 1.3. *The periodic function $\phi(t)$ is a solution of*

$$\frac{1}{2}\int_0^\infty [\phi(t) - \phi(t-s)]s^{-3/2}\,ds = -\pi^{1/2}\,\Phi[\phi(t) - f(t)] \tag{1.14}$$

and $\phi(t)$ is uniformly Lipschitz of order exceeding $1/2$. The integral equation (1.14) has no other continuous periodic solution than $\phi(t)$. The average of $\Phi(\phi(t) - f(t))$ is zero.

2. It will be convenient to formulate several lemmas.

LEMMA 2.1. *If $p(s)$ is piecewise continuous for $0 < s < c$ and if $|p(s)| \leqslant m$ then*

$$q(t) = \int_0^t p(s)\,(t-s)^{-1/2}\,ds \tag{2.1}$$

is continuous on $0 \leqslant t \leqslant c$ *and indeed*

$$|q(t_2) - q(t_1)| \leqslant 4m|t_2 - t_1|^{1/2}$$

for t_1, t_2 *on* $[0,c]$.

PROOF. Let $h > 0$. The proof of this well known result follows from

$$\begin{aligned} |q(t+h) - q(t)| &\leqslant \int_0^t |p(s)| \, [(t-s)^{-1/2} - (t+h-s)^{-1/2}] \, ds \\ &+ \int_t^{t+h} |p(s)| \, (t+h-s)^{-1/2} \, ds \\ &\leqslant 2m [t^{1/2} - (t+h)^{1/2} + h^{1/2}] + 2mh^{1/2} \\ &\leqslant 4mh^{1/2} \end{aligned}$$

LEMMA 2.2. *Let* $p(s)$ *satisfy Lemma* 2.1 *and for some* α, $0 < \alpha < 1$, *let*

$$|p(s_2) - p(s_1)| \leqslant K|s_2 - s_1|^{\alpha}$$

on $0 < s_1, s_2 < c$. *Then for* $0 < t < c$ *and* $h > 0$

$$|q(t) - q(t-h)| \leqslant 4(m+K)h^{(1+\alpha)/2} \tag{2.2}$$

where $h < \min\left(1, (1/3\, t)^{1/(1-\alpha)}\right)$.

PROOF. Let $a = 2h^{1-\alpha}$. Clearly

$$\begin{aligned} |q(t) - q(t-h)| &\leqslant \int_0^{t-a-h} |p(s)| \, [(t-h-s)^{-1/2} - (t-s)^{-1/2}] \, ds \\ &+ \int_{t-a-h}^{t-a} |p(s)| \, (t-s)^{-1/2} \, ds \\ &+ \int_0^a |p(t-h-\sigma) - p(t-\sigma)| \sigma^{-1/2} \, d\sigma \\ &\leqslant 2mha^{-1/2} + 2Kh^{\alpha} a^{1/2} \\ &\leqslant 4(m+K)h^{(1+\alpha)/2}. \end{aligned}$$

LEMMA 2.3. *Let Φ satisfy* (1.10). *For some $c > 0$ let $\gamma(t)$ be continuous on $0 \leqslant t \leqslant c$. Then there exists $b > 0$ such that the integral equation*

$$g(t) = \gamma(t) - \pi^{-1/2} \int_0^t \Phi(g(s))\,(t-s)^{-1/2}\,ds \tag{2.3}$$

has a unique continuous solution on $0 \leqslant t \leqslant b$.

PROOF. Let $\max |\gamma(t)|$ on $[0,c]$ be m. Let $y_0 = 2m$ and let $K(y_0) = K_0$. Choose b so that $b < c$ and $2K_0(b/\pi)^{1/2} < \frac{1}{2}$.

Then it follows by a standard successive approximation procedure, starting with $g_0 = 0$, that (2.3) has a continuous solution $g(t)$. Uniqueness on $[0,b]$ also follows from (1.10).

LEMMA 2.4. *Let p and q be continuous for $0 \leqslant t < c$ and for each t, $0 < t < c$, let p satisfy a Lipschitz condition of order exceeding zero and for each t for some K_1, $\delta > 0$ and sufficiently small $h > 0$ let*

$$|q(t) - q(t-h)| \leqslant K_1 h^{1/2+\delta} \tag{2.4}$$

If (2.1) *holds for $0 \leqslant t < c$ then*

$$t^{-1/2} q(t) + \tfrac{1}{2} \int_0^t [q(t) - q(\sigma)]\,(t-\sigma)^{-3/2}\,ds = \pi p(t)$$

for $0 < t < c$.

PROOF. The usual analytic continuation procedure can be used. Let $z = x + iy$. For $\operatorname{Re} z > 0$ it follows from (2.1) for $t < c$

$$\int_0^t q(\sigma)\,(t-\sigma)^{z-1} d\sigma = \int_0^t p(s)\,ds \int_0^t (t-\sigma)^{z-1}\,(\sigma - s)^{-1/2}\,d\sigma$$

or multiplying by z

$$z \int_0^t q(\sigma)\,(t-\sigma)^{z-1}\,d\sigma = \frac{\Gamma(z+1)\,\Gamma(\frac{1}{2})}{\Gamma(z+\frac{3}{2})}\,(z+\tfrac{1}{2}) \int_0^t p(s)\,(t-s)^{z-1/2}\,ds.$$

This can be written as

$$q(t)t^z + z \int_0^t [q(\sigma) - q(t)]\,(t-\sigma)^{z-1}\,d\sigma$$

$$= \frac{\Gamma(z+1)\,\Gamma(\frac{1}{2})}{\Gamma(z+\frac{3}{2})} \left(p(t)t^{z+1/2} + (z+\tfrac{1}{2}) \int_0^t [p(s) - p(t)]\,(t-s)^{z-1/2}\,ds \right).$$

In view of (2.4) and the Lipschitz condition on p, the integrals above converge uniformly with respect to z for $\operatorname{Re} z \geqslant -\frac{1}{2}$ and hence both sides are analytic for $\operatorname{Re} z \geqslant -\frac{1}{2}$. By analytic continuation then the equation must be valid for $\operatorname{Re} z \geqslant -\frac{1}{2}$ and in particular for $z = -\frac{1}{2}$ which completes the proof.

3. Here proofs will be given for Theorems 1.1, 1.2 and 1.3.

PROOF of THEOREM 1.1. Suppose (1.8) has a continuous solution for $0 \leqslant t < a < \infty$ but not on any larger interval $0 \leqslant t < a + \delta$ where $\delta > 0$. By Lemma 2.3 it is clear that $0 < a$. Hence either $a < \infty$ or else a continuous solution exists for $0 \leqslant t < \infty$. Thus to prove the theorem it is only necessary to show that $a < \infty$ is impossible.

Let max $|f(t)|$ on $[0,a]$ be M_1. Let $H(t) = F(t) + f(t)$. Then it will be shown that $|H(t)| \leqslant M_1$ for $0 \leqslant t < a$. Suppose this is false. Then there exists $t_1 < a$ such that $|H(t_1)| > M_1$. Let max $|H(t)|$ on $[0,t_1]$ be M_0 and let it be assumed at $t_0 \leqslant t_1$. Let $t_2 = (t_1 + a)/2$ and let max $|H(t)|$ on $[0,t_2]$ be M_2. Clearly $t_0 < t_2$.

It is convenient to write (1.8) as

$$H(t) = -\pi^{-1/2} \int_0^t \Phi\big(H(s) - f(s)\big)\,(t-s)^{-1/2}\,ds \tag{3.1}$$

Since $|H - f| \leqslant M_1 + M_2$ on $[0,t_2]$ it follows from Lemma 2.1 that H satisfies a Lipschitz condition of order $\frac{1}{2}$ on $[0,t_2]$. Since f satisfies a Lipschitz condition of order $\beta > 0$, (1.10) and Lemma 2.2 simply that H satisfies a Lipschitz condition ot the type (2.2) at each t on $0 < t < t_2$. Hence by Lemma 2.4 for $0 < t < t_2$, (3.1) yields

$$H(t)t^{-1/2} + \tfrac{1}{2} \int_0^t [H(t) - H(\sigma)]\,(t-\sigma)^{-3/2}\,d\sigma = -\pi^{1/2}\Phi[H(t) - f(t)]. \tag{3.2}$$

Putting $t = t_0$ then $H(t_0) = M_0$ (or else $-M_0$ which is treated similarly). Thus

$$H(t_0) - H(\sigma) \geqslant 0, \qquad 0 \leqslant \sigma \leqslant t_0.$$

Thus the left side of (3.2) is positive. Since $H(t_0) = M_0 > M_1 > 0$, $\Phi(H(t_0) - f(t_0)) \geqslant \Phi(M_0 - M_1) \geqslant 0$ and so the right side is not positive. This is impossible. Hence on $[0,a)$, $|H(t)| \leqslant \max |f(t)|$. In Lemma 2.1 this implies that $H(t)$ is uniformly continuous on $[0,a]$, (and indeed uniformly Lipschitz $\frac{1}{2}$). Hence $H(a-0)$ exists and so $F(t)$ exists as a continuous bounded solution of (1.8) on $0 \leqslant t \leqslant a$.

Next let $t = \tau + a$. Then (1.8) becomes

$$F(\tau + a) + f(\tau + a) = -\pi^{-1/2} \int_0^a \Phi(F(\sigma))(\tau + a - \sigma)^{-1/2} d\sigma$$

$$- \pi^{-1/2} \int_0^\tau \Phi[F(a + \sigma)](\tau - \sigma)^{-1/2} d\sigma. \quad (3.3)$$

If

$$\gamma(\tau) = -f(\tau + a) - \pi^{-1/2} \int_0^a \Phi(F(\sigma))(\tau + a - \sigma)^{-1/2} d\sigma$$

then by Lemma 2.3, $F(\tau + a)$ exists as a continuous solution for some $b > 0$, $0 \leqslant \tau \leqslant b$. Hence $F(t)$ is continuous beyond $t = a$. This proves that $a < \infty$ is impossible.

The uniqueness also follows from the formulation (3.3) since if a unique continuous solution exists for $0 \leqslant t \leqslant a$ (here $a = 0$ is allowed) then Lemma 2.3 applied to (3.3) shows that it is also unique on some interval to the right of a. Hence the assumption that uniqueness holds only on a finite interval leads to a contradiction. This proves Theorem 1.1.

Proof of Theorem 1.2. Since $|f|$ is periodic it is bounded on $[0, \infty]$. Denoting its l.u.b. by M_1, $|H| \leqslant M_1$ as was shown in the proof of Theorem 1.1. Let i and j be integers and suppose for $0 \leqslant t \leqslant 2\omega$ and $j \geqslant i \to \infty$

$$\limsup\, [H(t + j\omega) - H(t + i\omega)] = \lambda > 0.$$

Clearly $\lambda \leqslant 2M_1$. Given $\varepsilon > 0$ there exists (t_n, i_n, j_n), $n = 1,2,\ldots$ such that

$$0 \leqslant t_n \leqslant 2\omega, \qquad j_n \geqslant i_n \to \infty \qquad \text{as} \qquad n \to \infty$$

and, for $\varepsilon < \lambda$,

$$H(t_n + j_n\omega) - H(t_n + i_n\omega) > \lambda - \varepsilon > 0. \quad (3.4)$$

For $t \geqslant 0$, there exists $i_0(\varepsilon)$ such that for $j \geqslant i \geqslant i_0(\varepsilon)$

$$H(t + j\omega) - H(t + i\omega) < \lambda + \varepsilon. \quad (3.5)$$

Using $t = t_n + j_n\omega$ and $t = t_n + i_n\omega$ in (3.2) and subtracting gives

$$A_1 + \tfrac{1}{2} \int_0^{t_n + j_n\omega} [H(t_n + j_n\omega) - H(t_n + j_n\omega - s)]s^{-3/2}\, ds$$

$$- \tfrac{1}{2} \int_0^{t_n + i_n\omega} [H(t_n + i_n\omega) - H(t_n + i_n\omega - s)]s^{-3/2}\, ds$$

$$= - \pi^{1/2}\Phi[H(t_n + j_n\omega) - f(t_n + j_n\omega)]$$

$$+ \pi^{1/2}\Phi[H(t_n + i_n\omega) - f(t_n + i_n\omega)] \tag{3.6}$$

where since $|H| \leqslant M_1$

$$|A_1| \leqslant 2M_1(i_n\omega)^{-1/2}. \tag{3.7}$$

Since $|\Phi(H - f)| \leqslant \Phi(2M_1)$ it follows from (3.1) and Lemma (2.1) that H is uniformly Lipschitz $\frac{1}{2}$ on $[0, \infty]$. Since f is periodic and uniformly Lipschitz $\beta > 0$ on any finite interval it is uniformly Lipschitz β on $[0, \infty]$. (It is no restriction to assume $\beta < \frac{1}{2}$.) Using Lemma 2.2 on (3.1) it now follows that there is a K_2 such that

$$|H(t) - H(t - s)| \leqslant K_2 s^{(1+\beta)/2} \tag{3.8}$$

for large t and $0 \leqslant s < 1$. Hence

$$\int_0^{\varepsilon} |H(t_n + j_n\omega) - H(t_n + j_n\omega - s|s^{-3/2}\, ds \leqslant 2K_2\varepsilon^{\beta/2}/\beta \tag{3.9}$$

and similarly with j_n replaced by i_n. Hence (3.6) gives

$$\tfrac{1}{2} \int_{\varepsilon}^{i_n\omega/2} \{H(t_n + j_n\omega) - H(t_n + i_n\omega)$$

$$- [H_n(t_n + j_n\omega - s) - H(t_n + i_n\omega - s\}s^{-3/2}\, ds$$

$$+ A_1 + A_2 + A_3 = - \pi^{1/2}\, \Phi[H(t_n + j_n\omega) - f(t_n + j_n\omega)]$$

$$+ \pi^{1/2}\, \Phi[H(t_n + i_n\omega) - f(t_n + i_n\omega)] \tag{3.10}$$

where by (3.9)

$$|A_2| \leqslant 4K_2\varepsilon^{\beta/2}/\beta \tag{3.11}$$

and A_3 is the sum of two terms each dominated by

$$\frac{1}{2}\int_{i_n\omega/2}^{t_n+j_n\omega} 2M_1 s^{-3/2}\,ds \leqslant 2M_1(i_n\omega)^{-1/2}$$

and hence

$$|A_3| \leqslant 4M_1(i_n\omega)^{-1/2} \tag{3.12}$$

If n is large enough then by (3.5)

$$H_n(t_n + j_n\omega - s) - H(t_n + i\omega_n - s) < \lambda + \varepsilon$$

for $s \leqslant i_n\omega/2$. Using this and (3.4), the integral on the left of (3.10) is greater than $-2\varepsilon^{1/2}$. Hence

$$\begin{aligned}&-2\varepsilon^{1/2} + A_1 + A_2 + A_3\\ &\leqslant -\pi^{-1/2}\{[\Phi(H(t_n + j_n\omega) - f(t_n + j_n\omega))\\ &-\Phi(H(t_n + i_n\omega) - f(t_n + i_n\omega))]\}.\end{aligned}$$

By (1.11) and (3.4) this gives

$$\begin{aligned}&-2\varepsilon^{1/2} + A_1 + A_2 + A_3\\ &\leqslant -\pi^{-1/2}k[H(t_n + j_n\omega) - H(t_n + i_n\omega)].\end{aligned}$$

By (3.4)

$$2\varepsilon^{1/2} + |A_1| + |A_2| + |A_3| \geqslant \pi^{-1/2}k(\lambda - \varepsilon).$$

By (3.7), (3.11) and (3.12)

$$2\varepsilon^{1/2} + 6M_1(i_n\omega)^{-1/2} + 4K_2\varepsilon^{\beta/2}/\beta \geqslant \pi^{-1/2}k(\lambda - \varepsilon).$$

Letting $n \to \infty$ and then noting that ε is arbitrary it follows that $\lambda \leqslant 0$. A similar procedure holds for lim inf. Hence for $0 \leqslant t \leqslant 2\omega$ and $j \geqslant i \to \infty$

$$\lim\,[H(t + j\omega) - H(t + i\omega)] = 0.$$

Hence there exists $\phi(t)$ such that

$$\lim_{j\to\infty} H(t + j\omega) = \phi(t) \qquad 0 < t \leqslant 2\omega. \tag{3.13}$$

Since H is continuous and since the convergence in (3.13) is uniform in t, $\phi(t)$ is continuous. It obviously has period ω.

PROOF of THEOREM 1.3. From (3.8) and the uniform convergence of $H(t + j\omega)$ to $\phi(t)$ it follows that ϕ is uniformly Lipschitz of order exceeding $\frac{1}{2}$. Setting $\sigma = t - s$ in (3.2) and letting $t \to \infty$ yields

$$\frac{1}{2}\int_0^\infty [\phi(t) - \phi(t-s)]s^{-3/2}\,ds = -\pi^{1/2}\Phi[\phi(t) - f(t)]$$

which is (1.14). The left side can also be written as

$$\frac{1}{2}\int_0^\omega [\phi(t) - \phi(t-s)] \sum_{j=0}^{\infty} (s + j\omega)^{-3/2}\,ds$$

To prove (1.14) has no other solution than ϕ, assume there is a continuous periodic solution ψ. Let $\phi - \psi$ have a positive maximum $\lambda > 0$ which is taken on at t_0. Using (1.14) for ϕ and ψ, subtracting and setting $t = t_0$ gives

$$\frac{1}{2}\int_0^\omega [\phi(t_0) - \psi(t_0) - (\phi(t_0 - s) - \psi(t_0 - s))] \sum_{j=0}^{\infty} (s + j\omega)^{-3/2}\,ds$$
$$= -\pi^{1/2}\Phi[\phi(t_0) - f(t_0)] + \pi^{1/2}\Phi[\psi(t_0) - f(t_0)].$$

The integral on the left is non-negative while the right side is negative for $\lambda > 0$. This is impossible. Similarly $\phi - \psi$ cannot have a negative minimum and this proves uniqueness.

Since (1.14) is absolutely integrable both sides can be integrated with respect to t from 0 to ω and the order on the left side reversed. Since

$$\int_0^\omega [\phi(t) - \phi(t-s)]\,dt = 0$$

the left side vanishes which proves that the average of $\Phi(\phi(t) - f(t))$ is zero. Thus in terms of (1.3), as $j \to \infty$,

$$\frac{\partial u}{\partial x}(0, t + j\omega) \to \Phi[\phi(t) - f(t)]$$

and hence as $t \to \infty$, $\partial u/\partial x(0,t)$ tends to a periodic function with average value zero.

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