On Adjoints and Dual Matroids

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Duality among matroids is a well-known and well-understood relation. Besides this “set-theoretical” version of duality, there is another one based on lattice-theoretical concepts, which has been introduced by A. Cheung (Canad. Math. Bull. 17 (1974), 363–365). These two concepts do not seem to fit into one another very well and their relationship (provided there is any) is more than unclear. In general, matroids may fail to have “duals” in the lattice-theoretical sense. Therefore, a natural question, posed by J. H. Mason (in “Algebraic Methods in Graph Theory” (L. Lovász and V. T. Sós, Eds.), North-Holland, Amsterdam, 1981), is the following: If M does have a dual in the lattice-theoretical sense, does M* (the set-theoretical dual of M) also have one? We present a counterexample, showing that the answer is negative.

1. INTRODUCTION

The reader is assumed to be familiar with basic concepts from matroid theory. In particular we will make free use of the 1–1 correspondence between (simple) matroids and geometric lattices. Throughout, M will denote a simple matroid. M* denotes its dual in the traditional, set-theoretical sense; i.e., the bases of M and M* are set-theoretical complements of each other. L will denote the geometric lattice corresponding to M.

The notion of a lattice-theoretical dual or “adjoint” of L is defined as follows (cf. [3, 4]):

DEFINITION. A geometric lattice L d is called an adjoint of L if rank (L d) = rank(L) and there exists an orderreversing, injective map φ: L → L d, taking the points of L into the copoints of L d and the copoints of L onto the points of L d.

Obviously, every modular geometric lattice L does have an adjoint L d. In fact, L d may be chosen to be the lattice-theoretical dual M' (arising from
M by reversal of the order relation). Thus, in particular, projective geometries do have adjoints. More generally, if L is a linear geometric lattice, i.e., its elements correspond to the set of linear subspaces spanned by a finite set of points in some projective space M, then the embedding \( L \rightarrow M \), followed by the antiisomorphism \( M \rightarrow M' \) and an appropriate retraction, gives rise to an adjoint of L:

\[
\varphi: L \rightarrow M \rightarrow M' \rightarrow L^\ast.
\]

In general, matroids (or geometric lattices, if you prefer) may fail to have an adjoint (cf. [3, 4]). According to the above, however, the class of matroids that do have an adjoint generalizes the class of linear matroids. This generalization is a proper one, since every rank 3 matroid does have an adjoint. This can be seen in the same way as above: For every rank 3 geometric lattice L there exists a (possibly infinite) modular extension M (which can be constructed by successively "intersecting" lines). Now again it is obvious how to define

\[
\varphi: L \rightarrow M \rightarrow M' \rightarrow L^\ast
\]

as required.

Thus, matroids that do have an adjoint define a proper generalization of linear ones. In fact, matroids with adjoint seem to share some important properties with linear matroids; i.e., they appear to be "nice" in some sense. In particular, if the notion of adjoint is carried over in a natural way to oriented matroids, one can see that oriented matroids that do have an adjoint (in the oriented sense) appear to have nice properties both from an "algorithmic" and a "structural" point of view [1, 6]. For the class of rank 3 oriented matroids, Cordovil [5] proved equally the general existence of a dual relationship to pseudoline arrangements.

Besides that, the notion of adjoints itself turns out to be of some interest in its own right. For example, it may be used for investigating the (linear) representability of a matroid [2]. To summarize, the concept of adjoint, based on a lattice-theoretical duality, has been shown to be of some interest in matroid theory. Its relation to the "traditional" notion of duality between matroids, however, has remained unclear. (One is even tempted to say that there is no such relation.) In [8], the following natural question has been asked: If \( M \) does have an adjoint, does \( M^* \) have one, too? Section 3 provides a negative answer by showing that the dual of the "non-Desargues" configuration fails to have an adjoint. Note that the non-desargues configuration itself is a rank 3 geometry and therefore does have an adjoint, according to our discussion above.
2. SOME PREPARATIONS

This section is to provide some tools for proving nonexistence of adjoints. Our main tool will be a result of [4] which we are going to explain in the following.

Let $L$ be a geometric lattice. It is well-known [10] that there is a 1–1 correspondence between point extensions (single element extensions) of $L$ and so-called modular filters of $L$. Recall that a modular filter is a subset $\mathcal{F} \subseteq L$ such that

1. $x \in \mathcal{F}$ and $y \geq x$ implies $y \in \mathcal{F}$
2. $x, y \in \mathcal{F}$ a modular pair implies $x \land y \in \mathcal{F}$.

If a new point is added $p$ to $L$, thus obtaining a point extension $L \cup p$, the associated modular filter $\mathcal{F}$ is formed by those elements which are to contain $p$ in the extended lattice $L \cup p$ (i.e., $\mathcal{F}$ is the set of all $x \in L$ such that $i(x) = x \lor p$, where $i$ denotes the inclusion map $L \to L \cup p$) [7, 9].

The set of all modular filters of $L$, ordered by inclusion, gives rise to a lattice again, which we call the extension lattice $E(L)$. This has a unique minimum element, $\mathcal{F} = \{1_L\}$ (corresponding to a point extension “in general position”), and a unique maximum element, $\mathcal{F} = L$ (corresponding to a “loop extension”). The minimal nonzero elements of $E(L)$ are the modular filters $\{H, 1_L\}$, where $H$ is a copoint of $L$. A minimal nonzero element of $E(L)$ will also be called a point of $E(L)$. Thus, points of $E(L)$ are in 1–1 correspondence with copoints of $L$. See [2] for a further discussion of extension lattices. In general, $E(L)$ may look quite odd. However, in case $L$ has an adjoint, things cannot be too bad:

**Proposition 2.1** (Cheung’s embedding theorem, cf. [4]). Let $L$ and $L^*$ be geometric lattices of the same rank. If $L^*$ is an adjoint of $L$, then there exists an injective, order-preserving map $\varepsilon: L^* \to E(L)$ taking the points of $L^*$ onto the points of $E(L)$.

(In [2], the converse has been proved.) Of course, the most important thing about Proposition 2.1 is that it indicates where to look for adjoints. In addition, it may be used to derive simple necessary conditions for the existence of adjoints as follows.

**Definition.** Let $L$ be a geometric lattice of rank $r$. Then $L$ is said to have Levi’s property, if the maximum element of $E(L)$ is not the join of fewer than $r$ points.

Stated in another way, this means that the modular filter $\mathcal{F} = L$ is not generated by fewer than $r$ minimal nonzero filters $\mathcal{F}_i = \{H_i, 1\}$. In
geometrical terms, this means that for every set \( \{H_1, ..., H_{r-1}\} \) of \((r-1)\) copoints of \(L\), one can add a point \(p\). Hence we obtain a point extension \(L \cup p\), such that \(p\) is contained in each of \(H_1, ..., H_{r-1}\). Thus, loosely speaking, \(L\) has the Levi property, if every \(r-1\) copoints of \(L\) can be "intersected".

**Corollary 2.2.** If \(L\) has an adjoint, then \(L\) has the Levi property.

**Proof.** Let \(L^d\) be an adjoint of \(L\) and let \(\varepsilon: L^d \to E(L)\) be as in Proposition 2.1. Let \(p_1, ..., p_{r-1}\) be any \(r-1\) points of \(E(L)\) and let \(q_i = \varepsilon^{-1}(p_i)\) \((i = 1, ..., r - 1)\) be the corresponding points of \(L^d\). Since \(L^d\) has rank \(r = \text{rank}(L)\), the join \(x := \sup(a_1, ..., a_{r-1}) < 1_{L^d}\). Hence \(\sup(p_1, ..., p_{r-1}) \leq \varepsilon(x) < \varepsilon(1_{L^d}) \leq 1_{E(L)}\). 

**3. The Counterexample**

Let \(M\) denote the non-Desargues matroid as sketched in Fig. 1.

Then \(M\) is a rank 3 geometry formed by 10 points \((p\) is the center of the perspective and \(\{x, y, z\}\) is the "broken" 3-point line). \(M\) is well-known to be nonlinear. Nonetheless, being a rank 3 geometry, it does not fail to have an adjoint (cf. Section 1). Now consider its dual, \(M^*\). This is a rank 7 matroid and hence, in order to show that \(M^*\) (i.e., the geometric lattice corresponding to \(M^*\)) fails to have an adjoint, we are to find 6 hyperplanes in \(M^*\) which cannot be intersected.

![Fig. 1. Non-Desargues matroid.](image-url)
For this purpose, consider the set $\mathcal{H}$ of 3-point lines in $M$. For every $H \in \mathcal{H}$ its complement $H^*$ is a hyperplane in $M^*$. In particular, let $H_1$, $H_2$, and $H_3$ denote the three lines of the triangle $abc$ and let $H_4$, $H_5$, and $H_6$ denote the three lines $paa'$, $pb'b'$, $pcc'$. We claim that the modular filter $\mathcal{F}^*$ generated by $H_1^*, ..., H_6^*$ already contains all flats of $M^*$. This may be restated as follows:

**Proposition 3.1.** Let $N^* := M^* \cup q$ denote the point extension of $M^*$ corresponding to $\mathcal{F}^*$, the modular filter generated by $\{H_1^*, ..., H_6^*\}$. Then $q$ is a loop in $N^*$.

**Proof.** Let $S$ denote the 10-element groundset of $M$ and let $r$ denote the rank function, indexed by the matroid it refers to. By the well-known relation between dual rank functions, we have

$$r_M(S \setminus A) = |S \setminus A| - r_{M^*}(S) + r_{M^*}(A) \quad \forall A \subseteq S.$$ 

The assumption on $N^*$ now states that the new point $q$ "lies on" each of the hyperplanes $H_1^*, ..., H_6^*$, i.e.,

$$r_{N^*}(H_i^* \cup q) = 6 \quad \forall i = 1, ..., 6.$$ 

By duality, this means that for all $i$

$$r_N(H_i) = |H_i| - r_{N^*}(S \cup q) + r_{N^*}(M_i^* \cup q) = 3 - 7 + 6 = 2.$$ 

Thus $N$ is such that $N/q = M$ and all the $H_i$ are (still) 3-point lines in $N$. From this one immediately gets that $\{p, a, a', b, b', x\}$ is contained in some plane $E$ of $N$. On the other hand, $N/q = M$ implies that $\{x, a', b', q\}$ is a plane $E'$ of $N$. Hence $E \cap E' = \{x, a', b'\}$ is a line in $N$. Similarly, we get that $\{y, b', c'\}$ and $\{z, a', c'\}$ are lines in $N$. Thus, in fact, all 3-point lines of $M$ remain 3-point lines in $N$. From this one immediately deduces that the whole groundset $S$ of $M$ is a coplanar set in $N$. In fact, considering the three lines of the triangle $abc$, we find that $\{a, b, c, x, y, z\}$ is a coplanar set and, similarly, $\{a', b', c', x, y, z\}$ is a coplanar set. Since $\{x, y, z\}$ span a plane in $M$ (and hence in $N$), we conclude that $S \setminus p$ is a coplanar set in $N$, which finally yields that $S$ is a plane in $N$. Thus $N = S \oplus q$, i.e., $q$ is a loop in $N^*$. 

**References**