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Note

Perfect Codes over Graphs

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We present a generalization of the notion of perfect codes: perfect codes over graphs. We show an infinite family of 1-perfect codes in second powers of graphs and we prove the nonexistence of nontrivial 1-perfect codes over complete bipartite graphs with at least three vertices.

The theory of perfect codes forms an interesting part of combinatorics. There are some generalizations known (e.g., Lee-error correcting codes [1, 4] or perfect codes in graphs [2, 5]). We introduce another generalization, originally suggested by J. Nešetřil: perfect codes over graphs, which correspond to perfect codes over structured alphabets. The classical perfect codes are then just perfect codes over complete graphs. In the sense of [2] a code over a graph G is a code in a power of G. In this paper we show an infinite class of 1-perfect codes in second powers of graphs and we prove the nonexistence of nontrivial 1-perfect codes over complete bipartite graphs with at least three vertices.

1. Definitions

All graphs considered are simple and undirected. Given graphs $G_i = (V_i, E_i)$, $i = 1, 2, \ldots, n$, with distance functions d_i, we define a metric function d on the cartesian product $\mathsf{X}_{i=1}^{n} V_i$ of vertex sets as follows

$$d(u, v) = \sum_{i=1}^{n} d_i(u_i, v_i).$$

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We call the graph

$$\bigtimes_{i=1}^{n} G_i = \left(\bigtimes_{i=1}^{n} V_i, \{\{u, v\} \mid d(u, v) = 1\} \right)$$

the (cartesian) product of graphs G_i. We write simply G^n (called the nth power of G) in the case of $G_i = G$, $i = 1, 2, \ldots, n$. It is not difficult to check that d is exactly the distance function of $\bigtimes_{i=1}^{n} G_i$. It is also clear from the definition that the product of graphs is associative.

Given the graph $G = (V, E)$, we call any subset C of V a *code* in G. We say that C corrects t errors iff the sets

$$S_t(c) = \{u \mid u \in V \,\&\, d(u, c) \leqslant t\}, \qquad c \in C$$

are pairwise disjoint. Moreover we call C a *t-perfect code* iff these sets form a partition of V. A t-perfect code C is called nontrivial iff $t > 0$ and card $(C) > 1$. A perfect code in a power of G is called a *perfect code over G*.

Remark 1.1. A 1-perfect code in a graph G is simply an independent set of vertices C such that every vertex of G is either in C or adjacent to exactly one vertex in C.

Remark 1.2. Codes over graphs (or more generally in products of graphs) can be interpreted as codes with words over structured alphabets, the structures of those alphabets being described by the graphs themselves. Note that codes over complete graphs are exactly the classical Hamming-error correcting codes and codes over cycles are the Lee-error correcting codes.

2. 1-Perfect Codes in Products of Graphs

Though our main interest is in codes in powers of graphs, one may also look for perfect codes in products of graphs. There are infinitely many such 1-perfect codes, more precisely:

PROPOSITION 2.1. *For any graph G there are infinitely many graphs H such that a 1-perfect code exists in the product $G \times H$.*

Proof. For any graph K the set $\{(v, v) \mid v \in V(K)\}$ is a 1-perfect code in $K \times \bar{K}$ ($\bar{K}$ being the complement of K). Thus for all graphs G and L a 1-perfect code exists in the graph $G \times [L \times \overline{(G \times L)}] = (G \times L) \times \overline{(G \times L)}$.

From the proof of Proposition 2.1 we derive an infinite family of 1-perfect codes in second powers of graphs:

PROPOSITION 2.2. *If G is a self-complementary graph (i.e., $G \cong \bar{G}$) then a 1-perfect code of size card $(V(G))$ exists in G^2.*

One-perfect codes in second powers of non-self-complementary graphs also exist, e.g., 1-perfect Lee-error correcting codes over cycles with $5k$ vertices. In view of this it is slightly surprising that the converse of Proposition 2.2 is also true:

PROPOSITION 2.3. *If a 1-perfect code of size* card $(V(G))$ *exists in the graph G^2, then G is a self-complementary graph.*

The proof appears in [3].

Remark 2.4. It is easy to see that a 1-perfect code in G^2 always contains at least card $(V(G))$ vertices. Thus Proposition 2.3 characterizes all graphs whose second powers contain a 1-perfect code of minimum possible size.

3. 1-PERFECT CODES OVER COMPLETE BIPARTITE GRAPHS

The main result will be proved in this section. Let $K_{a,b}$ be the complete bipartite graph with partite sets A and B of cardinalities a and b, respectively, $a > 1$, and suppose that there exists a 1-perfect code C in $K^n_{a,b}$. We will show that $n = b = 1$.

Let A_i, $i = -1, 0, ..., n+1$, denote the sets of vertices with exactly i coordinates in A (obviously $A_{-1} = A_{n+1} = \varnothing$, while other A_i are independent sets of vertices). Since C is a 1-perfect code, it follows that each vertex in A_i is either in C or is adjacent to exactly one vertex in $C \cap A_{i+1}$ or is adjacent to exactly one vertex in $C \cap A_{i-1}$. As card $(A_i) = \binom{n}{i} a^i b^{n-i}$, putting $x_i = \text{card}\,(C \cap A_i)$, we have

$$\begin{aligned} \binom{n}{i} a^i b^{n-i} &= x_i + x_{i+1}(i+1)b + x_{i-1}(n-i+1)a, \qquad i = 0, 1, ..., n, \\ x_{-1} &= x_{n+1} = 0. \end{aligned} \tag{1}$$

Suppose nonnegative integers $x_0, ..., x_n$ satisfy (1). One can prove by induction on k that x_i is divisible by a^k for $i \geq k$. So $y_i = x_i / a^i$ are nonnegative integers satisfying

$$\binom{n}{i} \cdot b^{n-i} = y_i + y_{i+1}(i+1)ba + y_{i-1}(n-i+1),$$

$$i = 0, 1, ..., n, \; y_{-1} = y_{n+1} = 0.$$

Similarly, one can show that $z_i = y_i/b^{n-i}$ are integers satisfying

$$\binom{n}{i} = z_{i-1}(n-i+1)b + z_i + z_{i+1}(i+1)a, \qquad i = 0, 1, \dots, n, \ z_{-1} = z_{n+1} = 0. \tag{2}$$

Putting $i=0$ in formula (2), we get

$$1 = z_0 + az_1,$$

from which $z_0 = 1$ and $z_1 = 0$ immediately follow. Now putting $i=1$, we get the equation

$$n = nbz_0 + z_1 + 2az_2 = nb + 2az_2,$$

which can be satisfied only if $b=1$. Thus for $a \geqslant b > 1$ there are no 1-perfect codes over the graph $K_{a,b}$.

The rest is for $b=1$. Then the system (2) has a solution for $n=1$, which corresponds to a trivial code of cardinality 1 in $K_{a,1}$.

Now suppose $n>1$. As the system (2) is in fact a recursion for z_i, any solution of (2) can be extended to an infinite sequence $z_{-1}, z_0, \dots$, satisfying

$$\begin{aligned} &z_{-1} = 0, \quad z_0 = 1, \\ &\binom{n}{i} = z_{i-1}(n-i+1) + z_i + z_{i+1}(i+1)a, \qquad i = 0, 1, \dots. \end{aligned} \tag{3}$$

We prove by induction that for any $k>2$ there exists an integer r_k such that

$$z_k = \frac{n(n-1)}{a^{k-2}k!}\,[ar_k - (-1)^k]. \tag{4}$$

(1) The formulas for z_3 and z_4 can be easily derived from (3) by putting $i=2$ and $i=3$, respectively.

(2) Let $k>3$ and suppose (4) is true for all $j = 3, \dots, k$. Putting $i=k$ in (3), we obtain

$$\binom{n}{k} = z_{k-1}(n-k+1) + z_k + z_{k+1}(k+1)a$$

and therefore

$$\begin{aligned} z_{k+1} = \frac{n(n-1)}{a^{k-1}(k+1)!}\Bigg[&\binom{n-2}{k-2} a^{k-2}(k-2)! \\ &- (n-k+1)ka(ar_{k-1} + (-1)^k) - ar_k + (-1)^k\Bigg], \end{aligned}$$

from which the formula (4) for z_{k+1} immediately follows.

So for any $n > 1$ we get

$$z_{n+1} = \frac{n(n-1)}{a^{n-1}(n+1)!} [ar_{n+1} + (-1)^n] \neq 0,$$

which is a contradiction to (2). Thus we have proved

THEOREM 3.1. *There are no nontrivial 1-perfect codes over complete bipartite graphs with at least three vertices.*

Though the case of general complete k-partite graphs seems to be more difficult, it is however possible to follow the above proof in the case of regular complete k-partite graphs.

THEOREM 3.2. *For $k > 1$, there are no 1-perfect codes over regular complete k-partite graphs with more than k vertices.*

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