On Degenerations and Extensions of Finite Dimensional Modules

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We derive a cancellation theorem for degenerations of modules that says in particular, that projective or injective common direct summands can always be neglected. Combining the cancellation result with the existence of almost split sequences we characterize the orbit closure of a module living on preprojective components by the fact that the dimension of the homomorphism space to any other module does not decrease. For representation-directed algebras, whence in particular for path algebras of Dynkin quivers, we provide an alternative proof which shows in addition that any minimal degeneration $N$ of $M$ comes from an exact sequence with middle term $M$ whose end terms add up to $N$. By a careful examination, the same is true for degenerations of matrix pencils. Having used so far the existence of certain extensions to obtain degenerations we then turn the tables and use degenerations to produce a lot of interesting short exact sequences. In particular, we show that any non-simple indecomposable over a tame quiver is an extension of an indecomposable and a simple.

1. INTRODUCTION

Throughout this article we fix an algebraically closed field $k$. We are interested in a geometry study of finite dimensional modules over some finite dimensional associative algebra $A$. Given a natural number $d$ and a basis $a_1, a_2, \ldots, a_s$ with corresponding structure constants $a_{ij}^k$, one has the well-known scheme $\text{Mod}^d_A$ of $d$-dimensional $A$-modules whose rational points consist in $s$-tuples of $(d \times d)$-matrices with coefficients in $k$ subject to the relations $m_1 = E_d$ ($= d \times d$-unit matrix) and $m_i m_j = \sum a_{ij}^k m_k$. Any such $s$-tuple $m$ corresponds to a $d$-dimensional $A$-module $M = M(m)$ in the

Note added in proof. Since this article was accepted, the author has deepened and generalized some selected results in the following two papers: Minimal singularities for representations of Dynkin quivers, Comment. Math. Helvetici 63 (1994), 575–611, and Degenerations for representations of tame quivers, Ann. Scient. Ec. Norm. Sup., 4e série, t. 28, 1995, pp. 647–688.
obvious way. This correspondence induces a bijection between the $\text{GL}_d(k)$-orbits on $\text{Mod}^d_A(k)$ under conjugation and the isomorphism classes of $d$-dimensional $A$-modules. By abuse of notation we write $M \leq_{\text{deg}} N$ if the orbit of $n$ belongs to the closure of the orbit of $m$. Then one says that $M$ degenerates to $N$, or that $N$ is a degeneration of $M$. Clearly, $\leq_{\text{deg}}$ is a partial order on the set of isomorphism classes of $d$-dimensional modules.

If $S_1, S_2, \ldots, S_r$ is a list of representatives of isomorphism classes of simple $A$-modules the dimension vector $\text{dim} M$ of an $A$-module $M$ is that vector in $\mathbb{Z}^r$ whose $i$th component equals the multiplicity of $S_i$ as a composition factor of $M$. The connected components $\text{Mod}^d_A(k)$ of $\text{Mod}^d_A(k)$ consist of the modules having dimension vector $d$ (cf. [19]). In the literature, e.g., in the papers [1, 2, and 32] one often studies degenerations of bound representations of the Gabriel-quiver of $A$. But this is equivalent to the study of the degenerations within $\text{Mod}^d_A(k)$ as is explained in [14]. There it is also shown that the various $\text{GL}_d$-varieties $\text{Mod}^d_A(k)$ determine the algebra they come from up to isomorphism.

The fact that two points on an irreducible variety can be joined by an irreducible curve leads to the following characterization of degenerations ([30, 31, 27, 23]): “The closure of the orbit of $m$ contains $n$ if and only if there is a discrete valuation ring with residue field $k$ and finitely generated quotient field $K$ of transcendence degree one as well as an $R$-valued point $l$ of the scheme $\text{Mod}^d_A$ such that $l$ and $m$ are conjugate under $\text{GL}_d(K)$ whereas the residue of $l$ modulo the maximal ideal of $R$ is $n$.”

This criterion appears to be of little practical importance. What one would like to have is a handy criterion in terms of representation theory. But this seems to be a very hard problem for general algebras (see, e.g., 7.1). However, there is the following simple sufficient condition for degenerations which generalizes slightly a fundamental observation of Artin in [3].

**Lemma 1.1.** Let $E:0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of $A$-modules and let $\varphi$ be an endomorphism of $M'$. Then $M$ degenerates to the middle term $N$ of the pushout sequence $\varphi^* E$. In particular, $M$ degenerates to the middle term $M' \oplus M''$ of the split-extension.

**Proof.** Look at the pushout under $\varphi - t \cdot \text{id}$ for $t$ in $k$. By the five-lemma, its middle term $N_t$ is isomorphic to $M$ as long as $t$ is not an eigenvalue of $\varphi$, whereas $N_0$ is $N$.

It follows easily that the middle term of any non-split exact sequence $0 \rightarrow D \rightarrow U \rightarrow M \rightarrow U \rightarrow 0$ degenerates to the middle term of the almost split sequence. Here and in the following we refer the reader to Gabriels’ survey article [20] or Ringels’ book [35] for basic notions of representation theory like almost split sequences.
Following Abeasis and del Fra we consider a second partial order $\leq_{\text{ext}}$ on the isomorphism classes of $d$-dimensional modules such that $M \leq_{\text{ext}} N$ is true if and only if there is a finite sequence $M = M_0, M_1, \ldots, M_r$ of modules such that $M_i$ is the middle term of a short exact sequence whose end terms add up to $M_{i+1}$. By Lemma 1.1, the relation $\leq_{\text{ext}}$ is weaker than $\leq_{\text{deg}}$, whence $\leq_{\text{ext}}$ is a partial order, indeed.

Besides the sufficient condition for degenerations discussed before there is also a simple and handy necessary condition in terms of representation theory which is Riedtmann’s elegant interpretation of some natural rank conditions introduced by Abeasis and del Fra. Namely, if $M$ degenerates to $N$ the inequality $\langle U, M \rangle \leq \langle U, N \rangle$ holds for all $A$-modules $U$. We abbreviate $\dim_k \text{Hom}_A(X, Y)$ by $\langle X, Y \rangle$, and more generally $\dim_k \text{Ext}^i_A(X, Y)$ by $\langle X, Y \rangle^i$. We simply write $M \leq N$ provided $\langle U, M \rangle \leq \langle U, N \rangle$ holds for all $U$, and provided the dimensions or, equivalently, the dimension vectors of $M$ and $N$ coincide. It is a remarkable fact that $M \leq N$ also implies the inequality $\langle M, U \rangle \leq \langle N, U \rangle$ for all $U$ [32]. More precisely, Auslander and Reiten have shown in [4] that for all non-injective indecomposable $U$ the formula $\langle N, U \rangle - \langle M, U \rangle = \langle \text{Tr } DU, N \rangle - \langle \text{Tr } DU, M \rangle$ holds provided $M$ and $N$ have the same dimension vector. Furthermore, Auslander has proved that $M \leq N$ and $N \leq M$ forces $M$ and $N$ to be isomorphic. This follows also from our next observation. In its proof we use the notation $(M, N)$ for the greatest common direct summand of $M$ and $N$ which can be determined—at least theoretically—without knowing the decomposition of $M$ and $N$ into indecomposables [12].

**Lemma 1.2.** Let $M$ and $N$ be $A$-modules with $M \leq N$. Then the following are equivalent:

(a) $M$ and $N$ are isomorphic.

(b) $\langle M, M \rangle = \langle N, N \rangle$.

(c) $\langle M, V \rangle = \langle N, V \rangle$ holds for all indecomposable direct summands of $M \oplus N$.

**Proof.** Recall that we have $\langle M, L \rangle \leq \langle N, L \rangle$ and $\langle L, M' \rangle \leq \langle L, N' \rangle$ for all $L$. In particular, we get $\langle M, M \rangle \leq \langle M, N \rangle \leq \langle N, N \rangle$ and $\langle M, M \rangle \leq \langle N, M \rangle \leq \langle N, N \rangle$. Thus, condition (b) implies $\langle M, M \oplus N \rangle = \langle N, M \oplus N \rangle$. This equality is equivalent to condition (c) because $\langle M, V \rangle \leq \langle N, V \rangle$ holds for all indecomposable direct summands of $M \oplus N$.

To show that (c) implies (a) we write $M = L \oplus M'$ and $N = L \oplus N'$ with $L = (M, N)$. Then we have to show that $M'$ (and hence $N'$) is zero. Of course, we have $M' \leq N'$ and $\langle M', V \rangle = \langle N', V \rangle$ for all direct summands $V$ of $M' \oplus N'$, whence $\langle M', M' \rangle = \langle N', M' \rangle$ and $\langle M', N \rangle = \langle N', N \rangle$.
We choose a basis \(f_1, f_2, \ldots, f_r\) of \(\text{Hom}_A(N', M')\) and look at the exact sequence
\[
0 \to K \to N' \to M' \to 0
\]
with \(f(n_1, n_2, \ldots, n_r) = \sum f_i n_i\).

By construction, the induced sequence
\[
0 \to \text{Hom}_A(N', K) \to \text{Hom}_A(N', N') \to \text{Hom}_A(N', M') \to 0
\]
is exact and we examine the corresponding sequence
\[
0 \to \text{Hom}_A(M', K) \to \text{Hom}_A(M', N') \to \text{Hom}_A(M', M') \to 0.
\]

The rank of \(g\) can be estimated as follows:
\[
\text{rank } g = \langle M', N' \rangle - \langle M', K \rangle = \langle N', N' \rangle - \langle M', K \rangle = \langle N', M' \rangle = \langle M', M' \rangle.
\]
Therefore, \(g\) is surjective. Thus \(M'\) is a direct summand of \(N'\) so that \(M' = 0\) by the definition of \((M, N)\) and the Krull--Remak--Schmidt theorem.

It follows from 1.2 that any chain of neighbors \(M = M_0 < M_1 < \cdots < M_n = N\) has at most \(\langle N, N \rangle - \langle M, M \rangle\) members. This answers a question raised in [32].

Finally, Riedtmann introduces between \(\leq_{\text{deg}}\) and \(\leq\) still another partial order by saying that \(M \leq_{\text{virt}} N\) provided that \(M \oplus X\) degenerates to \(N \oplus X\) for some module \(X\). For representation-finite algebras, the partial orders \(\leq_{\text{virt}}\) and \(\leq\) coincide by [32], and it seems to be an open question whether this holds in general. Since indecomposables can be degenerations of other modules (see 7.1), \(\leq_{\text{ext}}\) is strictly weaker than \(\leq_{\text{deg}}\) in general. Also, \(\leq_{\text{deg}}\) usually is not equivalent to \(\leq_{\text{virt}}\) by a nice example of Carlson cited in [32], which we analyze a little more in 7.2. Thus for arbitrary algebras there is a large gap between the sufficient conditions "\(\leq_{\text{ext}}\)" and the necessary conditions "\(\leq\)". But surprisingly enough, for path algebras of quivers of type \(A_n\) or \(D_n\), Abeasis and del Fra and Riedtmann have shown the coincidence of \(\leq_{\text{ext}}\) and \(\leq\). Their proofs are based on the classification of the indecomposables and the homomorphism spaces between them and, therefore, they are hard to read.

As our first main result we extend in Section 4 the equivalence of \(\leq_{\text{ext}}\) and \(\leq\) to a considerably larger class of algebras, namely to the so-called representation directed algebras [35]. This class contains all path algebras to Dynkin quivers \(A_n\), \(D_n\), \(E_6\), \(E_7\), and \(E_8\). In contrast to the proofs mentioned before, our proof is short and easy to follow. Furthermore we show that a module \(M\) over a representation-directed algebra degenerates to the direct sum of two-indecomposables \(U\) and \(V\) if and only if there is—up to symmetry in \(U\) and \(V\)—an exact sequence \(0 \to U \to M \to V \to 0\). In this way one gets an unexpected amount of short exact sequences (see 7.8). At the of paragraph 4 we consider some radical square zero algebras.
But we start in Section 2 with several elementary geometric observations which are of fundamental importance for the whole article. As an immediate consequence we obtain a cancellation result which says that $M \oplus X \leq \deg N \oplus X$ implies $M \leq \deg N$ provided that $\langle X, M \rangle$ equals $\langle X, N \rangle$. In particular, projectives or—dually—injectives can always be cancelled in degenerations.

In Section 3 we use the cancellation result to show by induction that a module whose indecomposable direct summands all belong to preprojective components degenerates to another module $N$ if and only if $M \leq N$ holds true. To start the induction we generalize a result of Happel and Ringel in [24] and show in particular that the orbits of indecomposable preprojectives are always dense in their connected component.

In Section 5 we prove that the partial orders $\leq \ext$ and $\leq$ coincide for modules over the path algebra of the double arrow. As far as I know this is the first non-trivial example of a representation-infinite algebra where the degeneration behaviour of the modules is completely analyzed. Since the representations of the double arrow are usually called matrix pencils, and since these pencils play a certain role in numerical analysis (see, e.g., [26]), Section 5 could be of slightly more general interest than the rest of this note. There is some evidence that the equivalence of $\leq \ext$ and $\leq$ extends to all tame concealed algebras, but the proof given for the matrix pencils certainly does not.

Finally, in Section 6, we combine the detailed knowledge of the module categories over tame concealed algebras with some basic geometric observations, and we show that any non-simple indecomposable is an extension of an indecomposable and a simple. This generalizes the corresponding statement for representation directed algebras which was obtained some years ago by tangent space arguments in [9]. The wish to generalize this result was my main motivation to study degenerations.

We conclude the present article with several examples. Some of these are of a more theoretical nature, some of a numerical one. The later ones are always taken from the thesis of U. Markolf. I thank him for the permission to include these examples which stimulated parts of Section 4.

2. SOME GEOMETRY AND A CANCELLATION RESULT FOR DEGENERATIONS

In this part, we use some elementary facts from algebraic geometry, but we rely most of the time only on basic properties of determinants. For the convenience of the reader, we include short proofs of some well-known facts. Recall that $k$ denotes an algebraically closed field always.
2.1. The basic geometric objects we will consider are the variety 
\[ \text{Hom}(d, e) \] of homomorphisms from \( d \)-dimensional modules to \( e \)-dimensional ones and some of its subsets. To be explicit, if \( k^{e \times d} \) denotes the set of \((e \times d)\)-matrices with coefficients in \( k \), the variety \( \text{Hom}(d, e) \) is the set of all triples \((m, n, \xi)\) in the product \( \text{Mod}_d(k) \times \text{Mod}_e(k) \times k^{e \times d} \) such that \( \xi \) belongs to \( \text{Hom}_d(M, N) \), i.e., such that \( \xi m = n \xi \) holds for all \( i \). Obviously, these conditions define a Zariski-closed subset. Furthermore, we can interpret these conditions as a system \( B \cdot \xi = 0 \) of \( d \cdot e \cdot s \) homogeneous linear equations in \( e \cdot d \) unknowns \( \xi_{ij} \), where the coefficients of the matrix \( B = B(m, n) \) depend linearly on the coefficients of \( m \) and \( n \). Clearly, the rank of \( B(m, n) \) equals \( e \cdot d - \langle M, N' \rangle \), and we obtain from the characterization of the rank via subdeterminants the upper-semicontinuity of the function \( (m, n) \mapsto \langle M, N' \rangle \). Thus for any \( t \) the set of points \((m, n)\) with \( \langle M, N' \rangle \geq t \) is closed.

In the special case \( d = e \), one can consider the intersection of the closed set above with the diagonal, and one finds again the fact that the union of the orbits of a fixed dimension is locally closed. Recall that its irreducible components are called sheets [16, 27]. Furthermore, for \( t \) and \( M \) fixed, the set of all module structures \( n' \) with \( \langle M, N' \rangle \geq t \) is closed, so that we have re-proved Riedtmann’s observation that \( N \leq \deg N' \) implies \( N \leq N' \). As noted by Schofield in [36], \( N \leq \deg N' \) also implies \( \langle M, N' \rangle \leq \langle M, N' \rangle \) for all \( i \) and \( M \). But unfortunately, these apparently “new” necessary conditions for degenerations are a consequence of the relation \( N \leq N' \). For by dimension shifting one has to look at the case \( i = 1 \) only, and there one gets from a projective resolution \( 0 \to K \to A'' \to M \to 0 \) of \( M \) the equality \( \langle M, N' \rangle^1 - \langle M, N' \rangle^1 = \langle M, N' \rangle - \langle M, N' \rangle = \langle K, N' \rangle - \langle K, N' \rangle \) by applying \( \text{Hom}(\cdot, N') \) and \( \text{Hom}(\cdot, N) \) respectively. The following lemma turns out to be crucial later on.

**Lemma 2.1.** Given any natural number \( t \), the canonical projection \( p \) from the set \( X \) of triples \((m, n, \xi)\) with \( \langle M, N' \rangle = t \) to the set \( Y \) of pairs \((m, n)\) having the same property is a vector bundle. In particular, \( p \) is open.

**Proof.** From the discussion above, both sets are locally closed, hence varieties. Given a point \((m_0, n_0)\) of \( Y \), one chooses \( t' = e \cdot d - t \) linearly independent columns of \( B(m_0, n_0) \). Then the set \( U \) of points \((m, n)\) where the same \( t' \) columns of \( B(m, n) \) are linearly independent is an open neighborhood of \((m_0, n_0)\). The map \( \varphi: p^{-1}(U) \to U \times k' \) which forgets the \( t' \) components of \( \xi \) corresponding to the \( t' \) columns chosen before is bijective and linear in the fibers, and its inverse is a morphism by Cramers’ rule. Thus we have the required trivialization over \( U \).

2.2. Let us derive from this lemma some properties of sheets. Thus, we take \( d = e \) and we look at the bundle \( p: p^{-1}(S) \to S \) induced by the
diagonal inclusion of a sheet $S$ in $Y$. Since our new $p$ is open again, any open subset $V$ of $k^{d 	imes d}$ gives rise to the open subset $p(pr^{-1}(V))$, where $pr$ is the obvious projection. For instance, we get that for any $i$ the modules in $S$ which decompose into at least $i$ direct summands form an open set $S_i$. Indeed, a module belongs to that set if and only if its endomorphism algebra contains an element $\xi$ with at least $i$ different eigenvalues. Looking at the characteristic polynomials, it has to be shown that the set $P_i$ of polynomials $X^d + a_{d-1}X^{d-1} + \cdots + a_0$ with at most $i - 1$ different roots is closed. This set is the image of a closed set under the quotient map from $k^d$ to itself which belongs to the natural action of the symmetric group on $k^d$. Because the quotient map is finite it is closed and so is $P_i$. It would be interesting to have defining equations for $P_i$. In characteristic 0, this is done in Webers algebra book. For $i = 2$ we find again the well-known fact that the indecomposables form a closed subset of each sheet [28], and one can pursue this road a little further. Namely the locally closed set $S_i - S_{i+1}$ can be analyzed using the elaborated theory of sheets of $k[X]$-modules [13]. For instance, one obtains that on an open set the dimensions of the $i$ indecomposable direct summands differ at most by one, whereas the set of structures with $i - 1$ one-dimensional summands is closed. But, of course, both sets can be empty.

Our second remark is that—at least locally—the assignment $M \mapsto \text{End} M$ can be made to a morphism from $S$ to the well-known variety of $e$-dimensional unital associative algebras. For given a point $(m, m)$ in an open set $U$ as constructed in the proof of Lemma 2.1, the canonical basis of $k^e$ produces via $g^{-1}$ a basis of $\text{End} M$ and the corresponding structure constants $s^n_k$. The morphism $m \mapsto s^n_k$ can be used to obtain degenerations of algebras in a natural manner. For instance, if $A$ is the polynomial algebra $k[X]$ and $S$ the open sheet of $\text{Mod}_d^d(k)$, the endomorphism rings are generically isomorphic to the direct product of $d$ copies of $k$, whereas the endomorphism rings of the indecomposables are $k[X]/(X^d)$. For the other sheets one gets more interesting degenerations of non-commutative algebras.

2.3. Let $U$ be a $d$-dimensional module which embeds into an $e$-dimensional module $M$ and set $f = d - e$. To get all possible isomorphism classes of quotients of $M$ by $U$ one has to consider bases $g_1, g_2, \ldots, g_d, g_{d+1}, \ldots, g_e$ of $k^e$ such that the first $d$ vectors generate a submodule of $M$ given by $u$. To be more concise, we look at all invertible matrices $g$ such that $g^{-1}m_g$ has triangular shape $\begin{bmatrix} u & \ast \\ 0 & \ast \end{bmatrix}$. The set $Q$ of points $v = (v_1, v_2, \ldots, v_s)$ in $\text{Mod}_d^d(k)$ occurring this way is the union of all orbits corresponding to all quotients of $M$ by $U$. Since $g^{-1}m_g$ has triangular shape as above if and only if the first $d$ columns of $g$ define a homomorphism from $u$ to $m$, the set $Q$ is an irreducible constructible subset. We call a module the generic quotient of $M$ by $U$ if its orbit is dense in $Q$. Of course, a generic
quotient need not exist. But it exists provided the number of isomorphism
classes of possible quotients is finite. This case occurs if \( A \)
is representation
finite or if \( M \) is a regular module over a tame quiver. We also need the
following fact which is implicitly contained in Proposition 3.4 of [32].

**Lemma 2.3.** The generic quotient of \( U \oplus M' \) by \( U \) is \( M' \).

**Proof.** Let \( \epsilon = (\epsilon_1, \epsilon_2) \) be the components of an embedding of \( U \) in
\( U \oplus M' \). Then the map \( \epsilon(\lambda) = (\epsilon_1 - \lambda \text{id}_{\epsilon_2}, \epsilon_2) \), \( \lambda \) in \( k \), is a section as long as
\( \lambda \) is not an eigenvalue of \( \epsilon_1 \). This holds true for an open subset of \( k \), and
the corresponding cokernels are in the orbit of \( M' \). Thus, when \( \lambda \) tends
to 0, \( M' \) degenerates to the cokernel of \( \epsilon \), which was an arbitrary quotient
of \( M \) by \( U \). As an immediate consequence we obtain Riedtmann's result just mentioned.

2.4. Since we want to see what happens to the quotients under
certain degenerations of \( M \), we cannot fix any longer the point \( m \) and we
introduce for arbitrary \( M \) and \( U \) the following varieties \( \mathcal{P} \) and \( \mathcal{Q} \):

\[ \mathcal{P} = \mathcal{P}(M, U) \text{ consists of all } m' \text{ in } \text{Mod}^A(k) \text{ which belong to the}
\text{closure of the orbit } O(m) \text{ of } M \text{ and satisfy } \langle U, M' \rangle = \langle U, M \rangle. \]

\[ \mathcal{Q} = \mathcal{Q}(M, U) \text{ consists of all pairs } (m', g) \text{ with } m' \text{ in } \mathcal{P} \text{ and } g \text{ in } k^{m' \times m'} 
\text{such that the first } d \text{ columns of } g \text{ define a homomorphism from } u \text{ to } m'. \]

It follows immediately from 2.1 that the canonical projection \( p \) from \( \mathcal{Q} \)
to \( \mathcal{P} \) turns \( \mathcal{Q} \) into a vector bundle over \( \mathcal{P} \). The locally closed basis \( \mathcal{P} \)
contains \( O(m) \) as a dense open subset so that \( \mathcal{P} \) is irreducible. Since \( p \)
is open and has irreducible fibers, \( \mathcal{Q} \) has to be irreducible too. Therefore the
inverse image \( \mathcal{Q}' \) of the orbit of \( m \) is an open dense subset of \( \mathcal{Q} \).

**Theorem 2.4.** Let \( M, N, \) and \( U \) be \( A \)-modules such that \( M \) degenerates
to \( N \) and \( \langle U, M' \rangle = \langle U, N' \rangle \). Then we have:

(a) If \( U \) embeds into \( N \), it embeds into \( M \) too.

(b) The closure \( \bar{Q} \) of the quotients of \( M \) by \( U \) contains all quotients of
\( N \) by \( U \). In particular, any quotient of \( N \) by \( U \) is a degeneration of the
generic quotient of \( M \) by \( U \), if this exists.

**Proof.** (a) We consider the vector bundle \( p: \mathcal{Q} \rightarrow \mathcal{P} \) introduced above.
By assumption, the open subset \( \mathcal{Q}' \) of points \( (l, g) \) in \( \mathcal{Q} \), where the deter-
minant of \( g \) does not vanish, is not empty. Since \( \mathcal{Q} \) is irreducible, \( \mathcal{Q}' \) and \( \mathcal{Q}' \)
meet each other. This means that \( U \) embeds into \( M \).

(b) On \( \mathcal{Q}' \), we can define a morphism \( \varphi \) to \( \text{Mod}^A_k \) which maps a
pair \( (l, g) \) to the point \( v \) given by the right lower corners of the matrices
Since the intersection of $\mathcal{D}'$ and $\mathcal{D}''$ is dense in $\mathcal{D}'$, its image $Q$ is dense in the image of $\mathcal{D}''$ which contains all quotients of $N$ by $U$. The rest is obvious.

The theorem gives some new necessary conditions for degenerations which are difficult to use in practice because the quotients usually are not so easy to determine. Nevertheless, one obtains some inductive arguments, say by fixing the dimension of some isotypical component of the socle. Unfortunately, the last implication in part (b) cannot be reversed in general because $M$ might not be the “generic extension.” We will come back to these extensions in Section 6.

One can see by examples that the condition $\langle U, M \rangle = \langle U, N \rangle$ is needed for both parts of 2.4 (see 7.2).

**Corollary 2.5.** Let $X$, $Y$, and $Z$ be $A$-modules such that $\langle X, Y \rangle$ equals $\langle X, Z \rangle$. If $X \oplus Y$ degenerates to $X \oplus Z$ then $Y$ degenerates to $Z$.

**Proof.** Apply Lemma 2.3 and Part (b) of 2.4 to $M = X \oplus Y$, $N = X \oplus Z$, and $U = X$.

This cancellation result and its dual show that projectives or injectives can always be cancelled in degenerations. Another interesting application of 2.5 is given in the next chapter.

**2.6.** Let us conclude this chapter with a somewhat curious representation theoretic consequence of Hilberts’ Basissatz.

**Lemma 2.6.** Given a $d$-dimensional module $M$, there are finitely many indecomposables $U_1, U_2, ..., U_t$ depending only on $M$ such that for any $d$-dimensional module $N$ the conditions $\langle M, U_i \rangle \leq \langle N, U_i \rangle$ for $i = 1, 2, ..., t$ imply $M \leq N$.

**Proof.** For any indecomposable $U$ the set of modules $N$ satisfying $\langle M, U \rangle \leq \langle N, U \rangle$ is closed. The set of all $N$ with $M \leq N$ is the intersection of all these sets. It is a finite intersection because $\text{Mod}^d_A(k)$ is a Noetherian topological space.

In general, there is no such finite set which works for all $M$. For in that case the map $X \mapsto (\langle X, U'_1 \rangle, \langle X, U'_2 \rangle, ..., \langle X, U'_t \rangle)$ embeds the isomorphism classes of $d$-dimensional modules into a finite subset of $\mathbb{N}^t$.

### 3. DEGENERATIONS OF PREPROJECTIVES

**3.1.** Throughout this article, we call a module preprojective if all of its indecomposable direct summands belong to a preprojective component $\mathcal{P}$ of the Auslander–Reiten quiver of $A$ [35]. This notion of preprojective is much more restrictive than the notion used by Auslander and Smalo in [5].
In particular, it can happen now that no module at all is preprojective. On the other hand, a lot of interesting algebras have preprojective components, e.g., the path algebras of quivers without oriented cycle or the representation directed algebras. Also, the important separation criterion of Bautista and Larrión ensures the existence of preprojective components for many algebras.

We use also the dual notion of a preinjective module. Modules which have no preprojective or preinjective direct summand are called regular. Of course, any module $M$ admits an essentially unique decomposition $M = M_p \oplus M_\rho \oplus M_i$ into its preprojective, regular, and preinjective parts.

On the set of (isomorphism classes of) indecomposable modules one has the relation $U \leq V$ which denotes the existence of a sequence of non-zero non-invertible homomorphisms $U = U_1 \rightarrow U_2 \rightarrow \cdots \rightarrow U_n = V$. In general, $\leq$ is not anti-symmetric, whence is not a partial order. However, it induces a partial order on the indecomposable preprojectives.

**Lemma 3.1.** Let $M$ and $N$ be $A$-modules satisfying $M \leq N$ and $\langle M, N \rangle = 0$. Moreover, let $U$ be an indecomposable preprojective. Then we have:

(a) $U$ is $\leq$-minimal with the property $\langle N, U \rangle > \langle M, U \rangle$ if and only if $U$ is a $\leq$-minimal direct summand of $N$.

(b) If (a) is satisfied and $0 \rightarrow U \rightarrow X \rightarrow \text{Tr} DU \rightarrow 0$ is the almost split sequence then we get:

\[ \langle N \oplus X, V \rangle \leq \langle M \oplus U \oplus \text{Tr} DU, V \rangle = \langle N, V \rangle - \langle M, V \rangle - \alpha(V) \]

where $V$ is any indecomposable and $\alpha(V)$ is 1 if $V$ is isomorphic to $U$ and 0 otherwise. In particular, $M \oplus U \oplus \text{Tr} DU \leq N \oplus X$ holds.

(ii) $\langle X, M \rangle = \langle X, N \rangle$.

**Proof.** (a) (Compare [32].) The proof is easy and rests on the following two facts: (1) Any module $M'$ with $\langle M', U \rangle > 0$ contains a direct summand $V \leq U$. (2) By the definition of almost split sequences $0 \rightarrow \text{Tr} U \rightarrow Y \rightarrow U \rightarrow 0$ induces for any module $Z$ an exact sequence $0 \rightarrow \text{Hom}(Z, \text{Tr} U) \rightarrow \text{Hom}(Z, Y) \rightarrow \text{Hom}(Z, U) \rightarrow k^z \rightarrow 0$ where $z$ is the multiplicity of $U$ as a direct summand in $Z$. Similarly, if $U$ is projective one gets an exact sequence $0 \rightarrow \text{Hom}(Z, \text{rad} U) \rightarrow \text{Hom}(Z, U) \rightarrow k^z \rightarrow 0$.

(b) Note that $U$ is not injective because $M$ and $N$ have the same dimension. Part (i) now follows immediately from property 2 above. Part (ii) can be derived from the dual of property 2 by going from the left to the right in the preprojective component that $U$ belongs to. Alternatively, one can use the nice formula $\langle V, N \rangle - \langle V, M \rangle = \langle N, D \text{Tr} V \rangle - \langle M, D \text{Tr} V \rangle$ of Auslander and Reiten cited in the Introduction. By assumption the right-hand side vanishes for all non-projective indecomposable direct summands $V$ of $X$. The same holds for projective $V$, because $M$ and $N$ have the same dimension vector.
3.2. Suppose now we are in the situation of Lemma 3.1 and we want to show that $M \leq N$ implies $M \leq_{\text{deg}} N$. By part (b.ii) and the cancellation theorem we only have to prove that $M \oplus X \leq_{\text{deg}} N \oplus X$. Of course, $M \oplus X$ degenerates to $L = M \oplus U \oplus \text{Tr } DU$. By 3.1(b.i), $L \leq N \oplus X$ also holds true. Even though these modules are bigger than $M$ and $N$, their “distance” is smaller in the sense that the inequality $\langle N, V \rangle - \langle M, V \rangle \geq \langle N \oplus X, V \rangle - \langle L, V \rangle$ holds for all indecomposables $V$ and it is strict for $U = V$. Of course, this observation should be the basis of an induction, but, to cite A. V. Roiter, the question is: “Induction on what?” If $A$ is representation-finite there is no problem.

**Proposition 3.2.** Let $A$ be representation directed. Then the partial orders $\leq_{\text{deg}}$ and $\leq$ coincide.

**Proof.** If $M \leq N$ is given we define $d(M, N)$ as the sum of all terms $\langle N, V \rangle - \langle M, V \rangle$ where $V$ runs through a representative system of isomorphism classes of indecomposables. If $d(M, N)$ is zero, $M$ and $N$ are isomorphic by Auslanders’ theorem (see 1.2). In the inductive step we can split off $(M, N)$ and assume that $M$ and $N$ have no summand in common. Then we choose $U$ as in Lemma 3.1 and proceed as in the preceding discussion. Induction applies because $d(L, N \oplus X)$ is one less than $d(M, N)$.

The above proof uses only the cancellation theorem and almost split sequences, but it does not give the equivalence of $\leq_{\text{ext}}$ and $\leq$. We will prove this in the next section by constructing other types of short exact sequences than almost split sequences.

3.3. The main result of this chapter is the following theorem which includes Proposition 3.2 as a special case.

**Theorem 3.3.** A preprojective module $M$ degenerates to another module $N$ if and only if $M \leq N$ holds.

The proof given in 3.4 rests on the cancellation theorem, the existence of almost split sequences, and the next result which is of interest for its own. It generalizes a result of Happel and Ringel in [24] and uses some of their ideas. We call a module stretched if in its decomposition $M = \oplus M_i$ into indecomposables there is no pair of not necessarily different indices $i$ and $j$ such that $M_i \leq U \leq \text{Tr } DU \leq M_j$ holds for some indecomposable $U$.

**Proposition 3.4.** Let $M$ be a stretched module. Then we have:

(a) All modules $N$ with the same dimension vector as $M$ satisfy $M \leq N$. In particular, there is up to isomorphism at most one stretched module for each dimension vector and, if there is an indecomposable, it is isomorphic to $M$. 


(b) If $M$ is in addition preprojective, it degenerates to all modules with the same dimension vector.

Proof. (a) We may assume that $A$ is basic and given by the path algebra of its Gabriel quiver with relations. We interpret $A$ as a finite category and look at the full subcategory $B$ of $A$ which lives on the convex hull of the support of $M$. We claim that $M$ considered as an $B$-module is stretched again. Indeed, if we have an indecomposable $B$-module $U$ with $M_i \leq U \leq Tr D' U \leq M_j$, where $Tr D'$ is the transpose of the dual with respect to $B$, we get the following commutative diagram with exact rows:

$$
\begin{array}{cccccc}
0 & \rightarrow & U & \rightarrow & Y & \rightarrow & Tr DU & \rightarrow & 0 \\
& & \downarrow{f} & & \downarrow{g} & & \\
0 & \rightarrow & U & \rightarrow & Y' & \rightarrow & Tr D' U & \rightarrow & 0.
\end{array}
$$

Here $U$, $Y'$, and $Tr D' U$ are considered as $A$-modules by extension with zero so that the lower almost split sequence remains exact. Thus $f$ exists by the defining property of almost split sequences and it induces $g$. If this would be zero the upper almost split sequence would split. Therefore we get $M_j \leq U \leq Tr DU \leq M_i$ and we can assume $A = B$ right from the beginning.

In that case the projective dimension of any indecomposable $U$ satisfying $U \leq M_i$ for some $i$ is at most 1. For otherwise we have $\langle I(x), D Tr U \rangle \neq 0$ for the indecomposable injective $I(x)$ corresponding to some point of the Gabriel quiver $Q$ of $A$ (see [35]). Since $A$ equals $B$ there is a path in $Q$ from $x$ to some point $y$ such that $\langle M_j, I(y) \rangle \neq 0$ holds for some $j$. We arrive at the contradiction $M_j \leq R(y) \leq I(x) \leq D Tr U \leq U \leq M_i$. Since all indecomposable projectives $P$ satisfy $P \leq M_i$ for some $i$, all direct summands of all rad $P$ have projective dimension at most 1, which implies that $A$ has global dimension at most 2. Thus we can consider Ringel's bilinear form on the Grothendieck group of mod $A$. For any indecomposable $U$ and any $N$ with the same dimension vector as $M$ we obtain because of $\langle M, U \rangle \geq 0$ the equality

$$
\langle M, U \rangle - \langle M, U \rangle^1 = \langle N, U \rangle - \langle N, U \rangle^1 + \langle N, U \rangle^2.
$$

If $\langle M, U \rangle$ does not vanish we have $\langle M, U \rangle^1 = 0$. Otherwise we would get $M_i \leq U \leq Tr DU \leq M_j$ for some $i$ and $j$ from the important Auslander–Reiten formula $Ext^1(M, U) = D Hom(Tr DU, M)$. Similarly, if $\langle M, U \rangle \neq 0$, we obtain $\langle Tr DU, A \rangle = 0$ from the impossibility of $M_i \leq U \leq Tr DU \leq P(x) \leq M_j$. This means that the injective dimension of $U$ is at most 1 which implies that $\langle M, U \rangle$ equals $\langle N, U \rangle - \langle N, U \rangle^1$. Our claim $M \leq N$ follows and it is clear by 1.2 that there is up to isomorphism at most one stretched module with the same dimension vector. If there is an indecomposable $U$
we get $M_i \leq U$ and $U \leq M_j$ for some $i$ and $j$ from $M \leq U$. This means that $U$ is stretched, and therefore it is isomorphic to $M$.

(b) It is easy to see that $M$ considered as a $B$-module is preprojective again. Therefore we can assume $A = B$, and we can use all the additional information that we have derived in the preceding proof. Suppose now that some $N$ with the same dimension vector is not a degeneration of $M$. Then we can take such an $N$ which is minimal with respect to $\leq_{\text{deg}}$ and we can assume that $\langle M, N \rangle = 0$. By part (a) we have $M \leq N$. In particular, $0 \neq \langle M, M \rangle \leq \langle N, M \rangle$ shows that $N$ contains an indecomposable preprojective $U$ which we can assume to be minimal with respect to $\leq$. Then we get $\langle M, U \rangle = 0$, whence $0 \neq \langle M, U \rangle - \langle M, U \rangle^1 = \langle N, U \rangle - \langle N, U \rangle^1 + \langle N, U \rangle^2$. We infer $\langle N, U \rangle^1 \neq \langle N, U \rangle = 0$. Since $U$ is preprojective it admits no non-trivial self extensions. Hence $N$ decomposes as $U \oplus N'$ and there is a non-split exact sequence $0 \to U \to X \to N' \to 0$. This gives us the contradiction that $N$ is a proper degeneration of $X$.

**Corollary 3.5.** If $M$ is a stretched preprojective module its connected component is an integral Cohen–Macaulay scheme.

**Proof.** With the help of 3.4 the proof of Proposition 2 in [14] carries over to the present situation.

### 3.4.

The proof of Theorem 3.2 will be by double induction on the following two natural numbers $i(M, N)$ and $d(M, N)$ attached to any pair of modules $M, N$ such that $M$ is preprojective. Clearly we can assume that $M$ and $N$ have no direct summand in common. By definition, $d(M, N)$ equals the sum of all terms $\langle N, U \rangle - \langle M, U \rangle$ where $U$ is an indecomposable satisfying $U \leq V$ for some indecomposable preprojective direct summand $V$ of $M \oplus N$. Using 3.1(a), one sees that $d(M, N) = 0$ is equivalent to the fact that $M$ and $N$ are isomorphic. Similarly, $i(M, N)$ is the sum of all $\langle N, U \rangle - \langle M, U \rangle$, where $U$ is an indecomposable satisfying $U \leq$ for some indecomposable injective preprojective $I$. Note that in contrast to the first sum the second sum runs over a fixed index set which is independent of $M$ and $N$.

If $i(M, N) = 0$ we proceed by induction on $d(M, N)$ starting with zero. In the inductive step we choose a preprojective $U$ as in Lemma 3.1(a), i.e., $\leq$-minimal with the property $\langle N, U \rangle \neq \langle M, U \rangle$. We have to distinguish two cases:

1. If there is such an $U$ satisfying $\text{Tr} DU \leq V$ for some indecomposable direct summand $V$ of $M$, then we look at the almost split sequence $0 \to U \to X \to \text{Tr} DU \to 0$ and we obtain $d(M \oplus U \oplus \text{Tr} DU, N \oplus X) = d(M, N) - 1$. Arguing as in 3.2, we conclude $M \leq_{\text{deg}} N$ from $M \oplus X \leq_{\text{deg}} N \oplus X$.
(2) If we are not in case 1, then $M$ is stretched and Proposition 3.3 applies. Indeed, given a sequence $M_i \leq V \leq TrDV \leq M_j$ for some $i$ and $j$ and some indecomposable $V$, we take a $\leq$-minimal indecomposable direct summand $U$ of $M \oplus N$ satisfying $U \leq M_i$. Then we have a sequence of irreducible maps between indecomposables $U \to U_1 \to \cdots \to U_r = M_i \to U_{i+1} \to \cdots \to U_r = V$. Since $\langle M, N \rangle = 0$, no indecomposable $U_i$ is injective and we find $TrDV \leq TrDV \leq M_j$, so that we are in case 1.

Finally, if $\langle M, N \rangle$ is not zero, we take an indecomposable with $U \leq I$ for some injective $I$ and with $\langle N, U \rangle \neq \langle M, U \rangle$. This time we have $\langle M \oplus U \oplus TrDV, N \oplus X \rangle = \langle M, N \rangle - 1$ so that the induction shows $M \oplus X \leq_{eq} N \oplus X$. The cancellation theorem ends the proof.

4. DEGENERATIONS TO PREPROJECTIVES

4.1. The first result in this chapter reads as follows:

THEOREM 4.1. Let $M$ and $N$ be $A$-modules of the same dimension and suppose that $N$ is preprojective. If $M \leq N$ holds then $M$ is preprojective and there is an exact sequence $0 \to L_1 \to M \to L_2 \to 0$ such that $M \leq L_1 \oplus L_2 \leq N$.

Proof. Any indecomposable direct summand $U$ of $M$ satisfies $0 \leq \langle U, M \rangle \leq \langle U, N \rangle$. Therefore, $U$ admits a non-zero homomorphism to some direct summand of $N$, whence $U$ is preprojective and so is $M$.

Next, we decompose $M = \oplus M_i$ into indecomposables $M_i$ with $1 \leq i \leq r$ and $1 \leq j \leq n_i$ such that $M_{ij}$ is isomorphic to $M_{kl}$ if and only if $i = k$. We often identify $M_{ij}$ with $M_{i1}$ in the following proof. Furthermore, we number the $M_{ij}$'s in such a way that $\langle M_{i1}, M_{j1} \rangle > 0$ implies $i \geq j$. Thus, in view of $\text{End}_A M_{ij} = k$, we have for all $i$ the inequality:

$$\langle M_{i1}, N \rangle \geq \langle M_{i1}, M \rangle = \langle M_{i1}, M_{i} \rangle = \langle M_{i1}, M_{i-1} \rangle + n_i.$$  \hspace{1cm} (*)

Here, $M_i$ is the direct sum of all $M_{ij}$ for $k \leq i$ and $M_0 = 0$.

We are going to define a map $f$ from $M$ to $N$ such that none of its components $f_{ij}: M_{ij} \to N$ factors through the others. These components will be constructed by induction on $i$. For $i = 1$, we choose $f_{11}, f_{12}, \ldots, f_{1n_1}$ to be linearly independent in $\text{Hom}_A(M_{11}, N)$, which is possible by (*). In the induction step we denote by $f_{i-1}$ the homomorphism from $M_{i-1}$ to $N$ which is defined by the $f_{ij}$'s for $k \leq i-1$. By (*), again, there are elements $f_{i1}, f_{i2}, \ldots, f_{in_i}$ in $\text{Hom}_A(M_{i1}, N)$ which are linearly independent modulo $f_{i-1} \text{Hom}_A(M_{i1}, M_{i-1})$. Assume that some $f_{ik}$ factors through the others, i.e., that we have an equation $f_{ik} = \sum f_{ji}g_{ji}$ for some $g_{ji}$ in $\text{Hom}_A(M_{ki}, M_{ji})$.\n

where the summation runs through all pairs \((i, j)\) different from \((k, l)\). Then the \(g_{ij}\)'s are zero for \(i > k\) and scalars for \(i = k\). Hence, modulo \(f_{k+1} \cdot \Hom_A(M_{k+1}, M_{k+1})\), the component \(f_{kl}\) is a linear combination of the \(f_{ij}\)'s with \(j \neq 1\) which contradicts our construction.

Now, we look at the kernel \(K\) of the map \(f\) defined before, which is preprojective. Since \(M\) and \(N\) have the same dimension and are not isomorphic, \(K\) is not zero, and we can take the inclusion \(s\) of a \(\ll\)-maximal indecomposable direct summand \(K'\) of \(K\). By the Snake Lemma we obtain the following commutative diagram with exact rows and columns:

\[
\begin{array}{cccccc}
0 & & & & & \\
\downarrow & & & & & \\
K' & \rightarrow & M & \rightarrow & C & \rightarrow & 0 \\
\downarrow s & & & & & \\
0 & \rightarrow & K & \rightarrow & M & \rightarrow & N \\
\downarrow & & & & & \\
 & & & & & \\
 & & & & & \\
K' & \rightarrow & K & \rightarrow & 0.
\end{array}
\]

We claim that \(0 \rightarrow K' \rightarrow M \rightarrow C \rightarrow 0\) is the wanted exact sequence. Indeed, for any indecomposable \(V\) we have exact sequences \(0 \rightarrow \Hom_A(V, K') \rightarrow \Hom_A(V, M) \rightarrow \Hom_A(V, C) \rightarrow \Ext_A(V, K')\) and \(0 \rightarrow \Hom_A(V, K') \rightarrow \Hom_A(V, M) \rightarrow \Hom_A(V, N)\). If \(\Ext_A(V, K')\) vanishes we have \(\langle V, K' \rangle = \langle V, K' \oplus C \rangle \ll \langle V, N \rangle\). If \(\Ext_A(V, K')\) is not zero, we have \(K' \ll V\). This implies \(\langle V, K' \rangle = 0\) and \(\langle V, K' \oplus C \rangle = 0\) by the choice of \(K'\). Thus we obtain in this case \(\langle V, M \rangle \ll \langle V, K' \oplus C \rangle = \langle V, C \rangle \leqslant \langle V, N \rangle\). So it only remains to exclude that \(M\) and \(K' \oplus C\) are isomorphic. But in this case the sequence we are interested in splits as is easily seen by applying \(\Hom_A(V, K')\) to it and counting dimensions. Thus there is some \(h\) in \(\Hom_A(M, K')\) such that \(hgs = \text{id}_C\). Hence there is some index \((k, l)\) such that \(g_{kl}^{-1}\) is invertible. By the second row of our commutative diagram we have \(\sum f_{ij}g_{ij} = 0\). We infer that \(f_{kl} = (\sum f_{ij}g_{ij}s)(g_{kl}s)^{-1}\) where the summation runs through all pairs \((i, j)\) different from \((k, l)\). This contradicts the construction of \(f\) and completes the proof of Theorem 4.1.
4.2. An obvious induction shows:

**Corollary 4.2.** The partial orders $\leq_{\text{ext}}$ and $\leq$ coincide on preprojective modules.

The next lemma describes the minimal degenerations to a certain extent.

**Lemma 4.3.** Let $C$ be a full subcategory of some module category which is closed under extensions and direct summands. Assume that the partial orders $\leq_{\text{ext}}$ and $\leq$ coincide on the isomorphism classes of modules belonging to $C$. If a module $N$ in $C$ is a minimal degeneration of some module $M$ in $C$, then there are indecomposable modules $U$ and $V$ in $C$ and an exact sequence $0 \rightarrow U \rightarrow M' \rightarrow V \rightarrow 0$ such that $M = (M, N) \oplus M'$ and $N = (M, N) \oplus U \oplus V$.

**Proof.** Writing $M = (M, N) \oplus M'$ and $N = (M, N) \oplus N'$, we have by minimality an exact sequence $0 \rightarrow L_1 \rightarrow M' \rightarrow L_2 \rightarrow 0$ such that $N' = L_1 \oplus L_2$. If $L_1$ is decomposable we take a retraction $r$ onto a proper direct summand $L'$. Then we have the following commutative diagram with exact rows and columns:

```
  0 0
 0 -> L'' -> L'' -> 0
 0 -> L_1 -> M' -> L_2 -> 0
 0 -> L' -> M' -> L_2 -> 0
  0 0
```

We infer that $M' \leq L'' \oplus M'' \leq L'' \oplus L' \oplus L_2 \leq L_1 \oplus L_2 = N'$. Since $0 \neq L''$ is not a direct summand of $M'$, we have $M' < L'' \oplus M''$ so that $L'' \oplus M''$ is isomorphic to $N'$ by minimality. The theorem of Krull–Remak–Schmidt implies that $M''$ is isomorphic to $L' \oplus L_2$. Therefore, the lower exact row splits. Consequently, $L'$ is a direct summand of $M'$, a contradiction. Thus, $L_1$ is indecomposable and so is $L_2$ by duality.

There are some interesting and important full subcategories of module categories to which Theorem 4.1 and its corollaries generalize. For instance, if $S$ is a finite partially ordered set one adds a maximal point to
its Hasse diagram and obtains a quiver $Q$. Then one can interpret the
category of representations of $S$ (see, e.g., [35] for the definitions) as the
full subcategory $C$ of representations of $Q$ such that all diagrams commute
and all arrows are represented by injections. Thus, these representations of
a fixed dimension form a locally closed $GL_n$-stable subset $X$ of the variety
of all representations. Now, $C$ has almost split sequences by [6] and the
Auslander–Reiten quiver of $C$ contains a preprojective component by [10]
or [35]. Since $C$ is closed under subobjects and extensions, the proof of
Theorem 4.1 still works in the present situation. In particular, it follows for
a representation-finite $S$ that $M$ degenerates to $N$ in $X$ provided $\langle U, M \rangle \leq \langle U, N \rangle$ holds for all $U$ in $C$, which is in general much smaller than the
whole module category.

For more details and more general situations of the above type we refer
the reader to [10]. However, as was pointed out to me by Dräxler, the
Proof of Theorem 1 given there has to be modified slightly in order to
work.

4.4. The last lemma shows that to understand the minimal
degenerations one “only” has to study certain extensions between indecom-
posables. To this aim we first formulate a variation of 2.4 which will also
be needed in Section 6.

**Lemma 4.4.** Let a module $M$ and an exact sequence $0 \to U \to N \to V \to 0$
be given such that the following three conditions are satisfied:

(i) The orbit of $V$ is open,

(ii) $\langle U, N \rangle$ equals $\langle U, M \rangle$, and

(iii) $M$ degenerates to $N$.

Then there is an exact sequence $0 \to U \to M \to V \to 0$.

**Proof.** By 2.4(a), $U$ embeds into $M$. The set of quotients of $M$ by $U$
contains the orbit of $V$ in its closure by 2.4(b). Since this orbit is open,
$V$ is a quotient of $M$ by $U$.

Using this lemma we get:

**Theorem 4.5.** Let $U$ and $V$ be two indecomposable non-regular modules
with $V \not\cong U$. Then we have:

(a) $M$ degenerates to $U \oplus V$ if and only if $M$ is an extension of $U$
and $V$.

(b) If $U$ and $V$ belong to the same component of the Auslander–Reiten
quiver then $U \oplus V$ is a smooth point to the scheme $\text{Mod}_d A$. In particular,$U \oplus V$ belongs to one irreducible component only which has a dense orbit.
The codimension of the orbit of $U \oplus V$ is $\langle V, U \rangle^1$. Finally, the set of all extensions is open.

Proof. (a) We want to apply 4.4 to $N = U \oplus V$. By 3.4 the orbit of $V$ is open. Up to duality, we can assume that $U$ is preprojective. Since $\langle U, N \rangle - \langle U, M \rangle$ equals $\langle N, D Tr U \rangle - \langle M, D Tr U \rangle = 0$ by a formula mentioned in the Introduction, condition (ii) is also satisfied so that we get the wanted exact sequence.

(b) Suppose now that $V$ is preprojective too, and choose a deg-minimal module $M$ degenerating to $U \oplus V$. Then $M \leq U \oplus V$ holds, and $M$ is therefore preprojective. Hence the indecomposable direct summands of $M$ have no proper self-extensions, and we conclude $\langle M, M \rangle^1 = 0$ from the minimality of $M$. By part (a), there is an exact sequence $0 \to U \to M \to V \to 0$. Using well-known properties of preprojective modules we obtain the following induced exact sequences:

$$0 \to \text{Hom}(V, M) \to \text{Hom}(V, V) \to \text{Ext}(V, U) \to \text{Ext}(V, M) \to 0$$

and

$$0 \to \text{Hom}(V, M) \to \text{Hom}(M, M) \to \text{Hom}(U, M) \to \text{Ext}(V, M) \to 0$$

We calculate

$$\langle U \oplus V, U \oplus V \rangle^1 = \langle V, U \rangle^1$$

$$= \langle V, M \rangle^1 + \langle V, V \rangle - \langle V, M \rangle$$

$$= \langle U, M \rangle + \langle V, V \rangle - \langle M, M \rangle$$

$$= \langle U, V \rangle + \langle U, U \rangle + \langle V, V \rangle - \langle M, M \rangle.$$
an extension of an indecomposable and a simple. The present proof consists in dividing out in a generic way, whereas in my original proof the middle term was glued together from the end terms.

Note that the exact sequences given by 4.5 in general do not exist over the prime field, whereas those corresponding to minimal degenerations always do by the constructive proof of 4.1. For instance, if \( k \) has only \( q \) elements, one can take the quiver with \( q + 2 \) points \( 0, 1, \ldots, q + 1 \) and \( q + 1 \) arrows starting at \( 1, 2, \ldots, q + 2 \) and leading to \( 0 \). Then \( k^2 \) admits \( q + 1 \) one-dimensional subspaces and the representation \( U \), where the arrows are represented by the corresponding inclusions, degenerates to the sum of the indecomposable projective and the injective to the point \( 0 \). But of course, \( U \) is not an extension of these two modules over \( k \).

A consequence of 4.5(b) is an algorithm to calculate dimensions of extension groups for modules over representation-directed algebras in a rather round-about way. But the useful Auslander–Reiten formula expressing \( \text{Ext}(V, U) \) as a quotient of \( D \text{Hom}(D \text{Tr} U, V) \) does not seem to be easy to apply, whereas it is easy to find the generic extension \( M \) with the help of a computer. Of course this algorithm is far from being optimal. I know a slightly better way to compute \( \langle V, U \rangle \) which is based on Lemma 3.1 of [10], but also this method is not completely satisfactory.

4.5. Given preprojective indecomposables \( U \) and \( V \), we denote by \( S \) the set of isomorphism classes of modules degenerating to \( U \otimes V \). This is a finite set on which the partial orders \( \leq_{\text{ext}}, \leq_{\text{deg}} \) and \( \leq \) all agree. By part (b) of 4.5, \( S \) has always a greatest element \( M \) and a smallest one \( U \otimes V \). To analyze the partially ordered sets \( S \) occurring this way seems to be a hard problem. This is in contrast to the fact that for any concretely given \( U \) and \( V \) the set \( S \) and its partial order can be very effectively determined by a computer. This has been done for various cases by Markolf in his Diplomarbeit and we refer to 7.8 for a large example. For representation finite hereditary algebras he has found the following two facts:

1. If \( U \otimes V \) is a minimal degeneration of \( N \), the orbit dimensions differ only by 1.
2. The number of isomorphism classes of such \( N \) is \( 1/2 \cdot r \cdot (r + 1) \) with \( r = \langle V, U \rangle \).

Statement 2 is wrong in general even for representation directed algebras (see 7.8) for which 1 remains an open question. However, statement 1 is false for tame quivers as follows from Section 5 or from [29], where the minimal preprojective degenerations for a quiver of type \( D_4 \) are classified. To prove or disprove property 1 for representation directed algebras the next result might be useful.
Lemma 4.6. Let \( A \) be a representation directed algebra with two indecomposables \( U \) and \( V \) such that \( U \leq V \). If \( U \oplus V \) is a minimal degeneration of \( N \) of codimension \( \geq 1 \), then there exists a direct summand of \( X \) of \( N \) and a minimal degeneration \( X^2 \oplus Y < U' \oplus V' \) for some indecomposables \( U' \) and \( V' \).

We omit the proof. It follows from the lemma that the codimension 1 property holds provided all minimal degenerations are multiplicity free. Unfortunately this is false for general representation directed algebras (see 7.6), but it is true for hereditary algebras as can be shown using the well-known starting functions for these algebras [11].

4.6. To conclude this chapter we extend the equivalence of \( \leq \text{ext} \) and \( \leq \text{deg} \) to a larger class of algebras which includes all representation-finite algebras with radical square zero. So let us assume that the Gabriel quiver \( Q \) of \( A \) contains a point \( x \) such that all path of length 2 with middle point \( x \) are relations. Then one introduces a new quiver \( Q' \), where \( x \) is replaced by two points \( x' \) and \( x'' \). The arrows of \( Q \) not involving \( x \) remain unchanged. Each loop at \( x \) is replaced by an arrow from \( x' \) to \( x'' \), and each arrow \( y \to x \), resp. \( x \to z \), gives rise to an arrow \( y \to x'' \), resp. \( x' \to z \). Thus, \( x \) is divided into a source \( x' \) and a sink \( x'' \). Finally, the relations defining a new algebra \( A' \) with quiver \( Q' \) are just the relations of \( A \) not involving \( x \). For instance, if \( A \) is the algebra \( \mathbb{k}[X, Y]/(X^2, XY, Y^2) \) then \( A' \) is the path algebra of the double arrow studied in the next chapter. It is easy to see that \( A \) and \( A' \) are stably equivalent (see, e.g., [15]) so that their module categories are closely related. There is also a close connection between the degeneration behaviour of the modules as follows:

Proposition 4.7. We keep the notations and assumptions introduced before. Then we have:

(a) If \( \leq \text{deg} \) and \( \leq \) are equivalent for \( A' \) they are so for \( A \).

(b) If \( \leq \text{ext} \) and \( \leq \) are equivalent for \( A' \) the same holds for \( A \).

Proof. Let \( M \) and \( N \) be two \( A \)-modules with dimension vector \( d \) satisfying \( M \leq N \). Let us denote the simple corresponding to the point \( x \) by \( S \). Then we have \( \dim \, \text{soc} \, M(x) = \langle S, M \rangle \leq \langle S, N \rangle = \dim \, \text{soc} \, N(x) \) and also \( \dim \, \text{top} \, M(x) = \langle M, S \rangle \leq \langle N, S \rangle = \dim \, \text{top} \, N(x) \). Since \( \text{rad} \, M(x) \subseteq \text{soc} \, M(x) \) holds under our assumptions, we can perform a suitable base change and assume that the following inclusions hold: \( \text{rad} \, N(x) \subseteq \text{rad} \, M(x) \subseteq \text{soc} \, M(x) \subseteq \text{soc} \, N(x) \). We denote \( \text{rad} \, M(x) \) by \( U \) and we choose a complement \( L \) of \( U \) in \( M(x) = k^{\text{dim} \, L} \). With respect to this decomposition the arrows leading to \( x \), resp. coming from \( x \) are represented in \( M \) and \( N \) by matrices having zero entries in the last \( \dim L \) rows, resp. in the first \( \dim U \) columns. Thus, forgetting these zero parts, we associate to \( M \) and \( N \) two
representations $M'$ and $N'$ of $A'$ with the same dimension vector, whose $x'$-component is $\dim L$ and whose $x''$-component is $\dim U$. To show that $M \leq N$ is equivalent to $M' \leq N'$ we use Lemma 2.1 of [32]. Indeed, in the notations introduced there one only has to use $\varphi_{p,q}$ in the radical as soon as $j_i$ equals $i_k$, because only such elements occur in minimal projective resolutions. From this remark the wanted equivalence follows.

Now we prove the two parts of 4.7 separately. In part (a) we know that $M'$ degenerates to $N'$. This means that $N$ already belongs to the closure of the orbit of $M$ under the obvious block-diagonal subgroup so that $M$ degenerates to $N$ a fortiori.

In part (b), we have $A'$-modules $M_i = U_i \oplus V_i$ and short exact sequences $0 \to U_i \to M_i \to V_i \to 0$ for $i \geq 1$ satisfying $M_0 = M'$ and $M_1 = N'$. The obvious functor $F$ from $\text{mod } A'$ to $\text{mod } A$ with $FX(x) = X(x') \oplus X(x'')$ is exact and satisfies $FM' = M$ and $FN' = N$. Therefore, $M \leq_{\text{ext}} N$ also holds true.

We infer from Theorems 4.1 and 5.1 the following consequence.

**Corollary 4.8.** If $A$ is representation-finite with radical square zero, then the partial orders $\leq_{\text{ext}}$ and $\leq$ are equivalent. The same holds true for $k[XY]/(X^2, XY, Y^2)$.

**Proof.** One only has to observe that any representation finite algebra as above can be reduced to a product of path algebras of Dynkin quivers by applying the procedure described before several times (see [18]).

5. DEGENERATIONS OF MATRIX PENCILS

5.1. First of all recall the classical notion of a matrix pencil of type $(m, n)$ over a field $k$ (see [21]). By definition, such a pencil consists of a pair $(L, R)$ of two $(m \times n)$-matrices with coefficients in $k$. Furthermore, two pencils $(L, R)$ and $(L', R')$ are equivalent if there are invertible matrices $g$ and $h$ such that $gLh^{-1} = L'$ and $gRh^{-1} = R'$ hold simultaneously. In modern language, the set of matrix pencils of type $(m, n)$ can be identified with the variety $\text{Mod}_d^2(k)$, where $A$ is the path algebra of the quiver $\xymatrix{1 \ar[r]^1 & 1 \ar[r]^0 & 0} \ar[l]_0$ and $d$ the dimension vector $(n, m)$. Of course the equivalence classes correspond to the orbits.

Our main result in this section will be that the partial orders $\leq_{\text{ext}}$ and $\leq$ coincide on the isomorphism classes of $A$-modules. To prove this we follow the strategy developed by Abeasis and del Fra and Riedtmann, and we proceed in two steps:
(1) For all pairs $U, V$ of indecomposables we determine the possible middle terms $M$ in short exact sequences $0 \to U \to M \to V \to 0$.

(2) If $M \leq N$ are neighbors with respect to $\leq$, and if $(M, N) = 0$, then we show that $N$ is the direct sum of some appropriate $U$ and $V$ as in Step 1.

The proof of the first part is straightforward using the well-known structure of the indecomposables and the homomorphism spaces between them. The second part is harder to prove, but as a reward one obtains also a precise recipe of how to degenerate a given module into another one provided their decompositions into indecomposables are known.

The indecomposable $A$-modules are divided into preprojective, regular, and preinjective modules. The defect of a module $M$, which is the difference between $\dim M(1)$ and $\dim M(2)$, is $-1$ on preprojective, $0$ on regular, and $+1$ on preinjective indecomposables. Regular modules are called non-degenerate pencils in the classical terminology. Their classification in terms of elementary divisors is due to Weierstrass, whilst the more delicate degenerate case was solved by Kronecker, who also provides an algorithm to separate the regular part from the rest. Ironically enough, the classification of preprojectives and preinjectives is nowadays a trivial matter thanks to the existence of almost split sequences, whereas the regular modules still need some extra arguments.

To be more precise and to fix the notation, we denote by $P(i)$ the indecomposable preprojective with dimension vector $(i, i+1)$. Choosing bases $e_1, e_2, \ldots, e_i$ and $f_1, f_2, \ldots, f_{i+1}$ of $P(i)(1)$ and $P(i)(2)$ respectively, the action of $\ast$ and $\backslash$ on these vectors can be illustrated conveniently by the following diagram:

\[
\begin{array}{cccccccc}
e_1 & e_2 & e_3 & \cdots & e_i \\
f_1 & f_2 & f_3 & f_4 & f_i & f_{i+1}
\end{array}
\]

Here, $\ast$ always goes to the left and $\backslash$ to the right. Dually, $I(i)$ has dimension vector $(i+1, i)$ and it can be described by the dual diagram. Finally, the indecomposable regular modules $U(n, e)$ depend on two parameters $n = 1, 2, \ldots$ and $e$ in $P_1(k)$. For any $e$, the dimension vector of $U(n, e)$ is $(n, n)$. For $e \neq \infty$, $\lambda$ is given by the identity matrix $E_n$ and $\rho$ by $e \cdot E_n + N_n$, where $N_n$ is an upper triangular nilpotent Jordan block, while for $e = \infty$ one has $\lambda = N_n$ and $\rho = E_n$.

Clearly, $GL_2(k)$ acts on $A$ by algebra automorphisms fixing the points and mapping the arrows on linear combinations. Consequently, $GL_2(k)$ operates on the modules by scalar extension. Since dimension vectors and
decompositions are preserved, the isomorphism classes of the preprojectives and the preinjectives are fixed, while the $U(n,e)$'s are permuted transitively. This allows us to restrict ourselves to the case $e = 0$ in Step 1.

Finally, we recall the categorical structure of $\text{mod} \ A$ as derived in Ringel's book. There are no non-zero maps from regular or preinjective modules to preprojectives, or from preinjectives to regular modules. Furthermore, the full subcategory of the regular modules is a direct sum of uniserial categories $T(e)$ having $U(1,e)$ as the only simple.

5.2. To formulate our findings in step 1, we associate to any exact sequence $0 \to U \to M \to V \to 0$ a function $\delta = \delta_{\Sigma}$ from the points of the Auslander-Reiten quiver to the natural numbers by $\delta(W) = \langle U \oplus V, W \rangle - \langle M, W \rangle$.

**Proposition 5.2.** Let $U$ and $V$ be indecomposables with $U \leq V$. Then $M$ is an extension of $U$ and $V$ if and only if $M \leq U \oplus V$. Moreover, the functions attached to the neighbors $M \leq U \oplus V$ are given in Table I for all cases occurring up to duality and up to $\text{Gl}_2(k)$-action. Only the non-zero values of $\delta$ are written down.

**Proof.** We have to consider the various cases listed in Table I. First, let $U = P(k)$ and $V = P(l)$ be preprojective with $k \leq l$. Then $M \leq U \oplus V$ forces $M$ to be preprojective too. Looking at the defect, we get $M = P(i) \oplus P(j)$, and using $M \leq U \oplus V$ once more we obtain $k \leq i \leq j \leq l$. Comparing the dimension vectors we see $j = l + k - i$. Thus $M$ and $U \oplus V$ are isomorphic for $l \leq k + 1$, and we have to exhibit the wanted exact sequences in the remaining cases. To do so we simplify our problem using the well-known reflection functors. We can reduce it to the case $U = P(0)$. A glance at the

<table>
<thead>
<tr>
<th>Type</th>
<th>$(U, V)$</th>
<th>$M$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type 1</td>
<td>$(P(k), P(l))$, $1 \geq k + 2$</td>
<td>$P(k + 1) \oplus P(l - 1)$</td>
<td>$\delta = 1$ on $P(j), k \leq j \leq l - 2$.</td>
</tr>
<tr>
<td>Type 2</td>
<td>$(P(k), U(n, 0))$ $k \geq 0$ and $n \geq 1$ with $U(0, 0) = 0$</td>
<td>$P(k + 1) \oplus U(n - 1, 0)$</td>
<td>$\delta = 1$ on $P(j), k \leq j$, and on $U(l, 0), n \leq l$.</td>
</tr>
<tr>
<td>Type 3</td>
<td>$(P(k), I(l))$ $k \geq 0$ and $l \geq 0$ $\oplus U(n, e_i)$ with $e_i \neq e_j$ for $i \neq j$.</td>
<td>$\delta = j + 1$ on $P(k + j)$, $j \geq 0$</td>
<td></td>
</tr>
<tr>
<td>Type 4</td>
<td>$(U(r, 0), U(s, 0))$ $1 \leq r \leq s$ $\oplus U(r - 1, 0) \oplus U(s + 1, 0)$</td>
<td>$\delta = 1$ on $U(j, 0)$ for $r \leq j \leq s$.</td>
<td></td>
</tr>
</tbody>
</table>
diagrammatic presentation of the preprojectives shows that $P(l)$ is an amalgam of $P(1)$ and $P(l−1)$. Furthermore, the $δ$-function given in the table is correct. The fact that this case corresponds to the only minimal degeneration will follow from the next theorem, but of course it can also be shown directly. Indeed, the partially ordered set $S$ introduced in 4.5 is a chain. Moreover, as soon as $l > k + 2$ holds, the codimension of the minimal degeneration is 2.

Similarly, as above we infer from $M \leq P(k) \oplus U(n, 0)$ that $M$ equals $P(k + j) \oplus U(n − j, 0)$ for some $0 ≤ j ≤ n$. To see the desired exact sequence one reduces again to the case $k = 0$ where everything is obvious. Once more, $S$ is a chain, and the $δ$-function is easily checked.

In the third case, let $M \cong P(k) \oplus I(l)$ be given. If $M$ has a preprojective summand $P(i)$ then $k ≤ i$ follows. Since the defect of $M$ vanishes there is also a preinjective direct summand $I(j)$ with $l ≤ j$. Counting dimensions we see that $M$ is isomorphic to $P(k) \oplus I(l)$. Thus $M$ is regular in the remaining cases. The inequality $\langle M, U(1, e) \rangle ≤ \langle U \oplus V, U(1, e) \rangle = 1$ shows that $M$ contains at most one summand from each uniserial category $T(e)$. In other words, $M$ belongs to the open sheet. Taking into account the $GL_2$-action we can assume that $M(\lambda)$ is bijective, whence represented by $E_n$. Accordingly, $M(p)$ can be brought to its rational normal form so that it is a companion matrix having non-zero entries only in the last row and in the first upper diagonal. Forgetting the last row, we have divided the middle term by $P(0)$ and we obtain $I(l)$. Thus, this time $S$ is an antichain.

In the last case, $M$ clearly belongs to $T(0)$, too. From $\langle M, U(1, 0) \rangle ≤ \langle U \oplus V, U(1, 0) \rangle = 2$ we infer that $M = U(i, 0) \oplus U(r + s − i, 0)$ with $0 ≤ i ≤ r + s$. Using $\langle M, U(r + 1, 0) \rangle ≤ \langle U \oplus V, U(r + 1, 0) \rangle = r + \min(r + 1, s)$ we conclude $i ≤ r$. The existence of the desired exact sequences is obvious.

5.3. As Riedtmann does in [32], we consider for any pair of modules $X ≤ Y$ the $δ$-function $δ_{X, Y}(W) = \langle Y, W \rangle − \langle X, W \rangle$ from the indecomposables to the natural numbers. We recall without proof the next simple but useful result in a slightly generalized form (see [32]).

**Lemma 5.3.** Let $X ≤ Y$ and an exact sequence $Σ : 0 → U → M → V → 0$ be given. Suppose that $δ_Σ(W) ≤ δ_{X, Y}(W)$ is true for all $W$. Then we have:

(a) If $X$ equals $X' \oplus M$ then $X ≤ X' \oplus U \oplus V ≤ Y$ holds.

(b) If $Y$ equals $Y' \oplus U \oplus V$ then we have $X ≤ Y' \oplus M ≤ Y$.

Besides this lemma we need the following remark. If $0 → A → ⊕ B_i → C → 0$ is an almost split sequence, one applies $\text{Hom}(X, −)$ and
$\text{Hom}(Y, \_ \_ \_ \_)$ to it and obtains $\mu(X, C) - \mu(X, C) = \delta X, Y (A) + \delta X, Y (C) - \sum \delta X, Y (B)$. Here, $\mu(Z, C)$ denotes the multiplicity of the indecomposable $C$ as a direct summand of $Z$. The above equation shows in particular that we can recover $X$ and $Y$ from $\delta X, Y$ provided that $(X, Y) = 0$.

To illustrate the general strategy for the proof of our main result by a simple example we first treat the case that $Y$ is regular, too. If $n$ is larger than dim $X$, then $U(n, e)$ behaves like an injective with respect to $X(e)$ and $Y(e)$, i.e., we get $\langle X, U(n, e) \rangle = 1/2 \dim X(e) < 1/2 \dim Y(e)$. This shows that $X = \bigoplus X(e)$ with $X(e) \leq Y(e)$, whence we can assume that $X$ and $Y$ belong to $T(0)$ right from the beginning.

We want to show that for neighbors $X \leq Y$ without common direct summand we have $X = U(r - 1, 0) \oplus U(s + 1, 0)$ and $Y = U(r, 0) \oplus U(s, 0)$ for some $1 \leq r \leq s$ (see Table I). As observed before, $\delta = \delta_{X, Y}$ vanishes on $U(n, 0)$ for large $n$, so that we can look at the greatest natural number $s$ such that $\delta(U(s, 0)) > 0$. Remembering that the almost split sequences in $T(0)$ correspond to the case $r = s$ in Table I, we obtain $\mu(Y, U(s + 1, 0)) - \mu(X, U(s + 1, 0)) = 2\delta(U(s + 1, 0)) - \delta(U(s + 2, 0)) - \delta(U(s, 0)) < 0$. Therefore, $U(s + 1, 0)$ is a direct summand of $X$. If $\delta(U(r, 0))$ does not vanish for $j \leq s$ we can apply 5.3(a) to $\Sigma: 0 \to U(1, 0) \to U(s + 1, 0) \to U(s, 0) \to 0$. In the remaining case, let $r - 1$ be the greatest integer smaller than $s$ with $\delta(U(r - 1, 0)) = 0$. Then we get $\mu(Y, U(r - 1, 0)) - \mu(X, U(r - 1, 0)) \leq 0$ so that we can apply 5.3(a) to $0 \to U(r, 0) \to U(r - 1, 0) \oplus U(s + 1, 0) \to U(s, 0) \to 0$.

The same proof as above works for the well-known degenerations of nilpotent matrices, thereby giving an alternative approach to the Gerstenhaber–Hesselink–Flanigan theorem [22, 25, 16].

5.4.

**Theorem 5.4.** The partial orders $\leq_{\text{ext}}$ and $\leq$ coincide for matrix pencils. More precisely, if $X \leq Y$ are neighbors without common direct summand, then we are in one of the situations described in 5.2.

**Proof.** The strategy to prove this is clear. We start with a pair $X \leq Y$ as in the theorem and show that 5.3(a or b) applies to an exact sequence contained in Table I.

By the preceding discussion we can assume that $Y$ contains a preprojective direct summand. Let $k$ be the smallest integer such that $P(k)$ occurs in $Y$. Then we have $\delta(P(k)) > 0$. If $\delta(P(l)) = 0$ holds for some $l > k$, we can take $l$ minimal with that property, and we look at the values of $\delta$ at $P(l - 1)$, $P(l)$ and $P(l + 1)$. If the preprojective parts of $Y$ and $X$ are given by $\bigoplus P(i)^n$ and $\bigoplus P(i)^n$, we have $0 > \delta(P(l)) - \delta(P(l - 1)) = \sum y_i - x_i$, and $0 \leq \delta(P(l + 1)) - \delta(P(l)) = (y_i - x_i) + \sum y_j - x_j$, where both sums run
over all $j \leq l$. Thus we can apply 5.3(b) to $0 \rightarrow P(k) \rightarrow P(k + 1) \oplus P(l) \rightarrow P(l + 1) \rightarrow 0$. From now on we can suppose that $\delta(P(l)) > 0$ for all $l \geq k$. Therefore, if $y_j > 0$ for some $l > k + 1$, we can use 5.3(b) for $0 \rightarrow P(k) \rightarrow P(k + 1) \oplus P(l - 1) \rightarrow P(l) \rightarrow 0$ so that the preprojective part of $Y$ satisfies $y_k > 0$. $y_{k+1} > 0$ and $y_l = 0$ otherwise. Using almost split sequences and 5.3(a) one infers $x_j < 1$ for all $j$. Similarly, $x_j \neq 0 \neq x_l$ for some $j < l$ can be excluded by looking at $0 \rightarrow P(j - 1) \rightarrow P(j) \oplus P(l) \rightarrow P(l + 1) \rightarrow 0$. Consequently, $X_k$ equals 0 or $P(l)$ for some $l \geq k + 1$, and we will deal with the last case first. These are all the reductions we can achieve by using exact sequences of type 1. Applying the dual arguments to the preinjective part of $Y$ we can suppose that it is also of a very special shape if it is non-zero at all.

Next, we will use the sequences of type 2 to simplify the regular parts $\bigoplus X(e)$ and $\bigoplus Y(e)$ of $X$ and $Y$. We set $X(e) = \bigoplus U(n, e)^{\chi(n,e)}$ and $Y(e) = \bigoplus U(n, e)^{\upsilon(n,e)}$. Suppose now that there is an $\varepsilon$ with $y(\varepsilon) = \dim Y(e) > x(\varepsilon) = \dim X(e)$. Let $U(m, e)$ be a direct summand of $Y(e)$ of maximal dimension. Then one has for $n \geq m$ that $\langle Y, U(n, e) \rangle = n \cdot (y_k + y_{k+1}) + 1/2 \cdot x(\varepsilon) > 1/2 \cdot x(\varepsilon) + n \geq \langle X, U(n, e) \rangle$. Thus 5.3(b) applies to $0 \rightarrow P(k) \rightarrow P(k + 1) \oplus (m - 1, e) \rightarrow U(m, e) \rightarrow 0$, and we can assume $x(\varepsilon) \geq y(\varepsilon)$ for all $\varepsilon$ from now on.

If $y_k + y_{k+1}$ equals one we have $\delta(U(n, e)) = \langle Y, U(n, e) \rangle - \langle X, U(n, e) \rangle > 0$ for all $n$ and $\varepsilon$. In case $Y(e)$ is not zero for some $\varepsilon$ we infer $x(\varepsilon) = y(\varepsilon)$ and also $X(e) \leq Y(e)$ so that we are done by the case analyzed in 5.3. If the regular part of $Y$ is zero the same holds for $X$, and we arrive for large $j$ at the contradiction $\langle Y, l(j) \rangle < \langle X, l(j) \rangle$. Thus we have $y_k + y_{k+1} > 1$. If $X(e)$ is not zero for some $\varepsilon$ we choose a direct summand $U(m, e)$ of $X(e)$ of maximal dimension. Then 5.3(a) applies to $0 \rightarrow P(l - 1) \rightarrow P(l) \oplus U(m, e) \rightarrow U(m + 1, e) \rightarrow 0$. Namely for $n \geq m$ the value of $\langle X, U(n, e) \rangle$ increases in each step by 1, whereas the value of $\langle Y, U(n, e) \rangle$ increases at least by $y_k + y_{k+1}$. We are left with the case where the regular part of $X$ vanishes so that $Y(e)$ vanishes too by the relation $x(\varepsilon) \leq y(\varepsilon)$. Since the defect of $X$ is $-1$ or 0, the preinjective part of $Y$ is not zero and we can assume that it is already normalized as $I(k + 1)^{\varepsilon} \oplus I(k')^{\varepsilon}$. Then 5.3(b) can be applied to $0 \rightarrow P(k) \rightarrow U(k + k' + 1, 0) \rightarrow I(k') \rightarrow 0$ as one verifies easily. One only has to observe that $y_k' + y_{k+1}' > 1$ holds if the defect of $X$ is zero.

So far, we have treated all cases with $X \neq 0$. Dually, we can argue for $X \neq 0$ so that we can consider $X$ to be regular from now on. Let us denote by $t(\varepsilon)$ the maximum of all dimensions of all indecomposables $U(n, e)$ occurring in $X$ and $Y$. First, suppose $U(t(\varepsilon), e)$ occurs for some $\varepsilon$ in $Y$. Then 5.3(b) applies to $0 \rightarrow P(k) \rightarrow P(k + 1) \oplus U(t(\varepsilon) - 1, e) \rightarrow U(t(\varepsilon), e) \rightarrow 0$. Namely, $\langle Y, U(t(\varepsilon) - 1, e) \rangle \geq \langle X, U(t(\varepsilon) - 1, e) \rangle = 1/2 \cdot x(\varepsilon)$ holds by assumption, and the left-hand side increases strictly for $t \geq t(\varepsilon) - 1$, whilst
the right side is constant. Therefore, $U(t(e), e)$ always belongs to $X$ and we obtain:

$$\langle Y, U(t(e), e) \rangle = (y_k + y_{k+1}) \cdot t(e) + 1/2 \cdot y(e) \geq \langle X, U(t(e), e) \rangle = 1/2 \cdot x(e).$$

Summing up over all $e$ we get:

$$\begin{align*}
(y_k + y_{k+1}) \cdot \sum t(e) + 1/2 \cdot \text{dim } Y_R & \\
\geq 1/2 \cdot \text{dim } X_R = 1/2 \cdot \text{dim } X & \\
= 1/2 \cdot (\text{dim } Y_R + y_k(2k+1) + y_{k+1}(2k+3) & \\
+ y_k'(2k'+1) + y_{k'+1}'(2k'+3)).
\end{align*}$$

Because of $y_k + y_{k+1} = y_k' + y_{k'+1}'$, we finally obtain $\sum t(e) \geq k + k' + 1$. Therefore we can define $j$ by $k + j + 1 = \sum t(e)$ and consider the exact sequence $0 \rightarrow P(k) \rightarrow \bigoplus U(t(e), e) \rightarrow R(j) \rightarrow 0$. It is easy to verify that 5.3(a) applies to it. This finishes the proof of Theorem 5.4.

5.5. The minimal degenerations of Table I can also be divided into three types depending on whether $\delta$ has finite or bounded or unbounded support. It is not surprising that the last case is the most difficult one to deal with in the foregoing proof.

For an arbitrary algebra the length of a chain $X = X_0 < X_1 < \cdots < X_n = Y$ of neighbors is bounded by $\langle Y, Y \rangle - \langle X, X \rangle$, but it depends highly on some choices. For a concrete example one may look at the nilpotent matrices. Nevertheless, any chain as above gives rise to the relation $\delta_{X, Y} = \sum \delta_{X_i, X_{i+1}}$. Coming back to our matrix pencils we see from Table I and this equation that the number of "difficult" degenerations with unbounded $\delta$ is uniquely determined by $X$ and $Y$, e.g., as $\delta(P(j+1)) - \delta(P(j))$ for $j$ larger than $\text{dim } X$.

Also, the remark above gives a $K$-theoretic interpretation to some of our results. Namely, let $K_0(A, \oplus)$ be the Grothendieck group of mod $A$ with respect to split-exact sequences. Denote by $\mathbb{Z}(A)$ the subgroup generated by all $X - Y + Z$ coming from short exact sequences $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ and by $\mathfrak{n}(A)$ the monoid generated by these expressions. A nice result of Auslander and Butler says that $A$ is representation finite if and only if $\mathbb{Z}(A)$ is generated by the terms $X - Y + Z$ coming from almost split sequences. Our results imply that for a representation directed algebra or for the matrix pencils $\mathfrak{n}(A)$ is generated as a monoid by the terms $U - M + V$ coming from minimal degenerations to the direct sum of two indecomposables.

It is possible that Theorem 5.4 generalizes to all tame concealed algebras. Indeed, I have proved it for regular modules over these algebras by showing the equivalence of $\leq_{\text{ext}}$ and $\leq$ for representations of an oriented cycle.
with an arbitrary number of points. I omit this proof because it is not so easy and because it answers only a very special case of the above problem. It is clear that the brute force method used to show 5.4 leads to tremendous combinatorial difficulties which one might hope to circumvent by a more direct construction as in 4.1. However, it is easy to see that $\leq_{\text{deg}}$ and $\leq$ are not equivalent for wild quivers (see 7.3). Finally, a closer look at our arguments shows that there is a finite algorithm which decides for a given pair of matrix pencils $X$ and $Y$ whether $X \leq Y$ holds or not provided that one knows the eigenvalues involved in $X$ and $Y$.

6. INDECOMPOSABLES OVER TAME CONCEALED ALGEBRAS
AS EXTENSIONS OF INDECOMPOSABLES

6.1. Our main in this chapter is to prove more precisely:

\textbf{Theorem 6.1.} Let $U$ be a non-simple indecomposable over a tame concealed algebra. Then there is an exact sequence $0 \to U_1 \to U \to U_2 \to 0$ with indecomposable $U_i$'s, one of which is simple.

Whether this statement remains true for an arbitrary algebra seems to be an open question. It is true for representation finite algebras by \cite{9}, and using Theorem 6.1 and the deep results of \cite{7}, it should be possible to extend the theorem with some effort to minimal representation infinite algebras. For arbitrary algebras however, I have no idea how to prove it.

In the case considered above the proof rests on the detailed knowledge of the corresponding module categories as developed in Ringels' book. Fortunately, we never have to use any explicit description of the modules involved, say as representations of quivers. Nevertheless it was the detailed study of the case $D_4$ which forced me to find out enough geometric arguments to succeed with a "readable" proof. These geometric considerations should be useful not only in the present situation, and we describe them in the next two sections. Roughly speaking, statement 6.1 is true because in the above exact sequence one of the modules has to be generic.

6.2. Throughout this section, $A$ is an arbitrary algebra.

\textbf{Lemma 6.2.} Let $U_1$ and the exact sequence $0 \to U'_1 \to U \to U_2 \to 0$ be given. Assume that the following holds:

(i) The orbit of $U_2$ is open,
(ii) $\langle U_1, U \rangle = \langle U'_1, U \rangle$,
(iii) $U'_i$ degenerates to $U'_1$.

Then there is an exact sequence $0 \to U'_1 \to U \to U_2 \to 0$.
Proof. We proceed similarly as we did in 2.4, but this time we vary the submodule and keep the module fixed. Thus, if $d_1$, $d_2$, and $d_3$ are the dimensions of $U_1$, $U$, and $U_2$, respectively, we look at the varieties $B$ and $Z$ defined as follows: $B$ consists of all $l$ in the closure of the orbit of $U_1$ satisfying $\langle L, U \rangle = \langle U_1, U \rangle$, and $Z$ of all pairs $(l, g)$ with $l$ in $B$ and $g$ in $k^{d_3 \times d_2}$ such that the first $d_2$ columns of $g$ define a homomorphism from $L$ to $U$.

As in 2.4, one gets that $Z$ is a vector bundle over $B$. Therefore it is irreducible with the inverse image $Z_w$ of the orbit of $U_1$ as a dense open subset. Since $U_1$ belongs to $B$, the open subset $Z_w$ where $g$ is invertible is not empty. Again, we have a morphism $\varphi$ from $Z_w$ to $\text{Mod}^{d_3}(k)$ which describes the possible quotients. By assumption, $\varphi(Z_w)$ meets the orbit of $U_2$. Therefore its open inverse image intersects $Z_w$ and $Z_w'$ in some point which gives us the desired exact sequence.

6.3. In the proof of 6.1, we will also have to glue the middle term together from the end terms. Clearly, for arbitrary $X$ and $Y$ the union of all orbits of modules $E$ occurring in an exact sequence $0 \to X \to E \to Y \to 0$ is a constructable irreducible set. For one only has to conjugate the modules given by $e_i = \begin{bmatrix} x_i \\ \cdot \cdot \cdot \\ y_i \end{bmatrix}$, where the $s$-tuple $(z_i)$ belongs to the linear space $Z(Y, X)$ of such tuples satisfying $x_i \cdot \zeta_j + \zeta_i \cdot y_j = \sum s_i \zeta_k$ for all $i$ and $j$. In particular, we can speak about generic extensions.

Lemma 6.3. Let $X$, $X'$, and $Y$ be $A$-modules. Suppose there is an epimorphism $\alpha: X \to X'$ inducing an epimorphism $\text{Ext}^1(Y, X) \to \text{Ext}^1(Y, X')$. If $E$ and $E'$ are the generic extensions of $Y$ by $X$, resp. by $X'$, then there is a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & X' & \longrightarrow & E' & \longrightarrow & Y & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & X' & \longrightarrow & E' & \longrightarrow & Y & \longrightarrow & 0.
\end{array}
\]

Proof. Let us denote by $B( Y, X)$ the subspace of $Z(Y, X)$ consisting of those $\zeta_i$'s which are of the form $\zeta_i = h \cdot y_i - x_i \cdot h$ for some $(\dim X) \times (\dim Y)$-matrix $h$. Then we have a commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \longrightarrow & B( Y, X) & \longrightarrow & Z(Y, X) & \longrightarrow & \text{Ext}(Y, X) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & B( Y, X') & \longrightarrow & Z(Y, X') & \longrightarrow & \text{Ext}(Y, X') & \longrightarrow & 0,
\end{array}
\]
where the vertical maps are all induced by $\alpha$. By assumption the extreme arrows are surjective so that the middle arrow $f$ has to be surjective, too. Now, the orbit of $E'$ intersects $Z(Y, X')$ in a dense open subset so that its inverse image under the morphism $f$ meets the orbit of $E$.

A similar statement holds for a monomorphism $Y' \to Y$ inducing a surjection $\text{Ext}(Y, X) \to \text{Ext}(Y', X)$. Furthermore, if an epimorphism $\alpha$ as above induces an epimorphism $\text{Hom}(U, X) \to \text{Hom}(U, X')$ one obtains a commutative diagram involving the generic quotients as follows:

$$
\begin{array}{ccc}
0 & \rightarrow & U & \rightarrow & X & \rightarrow & Y' & \rightarrow & 0 \\
| & & \downarrow \alpha & & \downarrow & & | & & | \\
0 & \rightarrow & U & \rightarrow & X' & \rightarrow & Y' & \rightarrow & 0.
\end{array}
$$

We omit the proofs since we have no need for these results.

6.3. We review briefly those aspects of the representation theory of tame concealed algebras that we will need for the proof of 6.1 given in 6.5–6.7 (see [35]).

As for matrix pencils, there are preprojective, regular, and preinjective indecomposables. The full subcategory $R$ of the regular modules is an abelian subcategory closed under extensions so that we may speak about simple regular modules, regular length, and so on, $R$ breaks up into a direct sum of uniserial categories $\mathcal{U}(\epsilon)$, $\epsilon$ in $\mathbb{P}(k)$. Most of these categories have only one simple object up to isomorphism, but there are at most three values of $\epsilon$, where $\mathcal{U}(\epsilon)$ has more than one simple. In these cases the simples $E_1, E_2, \ldots, E_n$ are conjugate under $D \text{Tr}$. We call an indecomposable in such a category reduced if its regular length is strictly smaller than $n$, and periodic if it is a multiple of $n$.

The global dimension of a tame concealed algebra is at most 2. All preprojective modules have projective dimension at most 1 and—all preinjectives have injective dimension at most 1. The regular and the faithful indecomposables have both projective and injective dimension $\leq 1$. The bilinear form on $\text{K}_0(A) = \mathbb{Z}^r$ which extends the map $(\dim M, \dim N) \rightarrow \langle M, N \rangle - \langle M, N \rangle^\perp + \langle M, N \rangle^\perp$ will play an important role. The roots of the associated quadratic form $q$ are the vectors $x$ in $\mathbb{N}^r$ with $qx = 1$ and $qx = 0$. They coincide with the dimension vectors of the indecomposables. The quadratic form is positive semi-definite and its radical admits a generator $h$ in $\mathbb{N}^r$ with strictly positive entries one of which is 1. Given any root $x$, there is at least one simple root $e_i$ such that $x - e_i$ is a root again. The dimension vectors of the simple regular modules admitting proper self-extensions all agree with $h$ and modules in the corresponding uniserial categories are called homogeneous. Furthermore, the dimension vectors of
the periodic modules are multiples of $h$. Finally, there is a linear function, called defect, with the property that its values on the indecomposables are negative for preprojectives, positive for preinjectives, and zero for regular modules.

We will often use the following consequence of the previous results: If $U$ is an indecomposable such that $x = \dim U - n \cdot h$ has positive coefficients and is non-zero, then there is an indecomposable $U'$ with dimension vector $x$. Moreover, $U$ is preprojective or regular if and only if $U'$ is so.

Let us conclude this section by deriving a simple geometric property of reduced modules.

**Lemma 6.4.** Any reduced module $V$ is dense in its connected component.

**Proof.** Let $M$ be a $\leq_{\text{deg}}$-minimal module which is not a degeneration of $V$. We will derive a contradiction. If $M_P$ is not zero we choose an indecomposable preprojective direct summand $U$ of $M$ and compute $0 \geq \langle V, U \rangle - \langle V, U \rangle^1 = \langle M, U \rangle - \langle M, U \rangle^1 + \langle M, U \rangle^2$. Because of $\langle U, U \rangle^1 = 0$ there is an indecomposable direct summand $M'$ of $M$ and a non-split exact sequence $0 \to U \to M \to M' \to 0$ contradicting the choice of $M$. Thus $M_P$ and, dually, $M_I$ have to be zero. Let $E_1, E_2, ..., E_i$ be the modules occurring as quotients in the descending regular composition series of $V$. Then we have $\langle V, E_j \rangle^1 = 0$ for $j \leq i$, whence $\langle V, X \rangle^1 = 0$ for all regular quotients $X$ of $V$. We infer $1 = \langle V, X \rangle - \langle V, X \rangle^1 = \langle M, X \rangle - \langle M, X \rangle^1$ for each $X$ of this type. Consequently, $E_1, E_2, ..., E_i$ all occur as regular composition factors of $M$. Our claim follows easily.

6.5. Now we are ready to prove Theorem 6.1. We treat first the cases where $U$ is homogeneous or periodic. In both cases it is sufficient to find a preprojective simple submodule $S$ with preinjective quotient $Q$. For $\dim Q$ is the dimension vector of a preinjective indecomposable $V$, and we will show that an exact sequence $0 \to S \to U \to V \to 0$ exists. To do so we only have to verify that we can apply the dual of 6.2 in the present situation. Now, the orbit of $S$ is open, and $V$ degenerates to $Q$ by 3.4(b). Also, we get $\langle U, Q \rangle = \langle U, Q \rangle - \langle U, Q \rangle^1 = \langle U, V \rangle - \langle U, V \rangle^1 = \langle U, V \rangle$, as desired.

The existence of $S$ and $Q$ as above is shown separately. First let $U$ be periodic with regular top $E$, say. If $E$ remains simple in the whole category, $U$ is an extension of an indecomposable and a simple for obvious reasons. In the remaining case, let $f$ be a non-zero map from some simple $S$ to $E$. Then $S$ is preprojective and the cokernel $C$ of $f$ is preinjective. The projection $U \to E$ induces a surjection $\Hom(S, U) \to \Hom(S, E)$ so that there is a commutative diagram with exact rows and columns as follows:
We infer \( Q \leq U' \oplus C \) and \( \langle Q, E \rangle \leq \langle U' \oplus C, E \rangle = 0 \). Therefore, \( Q \) is pre-injective because otherwise the regular top of \( Q_R \) coincides with the regular top of \( U \).

If \( U \) is homogeneous with regular composition factor \( E \), we choose again a preprojective simple \( S \) admitting a non-zero map to \( E \). We prove the existence of the desired preinjective quotient by induction on the regular length of \( U \). If \( U \) equals \( E \) any quotient will do. In the inductive step, we look at the exact sequence \( 0 \to E \to U \to U' \to 0 \) which induces a surjection \( \text{Hom}(S, U) \to \text{Hom}(S, U') \). Thus we obtain the following commutative diagram with exact rows and columns, where \( Q' \) is preinjective by induction:

Decomposing \( Q = Q_R \oplus Q_I \) we get \( \langle Q, X \rangle = \langle Q_R, X \rangle \leq \langle Q' \oplus E, X \rangle \leq 1 \) for all indecomposable homogeneous \( X \). Therefore, \( Q_R \) equals \( 0 \) or \( E \). In the later case the right column splits, whence also the middle one, a contradiction.
6.6. Now, we consider a regular non-homogeneous non-periodic indecomposable $U$. Let $0 \rightarrow U' \rightarrow U \rightarrow U_{\text{red}} \rightarrow 0$ be an exact sequence with periodic $U'$ and reduced $U_{\text{red}}$. By well-known properties of roots there is a simple $S$ and an indecomposable $V'$ such that $\dim U_{\text{red}} = \dim S + \dim V'$ holds.

If $S$ is regular, we denote by $E$ the regular top of $U_{\text{red}}$. We have $1 = \langle S_{\text{red}}, E \rangle - \langle S_{\text{red}}, E \rangle^1 = \langle S \oplus V', E \rangle - \langle S \oplus V', E \rangle^1$, whence $\langle S \oplus V', E \rangle \neq 0$. If $\langle V', E \rangle$ does not vanish we have a surjection $U_{\text{red}} \rightarrow V'$ whose kernel must be $S$. Thus $S$ is the regular socle of $U_{\text{red}}$ and therefore also of $U$. Obviously we obtain an exact sequence $0 \rightarrow S \rightarrow U \rightarrow V \rightarrow 0$ with an indecomposable $V$. If $\langle V', E \rangle = 0$, $S$ coincides with $E$ and we get a sequence $0 \rightarrow V \rightarrow U \rightarrow S \rightarrow 0$.

Up to duality, we can assume next that $S$ is preinjective so that $V'$ is preinjective. By Lemma 6.4 the assumptions of Lemma 4.4 are satisfied, so that we get an exact sequence $0 \rightarrow S \rightarrow U_{\text{red}} \rightarrow V' \rightarrow 0$. Since $\text{Hom}(S, U) \rightarrow \text{Hom}(S, U_{\text{red}}) \rightarrow 0 = \text{Ext}(S, U')$ is exact we find the diagram

```
0 \rightarrow S \rightarrow U_{\text{red}} \rightarrow V' \rightarrow 0
```

As in the first case of 6.5 it follows that $Q$ is preinjective. Since $\dim Q = \dim U' + \dim V'$ is the dimension vector of a preinjective indecomposable $V$, the existence of $0 \rightarrow S \rightarrow U \rightarrow V \rightarrow 0$ follows once more from the dual of Lemma 7.2.

6.7. Up to duality, the only case that remains to be treated is that $U$ is preprojective. By a remark made before there is a root $v$ and a simple root $s$ with $\dim U = s + v$. Clearly, there is exactly one simple $S$ with dimension vector $s$, but there might be many indecomposables $V$ with dimension vector $v$, and it really happens for bad choices of $V$ that $U$ is not an extension of $S$ and $V$. Nevertheless, this is always true for good choices.
We know from 3.4(b) that $U$ degenerates to $S \oplus V$ for any $V$. Thus we get $1 = \langle U, U \rangle \leq \langle U, S \oplus V \rangle$ and also $1 = \langle U, U \rangle \leq \langle S \oplus V, U \rangle$ so that only the cases $V < U < S$ and $S < U < V$ can occur. In the first case one has $\langle V, U \rangle = \langle V, U \rangle - \langle V, V \rangle^{1} = \langle V, S \oplus V \rangle - \langle V, S \oplus V \rangle^{1} = \langle V, S \oplus V \rangle$, whence $V$ embeds into $U$ by 2.4(a). The quotient is $S$, and we are done in this case. In the second case $S < U < V$, one has similarly $\langle S, U \rangle = \langle S, S \oplus V \rangle$. If $V$ is preprojective, preinjective, or reduced, Lemma 4.4 always applies and bears the wanted exact sequence $0 \rightarrow S \rightarrow U \rightarrow V \rightarrow 0$.

The remaining cases of regular $V$'s are treated separately by induction on the regular length. Note that there is always a generic extension $X$. Using the dual of 4.4, it is enough to prove that $X$ is preprojective. For $U$ degenerates to $X$ by 3.4(b) and $\langle X, V \rangle = \langle X, V \rangle - \langle X, V \rangle^{1} = \langle U, V \rangle - \langle U, V \rangle^{1} = \langle U, V \rangle$ holds because $X$ and $U$ are preprojective.

First, let $V$ be homogeneous with regular composition factor $E$. Then $D \text{Tr} V$ equals $V$ and we get from $\text{Ext}^{1}(V, U) = D \text{Hom}(U, D \text{Tr} V)$ that $\langle V, U \rangle^{1}$ equals $\langle V, V \rangle$ which is non-zero because of $\langle U, U \rangle \leq \langle U, S \oplus V \rangle$. Therefore, $-\langle V, V \rangle^{1} = \langle V, S \oplus V \rangle - \langle V, S \oplus V \rangle^{1} = \langle V, U \rangle - \langle V, U \rangle^{1} < 0$ implies $\langle E, S \rangle^{1} \neq 0$. Any non-split extension of $S$ and $E$ has preprojective middle term, thereby proving the start of the induction. In the inductive step we look at $0 \rightarrow V' \rightarrow V \rightarrow E \rightarrow 0$. Then $\text{dim } V' + \text{dim } S$ is the dimension vector of an indecomposable preprojective $U'$. By induction there is an exact sequence $0 \rightarrow S \rightarrow U' \rightarrow V' \rightarrow 0$. Moreover, $0 < \langle U', V' \rangle = \langle V', U' \rangle^{1}$ shows $\langle E, U' \rangle^{1} \neq 0$. Thus we obtain a non-split exact sequence $0 \rightarrow U' \rightarrow Y \rightarrow E \rightarrow 0$ whose middle term is preprojective because $E$ is simple regular. Using the dual of 4.4 we even get $0 \rightarrow U' \rightarrow Y \rightarrow E \rightarrow 0$. Now, $\Sigma$ induces the exact sequence $\text{Ext}^{1}(E, U') \rightarrow \text{Ext}^{1}(E, V') \rightarrow 0$. We infer from Lemma 6.3 the following commutative diagram with exact rows and columns, which contains the sequence $0 \rightarrow S \rightarrow U \rightarrow V \rightarrow 0$ we are aiming at:

```
  0  0
↓   ↓
S → S
↓   ↓
0 → U' → U → E → 0
↓   ↓
0 → V' → V → E → 0
↓   ↓
0  0
```
Next, let $V$ be regular non-homogeneous and non-periodic. We consider the exact sequence $0 \to V_{\text{red}} \to V \to V' \to 0$ with reduced $V_{\text{red}}$ and periodic $V'$. Then $\dim U - \dim V'$ is the dimension vector of a preprojective indecomposable $U$. We know already that $0 \to S \to U' \to V_{\text{red}} \to 0$ exists. Applying $\text{Hom}(-, S)$ to the first exact sequence we obtain a surjection $\text{Ext}^1(V, S) \to \text{Ext}^1(V_{\text{red}}, S)$, whence the next diagram with exact row and columns:

$$
\begin{array}{cccccc}
0 & 0 \\
\uparrow & \uparrow \\
V' & V' \\
\uparrow & \uparrow \\
0 & S & Y & V & 0 \\
\uparrow & \uparrow & \uparrow & \uparrow \\
0 & S & U' & V_{\text{red}} & 0 \\
0 & 0
\end{array}
$$

Decomposing $Y = Y_p \oplus Y_R$ and using $Y \leq U' \oplus V'$ as well as $Y \leq S \oplus V$, we conclude that the regular socle of $Y_p$ embeds into the regular socle of $V'$ as well as of $V$. But these two socles are different because $V$ is non-periodic. This shows that $Y$ is preprojective and ends the proof in the present case. In the only remaining case $V$ is periodic. Then we can choose a homogeneous indecomposable with the same dimension vector to obtain $U$ as an extension of an indecomposable and a simple. But we can even analyze the original situation completely.

**Lemma 6.6.** Let $V$ be a periodic module with regular socle $E$ and let $S$ be a preprojective simple. Then there is a preprojective indecomposable $U$ with $\dim U = \dim S + \dim E$, and the following two statements are equivalent:

(a) $\langle E, S \rangle^1 \neq 0$.

(b) There is an exact sequence $0 \to S \to U \to V \to 0$.

**Proof.** If the sequence in (b) exists, its pull-back under $E \to V$ does not split because $\langle E, U \rangle$ is 0. Conversely, the inclusion induces a surjection $\text{Ext}^1(V, S) \to \text{Ext}^1(E, S)$, whence we get:
Here, the lower row is a non-split exact sequence which exists by
assumption. Then $U'$ is preprojective and the regular socle of $X$ embeds
into that of $V$ and of $V'$ by the argument used before. Thus $X$ is preprojective
and the existence of the sequence $0 \to S \to U \to V \to 0$ follows.

Using the lemma it is easy to give examples of $U$, $S$, and $V$ such that $U$
degenerates to $S \oplus V$ but is not an extension of $S$ and $V$. For instance, one
can take a quiver of type $D_4$, where the point of order 4 is a source, and
describe $S$, $U$, $E_1$, and $E_2$ by their dimension vectors $(0, 1, 0, 0, 0)$,
$(2, 2, 1, 1, 1)$, $(1, 1, 1, 0, 0)$, and $(1, 0, 0, 1, 1)$. Then the periodic module $V$
with regular socle $E_1$ and regular length two provides an example.

6.8. For representation directed algebras one has a more general
statement than Theorem 6.1. Namely, if the dimension vector of an
indecomposable is the sum of the dimension vectors of two indecom-
posables $V$ and $W$, then $U$ is an extension of $V$ and $W$. Obviously this
statement is wrong for tame concealed algebras, because one can choose
$U$, $V$, and $W$ in different homogeneous subcategories. The condition one
has to add to avoid these phenomena is that $U \leq V \oplus W$ holds. This
condition is superfluous for preprojective $U$ by 3.4(b), thus also for
representation directed algebras. But Lemma 6.6 says that even with this
additional assumption the above generalization of 6.1 is not true. However,
the proof of 6.6 shows:

**Proposition 6.7.** Let $U$, $V$, and $W$ be indecomposables over a tame
concealed algebra such that $\dim U = \dim V + \dim W$ and $V \leq W$. If $U$
is preprojective then either there is an exact sequence $0 \to W \to U \to V \to 0$
or else $V$ is periodic with regular socle $E$ satisfying $\langle E, W \rangle^1 \neq 0$.

For regular $U$, the proofs given in the text do not generalize immediately
because one often uses the existence of an embedding $S \to U$ which is not
so clear for a non-simple $S$. However, for hereditary algebras one can use the well-known reflection functors and their exactness properties to get a proof, and for arbitrary tame algebras one should use the fact that for modules of small projective and injective dimension $D \text{Tr}$ and $\text{Tr} D$ can be described by $\text{Ext}$’s. In this context, the next lemma should be useful to embed indecomposable projectives.

**Lemma 6.8.** Let $M$ be a faithful module over a Schurian algebra $A$. Then any indecomposable projective $P$ embeds into $M$.

**Proof.** We can assume that $A$ is given by its Gabriel quiver with relations. That $A$ is Schurian simply means that all paths between two points are proportional modulo the relations. Now, let $P$ be the indecomposable projective corresponding to the point $x$. For any point $y$ with $P(y) \neq 0$ we choose a path $w(y)$ from $x$ to $y$ with non-zero residue. Since $M$ is faithful, the kernel $K(y)$ of the linear map $M(w(y))$ is a proper subspace of $M(x)$. Because $k$ is infinite there is a vector $z$ outside the union of all these $K(y)$’s. This $z$ generates a copy of $P$.

For instance, Lemma 6.8 applies to an indecomposable homogeneous module $M$. Namely, $M$ is clearly omnipresent, and the usual proof generalizes to show that $M$ is even faithful (see [8]).

7. **EXAMPLES**

7.1. This example shows that the decompositions into indecomposables behave as bad as one can imagine under degenerations. More precisely, given any sequence $n_1 \geq n_2 \geq \cdots \geq n_r \geq 0$ of natural numbers we construct indecomposables $U, V, \text{ and } U_i$ such that $U \bigoplus_i \text{ext } M = V$ holds.

To this end we consider the path algebra of the quiver $\begin{array}{c}
\bullet \\
\cdots \\
\bullet
\end{array}$. For any natural number $n$ we denote by $U(\ell)$ the indecomposable with simple top of dimension $4n$, where $\alpha$ and $\beta$ act as indicated in the following picture on base-vectors:
If we set \( m = \sum n_i \cdot 1 \), then \( U = U(m) \) is an iterated extension of \( M = \bigoplus U(i)^m \), which in turn splits into \( U(1)^m \). Note that \( U(1) \) is a module over the self-injective algebra \( A = k[X, Y]/(X^2, XY, Y^2) \). Now, let \( N \) be a nilpotent Jordan block of size \( 2m \), and let \( N' \) be the matrix mapping \( e_i \) to \((-1)^{i+1} e_{m+i} \) for \( i \leq m \) and all other standard base vectors \( e_i \) to 0. Then we have \( N^2 = 0 \) and \( N \cdot N' + N' \cdot N = 0 \). For arbitrary \( \varepsilon \) in \( k \) we set \( \alpha(\varepsilon) = [\begin{smallmatrix} \varepsilon & 0 \\ 0 & 0 \end{smallmatrix}] \) and \( \beta(\varepsilon) = \varepsilon [\begin{smallmatrix} 0 & 1 \\ N & 0 \end{smallmatrix}] + [\begin{smallmatrix} 0 & 0 \\ 1 & \varepsilon \end{smallmatrix}] \). Then we have \( \alpha(\varepsilon)^2 = 0 = \beta(\varepsilon)^2 \) and \( \alpha(\varepsilon) \cdot \beta(\varepsilon) = \beta(\varepsilon) \cdot \alpha(\varepsilon) = \varepsilon \cdot N' \). Thus for \( \varepsilon \neq 0 \) the socle of the corresponding \( A \)-module has dimension \( m \) so that the module is isomorphic to \( U(1)^m \) because \( A \) is self-injective. However, for \( \varepsilon = 0 \) the corresponding module is the wanted indecomposable \( V \).

7.2. Carlsons example mentioned before deals with the four-dimensional representations of the self-injective algebra considered in 7.1. All indecomposables except \( U(1) \) are annihilated by \( XY \), so that they are classified by Kroneckers’ results on matrix pencils. A short calculation then shows that \( U(1) \) is the smallest element with respect to \( \leq \). On the other hand, the closure \( C \) of the orbit of \( U(1) \) does not contain all representations as shown in [32]. To determine \( C \) explicitly one can use the following important general result of Steinberg which simplifies the computations a little bit (see [38]).

**Steinberg’s Lemma.** Let \( G \) be an affine algebraic group with Borel subgroup \( B \). If \( G \) acts on a variety \( V \), the \( G \)-saturation of the closure of any \( B \)-orbit is closed again.

In our example, the closure of the orbit of \( U(1) \) under the lower triangular matrices is given by the pairs of matrices \( \alpha = (\alpha_i) \) and \( \beta = (\beta_i) \) subject to the conditions \( \alpha_j = \beta_j = 0 \) for \( i \leq j \), \( \alpha_{12} = \beta_{12} = \beta_{32} = \beta_{43} = 0 \), and \( \alpha_{23} = \beta_{32} = \beta_{23} = \beta_{34} = \beta_{43} = 0 \). It is an easy matter then to describe the orbits occurring in that set. In particular, the modules \( V(\lambda) \oplus V(\mu) \) with \( V(\lambda) = A/(XY, X - \lambda \cdot Y) \) belong to the closure if and only if \( \lambda + \mu = 0 \). Nevertheless, \( U(1) \oplus S^2 \) degenerates to \( V(\lambda) \oplus V(\mu) \oplus S^2 \) for all \( \lambda \) and \( \mu \) as shown in [32]. Here \( S \) denotes the simple \( A \)-module. Observe that the generic quotient of the second direct sum by \( S^2 \) is not a degeneration of the generic quotient of the first direct sum. This shows that the assumptions in Theorem 2.4 are essential.

7.3. Quite often, the following fact that I learned from H. P. Kraft and P. Slodowy helps to calculate orbit closures.

**Lemma [37].** Let \( G \) be an affine algebraic group acting on two varieties \( X \) and \( Y \) related by a \( G \)-equivariant morphism \( \pi \). Suppose that \( Y \) is the orbit of a point \( y \) having stabilizer \( H \) and fiber \( F \). Then \( X \) is isomorphic to the
associated fiber bundle \( G \times HF \). Consequently, the map \( U \mapsto U \cap F \) induces a closure preserving bijection between \( G \)-stable subsets of \( X \) and \( H \)-stable subsets of \( F \).

In our context the lemma can be applied to \( G = GL_d(k) \) acting on \( \text{Mod}^d_A(k) \) and \( \text{Mod}^d_B(k) \) for some algebra \( A \) with subalgebra \( B \). The map \( \pi \) is the obvious restriction map, \( Y \) is a \( G \)-orbit in \( \text{Mod}^d_A(k) \) and \( X \) is the inverse image of \( Y \). Let us look at two concrete examples given by quivers.

First, let \( A \) be the path algebra of the quiver \( 1 \rightarrow 2 \) with three arrows \( \alpha, \beta, \) and \( \gamma \) and let \( B \) be the path algebra of the arrow \( \gamma \). We are interested in a geometric study of the \( G \)-stable subset \( X \) of representations of dimension type \((n, n)\), where \( \alpha \) is bijective. Then the above lemma implies that we can just as well study the action of \( GL_d(k) \) on two matrices by simultaneous conjugation. This shows in particular that Carlsons’ example occurs also for the wild quiver under consideration. A similar argument applies to the other wild quivers.

As another concrete example let us take the representations of the quiver

\[
\begin{array}{cccccc}
& 4 & 1 & \rightarrow & 2 & \rightarrow & 3 \leftarrow & 5 \leftarrow & 6 \\
\end{array}
\]

with given dimension vector \( d \).

Let us look at those representations where \( \alpha \) and \( \varepsilon \) are injective. The restrictions of these representations to the non-connected quiver \( 1 \rightarrow 2, 4, 6 \rightarrow 5 \) are all isomorphic and the fiber is given by an arbitrary \( d_1 \times (d_2 + d_3) \)-matrix. The stabilizer acts on these matrices by arbitrary row and certain column transformations. In other words, our reduced problem is a problem of representations of the partially ordered set \([a < b, c < d, e]\) in the sense of Nazarova–Roiter. Since we have “solved” the degeneration problem for \( E_{6} \), we also know the degenerations in the problem above. Of course, this example generalizes.

7.4. To see a concrete example of how strongly the isomorphism type of a quotient depends on the embedding, we consider again matrix pencils and we use the notations introduced in Section 5. We choose \( M = P(i) \) for some large \( i \) and \( U = P(0) \). Given any \((a_1, a_2, \ldots, a_i)\) in \( k^i \), the vector \( \sum a_i f_i + f_{i+1} \) generates a copy of \( U \) in \( M \). Clearly, the \( e_j \)'s and the \( f_j \)'s with \( j \leq i \) give us a basis of the quotient. With respect to that basis \( \lambda \) acts by the identity and \( \rho \) by a companion matrix. Taking into account the \( GL_2 \)-action on the matrix pencils, we see that any regular representation from the open sheet occurs as a quotient. It is not hard to see that these are the only quotients so that the set of all quotients is open in our example.
7.5. Let $A$ be the algebra with Gabriel quiver $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5$ and relation $\beta^2 = 0$. Then $A$ has seven indecomposables, and a straightforward but lengthy computation shows that $\leq_{\text{deg}}$ and $\leq_{\text{ext}}$ are equivalent whereas $\leq_{\text{ext}}$ and $\leq$ are not. Namely, as observed in [32], the projective indecomposable $P$ to the point 1 degenerates to another indecomposable $U$. Moreover, if $S$ denotes the simple to the point 2, then $P \oplus S$ degenerates to $U \oplus S$, but it is not an extension of these two modules.

7.6. Let $A$ be given by the quiver

$$
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\downarrow \\
4 \\
\downarrow \\
5 \\
\end{array}
$$

and all commutativity relations. Then the direct sum of the indecomposables with dimension vectors $(2, 1, 1, 1, 0)$ and $(0, 1, 1, 1, 2)$ is a minimal degeneration of $P^2$, where $P$ is the projective to the point 1.

7.7. Let us have a closer look at the set of extensions in the concrete example of the quiver

$$
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\downarrow \\
5 \\
\downarrow \\
6 \\
\downarrow \\
7 \\
\end{array}
$$

and the indecomposables $U$ and $V$ with dimension vectors $(0, 0, 1, 0, 0, 0, 1, 2, 3, 2, 3, 2, 1)$. It is not hard to see that any extension is isomorphic to a representation of the form

$$
\begin{array}{cccc}
& & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \\
& \cdot & k^2 & \cdot \\
& & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \cdot \\
& \cdot & k^4 & \cdot \\
& & \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} & \cdot \\
& \cdot & k^3 & \cdot \\
& & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \cdot \\
& \cdot & k^2 & \cdot \\
& & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \cdot \\
& & k & \cdot \\
\end{array}
$$

with $x$, $y$, $z$, and $t$ in $k$. The complement of the open orbit is described by the equation $x \cdot y \cdot z \cdot (x - y - t) \cdot (y + z + t) \cdot (x \cdot z + x \cdot t - y \cdot z) = 0$. Thus we obtain six irreducible components corresponding to six subgeneric orbits of codimension 1, as predicted by the results of Riedtmann and Schofield in [33]. There are 12 orbits of codimension 2 which are all given
by linear equations. But two of them, namely \( z = t = 0 \) and \( t = x = y = 0 \), are not intersections of orbit closures. Finally, the various non-trivial intersections of these 12 planes give rise to 10 orbits of codimension 3, in accordance with the formula found by Markolf via computer. Thus in this case any maximal chain in the partially ordered set \( S \) has the same length. This is wrong in general. I do not even know whether there is always a chain of length \( \langle V, U \rangle^3 \). I also do not know whether the closure of all orbits of some fixed codimension contains all orbits of higher codimension.

7.8. Our last example deals with the algebra \( A \) given by the following quiver and relations:

\[
1 \xrightarrow{\rho} 2 \xrightarrow{\sigma} \rho \delta \cdot \sigma = \rho^4, \quad 0 = \sigma \cdot \rho = \rho^2 \cdot \delta = \sigma \cdot \delta.
\]

As shown in [40], this algebra is representation-finite. The Auslander–Reiten quiver of the universal cover of \( A \) contains two points \( U \) and \( V \) such that the part of the Auslander–Reiten quiver lying between the two points has the shape indicated in the next figure. The generic extension of \( U \) and \( V \) has 6-dimensional endomorphism ring, whilst \( \text{End} U \oplus V \) has dimension 27. Therefore, \( \langle V, U \rangle^3 \) equals 21. We list the number of isomorphism classes of possible extensions with decreasing orbit dimension: 1, 5, 24, 75, 212, 501, 1,081, 2,083, 3,681, 5,929, 8,779, 11,902, 14,759, 16,531, 16,628, 14,736, 11,241, 7,131, 3,566, 1,247, 233, 1. Thus, there are 120,345 possible extensions up to isomorphism, and 233 of them have \( U \oplus V \) as a minimal degeneration. All these results are taken from [29].

References