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Procedia IUTAM 19 (2016) 187 – 192

Procedia IUTAM

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IUTAM Symposium Analytical Methods in Nonlinear Dynamics

Study of coupled oscillators with local and global nonlinear potentials

S. Charlemagne[a,*], C.-H. Lamarque[a], A. Ture Savadkoohi[a], M. Weiss[a,b], B. Vaurigaud[b]

[a]Univ Lyon, ENTPE, LTDS UMR CNRS 5513, Rue Maurice Audin, F-69518, Vaulx-en-Velin Cedex, France

[b]CEREMA - Direction territoriale Sud-Ouest, Département Laboratoire de Bordeaux, 24 rue carton - CS 41635, Bordeaux Cedex 33073, France

Abstract

Multi-scale dynamics of a linear oscillator coupled to a nonlinear energy sink with local and global potential is studied. Detection of the slow invariant manifold gives information on the system behavior, while equilibrium and singular points provide finer predictions, which are in agreement with numerical results.

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Peer-review under responsibility of organizing committee of IUTAM Symposium Analytical Methods in Nonlinear Dynamics

Keywords: Nonlinear energy sink, multi-scale dynamics, energy exchanges, local potential

1. Introduction

Passive control of structures can be carried out via endowing internal capabilities of structural members, e.g. hysteresis responses of semi-rigid joints during cyclic or seismic loads[1], and/or by coupling other oscillators to them[2]. Coupled passive controller oscillators can be divided in two general categories: they can be linear, e.g. Frahm devices[3], or nonlinear such as nonlinear energy sink (NES) systems[4,5]. There are some advantages in using NES systems over corresponding linear controllers: they are very light; they do not modify frequencies of main structures; they are valid and can trigger the energy of main systems for quite large frequency widths. The control process by NES devices can be carried out by trapping both coupled oscillators into periodic regimes and/or by performing strongly modulated responses that are accompanied by persisting back and forth bifurcations of systems[6,7]. Nonlinear potential of the NES can be cubic[8,9] or nonsmooth[10,11]. The nonsmooth potential of the NES is recognized by piece-wise linear[12,13,14] or vibro-impact[15,16] systems. Most of researches on passive control of structures by NES devices endow the global potential of the NES which performs direct interactions with the main structure. In this study we are interested to detect time multi-scale energy exchanges between a main structure and a NES with local and global potentials. The local potential of the NES which depends only on the behavior of the NES, is realized intentionally to see its effects on energy exchanges of both oscillators at different time scales. A step by step original analytical treatments are developed for preparing necessary design tools for tuning parameters of the NES with local and global potentials

* Corresponding author. Tel.: +33472047231
E-mail address: simon.charlemagne@entpe.fr

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Peer-review under responsibility of organizing committee of IUTAM Symposium Analytical Methods in Nonlinear Dynamics
doi:10.1016/j.piutam.2016.03.024

for the aim of passive control and/or vibratory energy harvesting of main structural systems. This paper is organized as it follows: model description, change of variables and scaled parameters are given in Sect. 2. In Sect. 3, analytical treatment of the equations at fast and slow time scales is carried out. These predictions are compared to numerical results in Sect. 4. Finally, the paper is concluded in Sect. 5.

2. Mathematical description of the considered system

Let us consider the model described in Eq. (1). Displacements of the main linear system and the NES are described by y and x, respectively, and ϵ is a small parameter ($0 < \epsilon \ll 1$) standing for the mass ratio of the NES and the main oscillator. The principal structure has the damping a and the angular frequency ω_0. It is subjected to the external force $f(t)$. The NES has the damping c and possesses two odd potential functions: the global one consists of linear and nonlinear parts (ω_c and $W(z)$, respectively) while the local one, $\tilde{g}(z)$, is purely nonlinear.

$$\begin{aligned} &\ddot{y} + a\dot{y} + \omega_0^2 y + c(\dot{y} - \dot{x}) + \omega_c^2(y - x) + W(y - x) = f(t) \\ &\epsilon\ddot{x} - c(\dot{y} - \dot{x}) - \omega_c^2(y - x) - W(y - x) + \tilde{g}(x) = 0 \end{aligned} \tag{1}$$

New variables of the system are given by:

$$v = \frac{y + \epsilon x}{1 + \epsilon} \ , \ w = x - y \tag{2}$$

Let us now introduce complex variables[17]:

$$\psi e^{i\omega t} = \dot{v} + i\omega v \ , \ \varphi e^{i\omega t} = \dot{w} + i\omega w \tag{3}$$

Multiple time scales $\tau_j = \epsilon^j t, \ j = 0, 1, \ldots$ are then introduced. Finally, following assumptions about scales of parameters are carried out so that we can study the system behavior around 1:1:1 resonance at fast and slow time scales:

$$f(t) = \epsilon f^0 \sin(\omega t) \ , \ \omega = \omega_0(1 + \sigma\epsilon) \ , \ c = \epsilon d \ , \ a = \epsilon a_0 \ , \ \omega_c^2 = \epsilon\Omega^2 \ , \ W(z) = \epsilon W^0(z) \ , \ \tilde{g}(z) = \epsilon\tilde{g}^0(z) \tag{4}$$

Local potential $\tilde{g}(z)$ and global potential $W(z)$ are assumed to be cubic:

$$W^0(z) = \mathcal{A}^0 z^3 \ , \ \tilde{g}^0(z) = \mathcal{B}^0 z^3 \tag{5}$$

3. Analytical treatment

We follow the method developed in references[12,18] for instance.
Assuming that ψ and φ do not depend on τ_0 (to be verified or assumed by asymptotic state $\tau_0 \to +\infty$), we obtain:

$$\begin{aligned} &\dot{\psi} + \left(\frac{i\omega_0(1+\sigma\epsilon)}{2} + \frac{a_0\epsilon}{2(1+\epsilon)} - \frac{i\omega_0}{2(1+\epsilon)(1+\sigma\epsilon)}\right)\psi - \frac{\epsilon}{2(1+\epsilon)^2}\left(a_0\epsilon - \frac{i\omega_0}{1+\sigma\epsilon}\right)\varphi + \frac{\epsilon}{1+\epsilon}F_{\tilde{g}^0} = -\frac{\epsilon}{1+\epsilon}\frac{if^0}{2} \\ &\dot{\varphi} + \left(\frac{i\omega_0(1+\sigma\epsilon)}{2} + \frac{a_0\epsilon^2}{2(1+\epsilon)} - \frac{i\omega_0\epsilon}{2(1+\epsilon)(1+\sigma\epsilon)} + \frac{d(1+\epsilon)}{2} - \frac{i\Omega^2(1+\epsilon)}{2\omega_0(1+\sigma\epsilon)}\right)\varphi - \\ &\left(\frac{a_0\epsilon}{2} - \frac{i\omega_0}{2(1+\sigma\epsilon)}\right)\psi + (1+\epsilon)F_{W^0} + F_{\tilde{g}^0} = \epsilon\frac{if^0}{2} \end{aligned} \tag{6}$$

with

$$\begin{aligned} F_{W^0} &= -\frac{3i\mathcal{A}^0}{8\omega_0^3}|\varphi|^2\varphi = -i\mathcal{A}|\varphi|^2\varphi \\ F_{\tilde{g}^0} &= -\frac{3i\mathcal{B}^0}{8\omega_0^3}|\psi+\varphi|^2(\psi+\varphi) = -i\mathcal{B}|\psi+\varphi|^2(\psi+\varphi) \end{aligned} \tag{7}$$

3.1. Study at ϵ^0 order

At fast time scale, the system (6) reads:

$$
\begin{aligned}
&D_0\psi = 0 \Rightarrow \psi = \psi(\tau_1, \tau_2, ...)\\
&D_0\varphi + \left(\frac{i\omega_0}{2} + \frac{d}{2} - \frac{i\Omega^2}{2\omega_0}\right)\varphi + \frac{i\omega_0}{2}\psi + F_{W^0} + F_{\tilde{g}^0} = 0
\end{aligned}
\tag{8}
$$

Considering asymptotic behavior ($\tau_0 \to +\infty$), we obtain equilibrium relation defining slow invariant manifold (SIM) of the system:

$$
\left(\frac{i\omega_0}{2} + \frac{d}{2} - \frac{i\Omega^2}{2\omega_0}\right)\phi + \frac{i\omega_0}{2}\psi - i\mathcal{A}|\phi|^2\phi - i\mathcal{B}|\psi + \phi|^2(\psi + \phi) = 0 \tag{9}
$$

Eq. (9) is solved by using following equations:

$$
\begin{aligned}
&\chi = \psi + \phi \Leftrightarrow \rho e^{i\theta} = N_1 e^{i\delta_1} + N_2 e^{i\delta_2}\\
&\frac{i\omega_0}{2}\chi + \left(d - \frac{i\Omega^2}{\omega_0}\right)\frac{\phi}{2} - i\mathcal{A}|\phi|^2\phi - i\mathcal{B}|\chi|^2\chi = 0\\
&\left(\frac{\omega_0}{2} - \mathcal{B}\rho^2\right)^2\rho^2 - \left(\frac{d^2}{4} + \left(\frac{\Omega^2}{2\omega_0} + \mathcal{A}N_2^2\right)^2\right)N_2^2 = 0\\
&N_1^2 = \rho^2 + N_2^2 - 2\rho N_2\cos(\theta - \delta_2)
\end{aligned}
\tag{10}
$$

One can notice that the third equation is a polynomial of degree three for ρ^2, that leads to the existence of three possible values of ρ (and N_1 consequently) for a given value of N_2. Thus, the SIM presents three branches.

3.2. Study at ϵ^1 order

Let us consider the first equation of (6) at ϵ^1 order:

$$
D_1\psi + \left(i\omega_0\left(\sigma + \frac{1}{2}\right) + \frac{a_0}{2}\right)\psi + \frac{i\omega_0}{2}\varphi - i\mathcal{B}|\psi + \varphi|^2(\psi + \varphi) = -\frac{if^0}{2} \tag{11}
$$

Equilibrium points around SIM are obtained by solving:

$$
\left(i\omega_0\left(\sigma + \frac{1}{2}\right) + \frac{a_0}{2}\right)\psi + \frac{i\omega_0}{2}\phi - i\mathcal{B}|\psi + \phi|^2(\psi + \phi) = -\frac{if^0}{2} \tag{12}
$$

After cumbersome algebra, one can reach:

$$
\begin{aligned}
\mathcal{H}_1 &= 0\\
\mathcal{H}_2 &= 0
\end{aligned}
\tag{13}
$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are real and imaginary parts of an equation obtained from Eq. (9) and Eq. (12). The points verifying (13) are equilibrium points of the system on its SIM.

Singular points suggest existence of strongly modulated response (SMR)[6] of the system, which is subjected in this case to persistent bifurcations. They occur along the SIM when implicit function theorem is not valid, i.e.:

$$
\begin{aligned}
\mathcal{F}_1 &= 0\\
\mathcal{F}_2 &= 0\\
\det(D) &= 0\\
\mathcal{H}_1 &= 0\\
\mathcal{H}_2 &= 0
\end{aligned}
\tag{14}
$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ correspond to real and imaginary parts of the SIM and matrix D to:

$$D = \begin{pmatrix} \frac{\partial \mathcal{F}_1}{\partial N_2} & \frac{\partial \mathcal{F}_1}{\partial \Delta} \\ \frac{\partial \mathcal{F}_2}{\partial N_2} & \frac{\partial \mathcal{F}_2}{\partial \Delta} \end{pmatrix} \quad \text{with} \quad \Delta = \delta_1 - \delta_2 \tag{15}$$

4. Numerical results

Analytical predictions are compared in this section with numerical results obtained via direct numerical integration of system (1) performed by the function *ode45* of Matlab. Simulations were run for following parameters: $\epsilon = 0.001$, $\omega_0 = 1$, $\Omega = 0.1$, $d = 0.1$, $a_0 = 0.1$, $\mathcal{A} = 1.5$, $\mathcal{B} = 0.5$, $\sigma = 1$, and $f^0 = 0.34$. Assumed initial conditions are $(y(0), \dot{y}(0), x(0), \dot{x}(0)) = (1.2, 0, 0, 0)$.

Fig. 1 depicts the SIM of the system, which presents as predicted three branches (named as branches 1, 2 and 3 respectively). Fig. 2 collects position of equilibrium points and singular points of the system. The latter has two equilibrium points, one stable (no. 1) and the other unstable (no. 3), and one singular point (no. 2), all on branch 1. One can note that the condition of singularity $\det(D) = 0$ is verified on the local extremums of branch 1. Fig 3 shows that the SIM is attracting the behavior with SMR. Predicted singular points generate bifurcations and repeated jumps between local extremums of the SIM with almost periodic regime at low energy levels. Analytical predictions and effective behaviors are in good agreement.

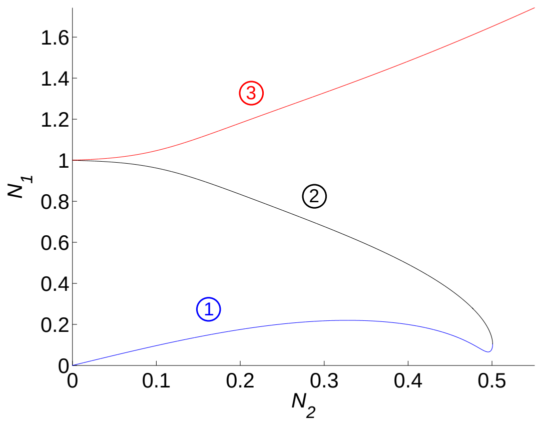

Fig. 1. SIM of the system for the following set of parameters: $\omega_0 = 1$, $\Omega = 0.1$, $d = 0.1$, $\mathcal{A} = 1.5$ and $\mathcal{B} = 0.5$

5. Conclusion

Interactions of two coupled oscillators during different scales of time are studied: one of the oscillators which is supposed to be the main structure and to be controlled is coupled to a nonlinear energy sink with local and global potentials. The global potential of the latter oscillator has direct interactions with the main oscillator while the local one depends only the behavior of the nonlinear energy sink. At fast time scale the slow invariant manifold of the system is detected, which provides the overall view about all possible behaviors during extreme energy exchanges between two oscillators. Detected equilibrium and singular points of the system at slow time scale provide more information about existence of all possible regimes. The system can be attracted by periodic regimes (not illustrated here) due to existence of equilibrium points and/or can present strongly modulated responses due to existence of

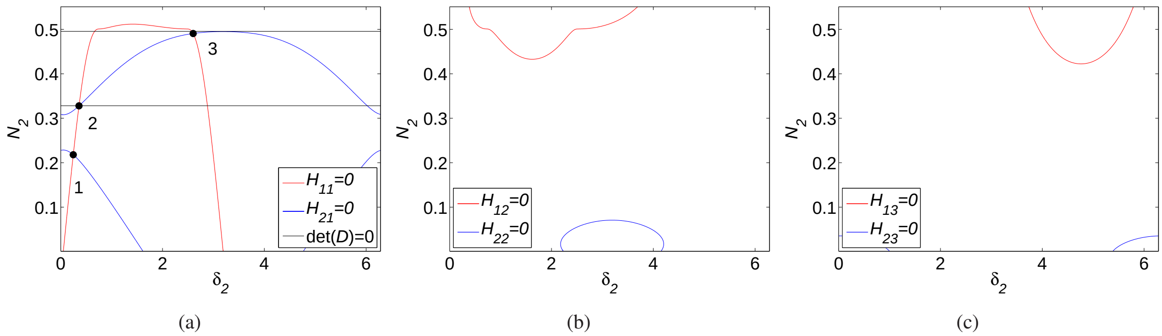

Fig. 2. Position of equilibrium and singular points of the system for the following set of parameters: $\epsilon = 0.001$, $\omega_0 = 1$, $\Omega = 0.1$, $d = 0.1$, $a_0 = 0.1$, $\mathcal{A} = 1.5$, $\mathcal{B} = 0.5$, $\sigma = 1$, and $f^0 = 0.34$. The system has two equilibrium points (No. 1 and No. 3) and one singular point (No. 2). (a) Branch 1, (b) Branch 2 and (c) Branch 3

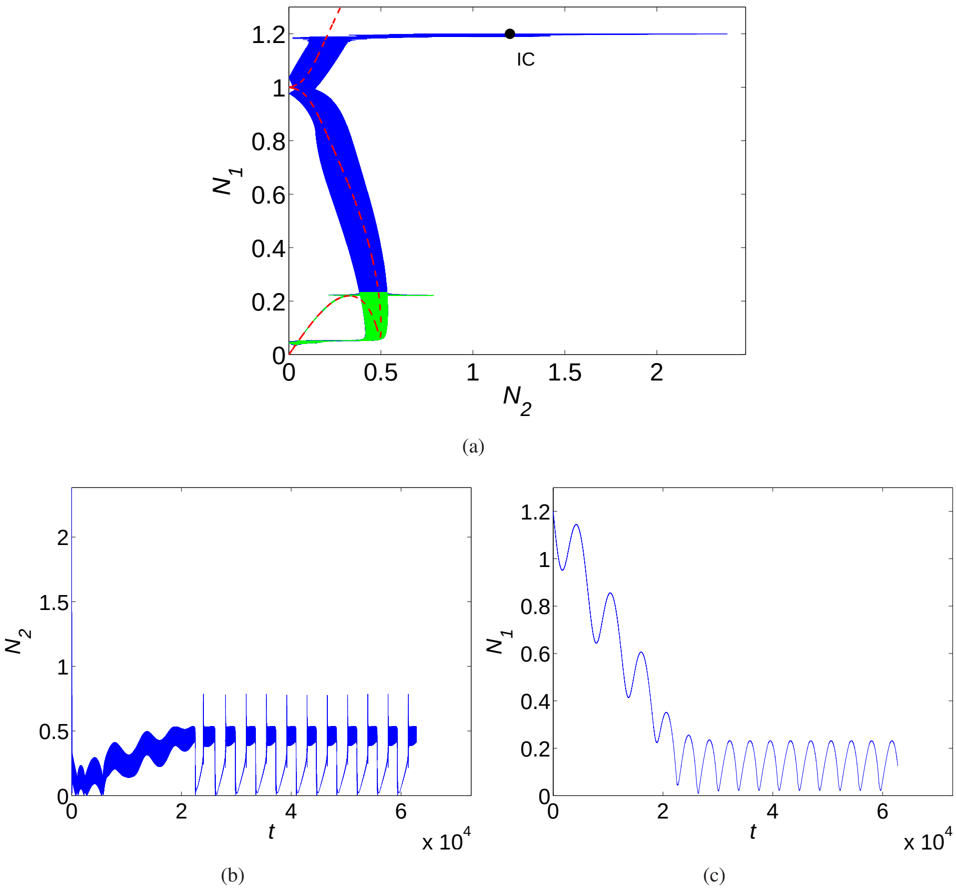

Fig. 3. Numerical simulations for the following set of parameters: $\epsilon = 0.001$, $\omega_0 = 1$, $\Omega = 0.1$, $d = 0.1$, $a_0 = 0.1$, $\mathcal{A} = 1.5$, $\mathcal{B} = 0.5$, $\sigma = 1$, and $f^0 = 0.34$. (a) N_1 vs N_2: SIM of the system (dotted red line) and corresponding numerical results in blue. The green line corresponds to the system behavior during SMR, (b) N_2 vs t and (c) N_1 vs t. *IC* stands for "Initial Conditions"

singular points. All developed techniques can be used to tune all possible periodic and strongly modulated regimes for having allowable or acceptable energy level(s) of the main structure during mentioned regimes. Tuning of these regimes leads to the design of nonlinear energy sink devices for passively controlling main structural systems and/or harvesting their vibratory energies.

Acknowledgements

The authors would like to thank following organizations for supporting this research: i) The "Ministère de l'Écologie, du Développement Durable et de l'Énergie" and ii) LABEX CELYA (ANR-10-LABX-0060) of the "Université de Lyon" within the program "Investissement d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

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