Dynamic model of a mobile robot

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Abstract

The contribution presents the dynamic model of a differential two-wheeled mobile robot in the joint frame, i.e. in the plane of angular position of both the right-hand side wheel and the left-hand side one. The aim is to create an appropriate MIMO plant model for the purpose of the mobile robot motion control synthesis. Based on the Lagrangian formalism, the complete description of the second order matrix differential equation (the dynamic model of the mobile robot) is provided, including the influence of inertial coupling forces. A pair of wheel driving torques represents the input of given model. Additionally, a decoupled dynamic model for the robust motion control implementation is presented.

Keywords: Euler-Lagrange’s equation; kinetic energy; instantaneous center of rotation; horizontal plane; decoupling

1. Introduction

The challenging ambition of a mobile robot control is the fast and accurate motion along the desired trajectory in the task plane given by an actual application (industry, agriculture, transport, household, prosthetics etc.). To keep both the high velocity of the motion and the reliability of the applied control, it is inevitable to take the dynamics of the moving mass of the mobile robot into account. The motion of the differential two-wheeled mobile robot is performed by the wheel rotation described via the angular position and it’s time derivatives in the joint frame. On the other hand, from the user’s point of view, the position and velocity of the mobile robot motion in the task frame (for

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the simplicity, assume the motion within the horizontal plane) – usually described by the planar Cartesian coordinates – is preferred. The homogeneous transformation [1] between the joint and task frames counts to time independent (non-dynamic) transformations resulting in non-linear forward and inverse kinematics (FKM – forward kinematic model, IKM – inverse kinematic model) of the robot. This implies that the structure of the mobile robot motion control (as well as the motion control of any kinematic chain in general) can consist of the inner (dynamic) open control loop (comprising the control algorithm and the controlled plant) performed in the joint frame, and the outer feedback loop with the non-dynamic, non-linear forward and inverse transformations between the task and joint frames (Fig. 1) [2]. In figure, \( \mathbf{p} \in \mathbb{R}^2 \) and \( \mathbf{p}_d \in \mathbb{R}^2 \) denote the position and the desired position vectors in the Cartesian task frame, \( \mathbf{q} \in \mathbb{R}^2 \) and \( \mathbf{q}_d \in \mathbb{R}^2 \) stand for the angular position and the desired angular position vectors in the joint frame, and \( \mathbf{e}_q \in \mathbb{R}^2 \) represents the error vector of the angular position in the joint frame. Therefore, the dynamic model of the mobile robot in the joint frame (resulting in a control-affine dynamic model [3, 4]) rather than in the task frame (non-linear dynamic model) has to be synthesized. This approach allows avoiding the problem of non-uniqueness and singularity [5] of the inverse kinematics solution in the inner control loop, transforming it to less complicated problem of the desired trajectory forming.

### Nomenclature

- **ICR**: instantaneous centre of rotation of the mobile robot
- **C**: centre of gravity (centre of the robot body)
- **\( r_0 \)**: instantaneous turning radius of the robot trajectory
- **\( \theta_C \)**: robot angular position with respect to ICR (robot orientation)
- **\( \dot{\theta}_C \)**: angular velocity of the robot orientation
- **\( m_C \)**: mass of the robot body
- **\( r_C \)**: radius of the robot (cylindrical) body
- **\( J_C \)**: moment of inertia of the robot body
- **\( m \)**: mass of each of the robot wheels
- **\( r \)**: radius of each of the robot wheels
- **\( J \)**: moment of inertia of each of the robot wheels
- **\( L \)**: distance between the wheels (wheel spacing, bias)
- **\( l_C \)**: position of the robot in the task plane
- **\( v_C \)**: velocity of the robot in the task plane
- **\( l_i \)**: position of the i-th robot wheel in the task plane
- **\( v_i \)**: velocity of the i-th robot wheel in the task plane
- **\( q_i \)**: angular position of the i-th robot wheel – an element of the angular position vector \( \mathbf{q} \)
- **\( \dot{q}_i \)**: angular velocity of the i-th robot wheel – an element of the angular velocity vector \( \dot{\mathbf{q}} \)
- **\( \tau_i \)**: driving torque for the i-th robot wheel – an element of the driving torque vector \( \mathbf{\tau} \)

![Fig. 1. Block diagram of the motion control in the task frame.](image-url)
2. Mobile robot description

Suppose the basic robot structure consisting of a cylindrical body and a pair of differentially driven side wheels (Fig. 2). Let $i = 1$ denotes the index of the right-hand side wheel and $i = 2$ of the left-hand side one.

It is evident, that the pair $(l_C, \theta_C)$ matches with the number of degrees of freedom of the mobile robot in the task frame, and equally the pair $(q_1, q_2)$ describes the robot DOFs in the joint frame. The upper notation yields the basic expression [6] for the robot wheel position

$$l_1 = r q_1 = \left( r_0 + \frac{L}{2} \right) \theta_C, \quad l_2 = r q_2 = \left( r_0 - \frac{L}{2} \right) \theta_C$$

(1)

as well as the robot body position and orientation

$$l_C = \frac{l_1 + l_2}{2} = \frac{r}{2} (q_1 + q_2), \quad \theta_C = \frac{l_1 - l_2}{L} = \frac{r}{L} (q_1 - q_2)$$

(2)

Consequently, we are able to express the corresponding velocities of wheels

$$v_i = r q_i \quad i = 1, 2$$

(3)

and of the robot body

$$v_C = \frac{v_1 + v_2}{2} = \frac{r}{2} (q_1 + q_2), \quad \dot{\theta}_C = \frac{v_1 - v_2}{L} = \frac{r}{L} (\dot{q}_1 - \dot{q}_2)$$

(4)

Underline that expressions (3) and (4) are the functions of elements $\dot{q}_1$ and $\dot{q}_2$ of the angular velocity vector $\dot{\mathbf{q}}$.

Assuming the cylindrical shape of the mobile robot body, its moment of inertia is given by

$$J_C = \frac{1}{2} m_C r_C^2$$

(5)
and similarly is the moment of inertia of each of the robot wheels

\[ J = \frac{1}{2} mr^2 \]  

(6)

We are now ready to get the dynamic model of mobile robot in the joint frame.

3. Dynamic model of the mobile robot

In MIMO dynamic system’s analysis, the straightforward and powerful tool for the dynamic model synthesis represents the Euler-Lagrange method [1, 7] based on the system’s total kinetic and potential energy concept. Lagrange’s energy function \( L \) stands for the difference between the total kinetic energy \( E_K \) and total potential energy \( E_P \). Due to above mentioned simplification – motion within the horizontal plane (i.e. the constant potential energy) – only the kinetic energy of a moving robot should be taken into consideration. Furthermore, the total kinetic energy \( E_K \) of the differentially-driven mobile robot is independent of its position in the task plane likewise in the joint frame. Thus, for the mobile robot dynamics description, the reduced version of the Euler-Lagrange’s equation for the \( i \)-th joint coordinate \( q_i \) is given by

\[ \frac{d}{dt} \frac{\partial E_K}{\partial \dot{q}_i} = Q_i \]

(7)

where \( Q_i \) stands for the generalized non-conservative (external) force in the direction of the \( i \)-th coordinate. This force comprises the driving torque \( \tau_i \) and the sum of load forces \( \tau_{Li} \) (i.e. dissipative forces like viscous friction and rolling friction). The resultant generalized force is then expressed by the difference

\[ Q_i = \tau_i - \tau_{Li} \]

(8)

The total kinetic energy of mobile robot consists of energy of both the translational and the rotational motion of the robot body and the robot wheels

\[ E_K = \frac{1}{2} m_C v_C^2 + \frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2 + \frac{1}{2} J_C \dot{q}_C^2 + \frac{1}{2} J_1 \dot{q}_1^2 + \frac{1}{2} J_2 \dot{q}_2^2 \]

(9)

Substituting expressions (1) – (6) to (9) yields

\[ E_K = \frac{3}{4} mr^2 (q_1^2 + q_2^2) + \frac{1}{4} m_C r^2 \left[ \frac{r_C^2}{L^2} (\dot{q}_1 - \dot{q}_2)^2 + \frac{1}{2} (\dot{q}_1 + \dot{q}_2)^2 \right] \]

(10)

As mentioned above, the total kinetic energy of the mobile robot is independent of the angular position vector \( \mathbf{q} \). Euler-Lagrange’s equation (7) should be solved for both degrees of freedom, i.e. for \( q_1 \) and \( q_2 \), giving the resultant system of two differential equations of second order

\[ j_1 \ddot{q}_1 + j_2 \ddot{q}_2 = \tau_1 - \tau_{L1} \]

\[ j_2 \ddot{q}_1 + j_2 \ddot{q}_2 = \tau_2 - \tau_{L2} \]

(11)

where
\[ j_i = \frac{r^2}{2} \left[ 3m + m_C \left( \frac{1}{2} + \frac{r^2}{L^2} \right) \right], \quad i = 1, 2 \]  

(12)

represents the moment of inertia of the \(i\)-th degree of freedom,

\[ j_{ij} = \frac{r^2}{2} m_C \left( \frac{1}{2} - \frac{r^2}{L^2} \right), \quad i, j = 1, 2, \quad i \neq j \]  

(13)

is the coupling moment of inertia between the DOFs (the product \(j_{ij} \ddot{q}_j\) corresponds to the undesirable mutual inertial coupling forces – Euler’s forces), and

\[ \ddot{q}_i = \frac{d^2 q_i}{dt^2} = \frac{d^2 q_i}{dt^2}, \quad i = 1, 2 \]  

(14)

stands for the angular acceleration in the \(i\)-th DOF. The angular position \(q_i\) and angular velocity \(\dot{q}_i\) are the elements of the \(i\)-th DOF’s phase vector \(\mathbf{q}_i\). Denote

\[ \mathbf{J} = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix} \in \mathbb{R}^{2 \times 2} \]  

(15)

the inertial matrix of the mobile robot. Now, the system of equations (11) can be rewritten in the form of matrix differential equation of second order

\[ \mathbf{J} \ddot{\mathbf{q}} = \mathbf{\tau} - \mathbf{\tau}_L \]  

(16)

System of differential equations (11) or (16) stands for the control-affine [3, 4] dynamic model of the mobile robot in the joint frame. The solution of this system yields the behavior of the robot wheel’s angular position in time. To assure the desired trajectory of the mobile robot, any control algorithm should generate the corresponding driving torque vector \(\mathbf{\tau}\).

4. Decoupled dynamic model

To guarantee the desired behavior (perfect accuracy and fast dynamics) of mobile robot in presence of load forces and variable inertial coupling forces (signal disturbances), the robust control techniques should be taken into account. In general, the aim of the robust control is to keep the quality of the controlled process despite the parametric and signal disturbances [8], preferably without any adaptation or on-line identification. The only information about the disturbances is their estimated or computed amplitude (maximal value).

In the case of mobile robot, the combination of the load torque and the inertial coupling force acts against the driving torque. Denote \(\tau_{Li\text{max}}\) the maximal value of this combination in the mobile robot working plane. This maximal value is identical also in the joint space

\[ \tau_{Li\text{max}} = \max_{q_i, q, q_j} \left\{ \tau_{Li} + j_{ij} \ddot{q}_j \right\} = \text{const}. \]  

(17)

The constant (independent of the system’s joint variables) maximal value in (17) implies, that using (11) and (17), the decoupled dynamic model [9, 10, 11] of the mobile robot for the purpose of the robust control algorithm synthesis can be created
\begin{align*}
    j_{1} \ddot{q}_1 &= \tau_1 - \tau_{L1\text{max}} \\
    j_{2} \ddot{q}_2 &= \tau_2 - \tau_{L2\text{max}}
\end{align*} \tag{18}

provided that once designed for the decoupled dynamic model (18), the resultant robust control \[9, 10, 11\] guarantees the desired quality of the real motion control system for any combination of signal disturbances under the condition

\[
|\tau_{Li} + j_{ij} \dot{q}_j| < |\tau_{L\text{max}}|
\] \tag{19}

regardless of the frequency spectra or the time history of disturbances.

We can conclude that from the point of view of the robust control design, the dynamic model of a mobile robot can be treated as a system of the pair of mutually independent double integrators (18). Similar approach to the dynamic system analysis has been applied also in [12] and [13].

5. Conclusions

The paper shows two specific features of the mobile robot dynamic model synthesis, the designer of the motion control algorithm should take into account. The first feature is that for the motion control in the task frame, the dynamic model in the joint frame can be applied. This yields the control-affine MIMO dynamic model in the form of the system of two differential equations of second order. The second feature deals with the decoupling of this MIMO model to the pair of independent (SISO) double integrator models, provided that the robust control algorithm for the mobile robot motion is under consideration. The outcome of this contribution can essentially simplify the design of a reliable robust control algorithm in originally complex and intricate non-linear MIMO control task.

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References