Coloring the square of the Kneser graph KG(2k + 1, k) and the Schrijver graph SG(2k + 2, k)

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\textbf{A B S T R A C T}

The Kneser graph \(KG(n, k)\) is the graph whose vertex set consists of all \(k\)-subsets of an \(n\)-set, and two vertices are adjacent if and only if they are disjoint. The Schrijver graph \(SG(n, k)\) is the subgraph of \(KG(n, k)\) induced by all vertices that are 2-stable subsets. The square \(G^2\) of a graph \(G\) is defined on the vertex set of \(G\) such that distinct vertices within distance two in \(G\) are joined by an edge. The span \(\lambda(G)\) of \(G\) is the smallest integer \(m\) such that an \(L(2, 1)\)-labeling of \(G\) can be constructed using labels belonging to the set \(\{0, 1, \ldots, m\}\). The following results are established. (1) \(\chi(KG^2(2k + 1, k)) \leq 3k + 2\) for \(k \geq 3\) and \(\chi(KG^2(9, 4)) \leq 12\); (2) \(\chi(SG^2(2k + 2, k)) = \lambda(SG(2k + 2, k)) = 2k + 2\) for \(k \geq 4\), \(\chi(SG^2(8, 3)) = 8\), \(\lambda(SG(8, 3)) = 9\), \(\chi(SG^2(6, 2)) = 9\), and \(\lambda(SG(6, 2)) = 8\).

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1. Introduction

Let \(G(V, E)\) be a simple graph. The distance \(d_C(u, v)\) between two vertices \(u\) and \(v\) of \(G\) is the length of a shortest path connecting them. The graph \(G^2\), called the square of \(G\), is defined on \(V(G)\) such that two vertices \(u\) and \(v\) are adjacent in \(G^2\) if and only if \(1 \leq d_C(u, v) \leq 2\). For \(n > 2k > 0\), the Kneser graph \(KG(n, k)\) is the graph whose vertex set consists of all \(k\)-subsets of the set \([n] = \{0, 1, \ldots, n - 1\}\), and two vertices \(A\) and \(B\) are adjacent if and only if \(A \cap B = \emptyset\). In his widely cited paper [4], Lovász determined the chromatic number \(\chi(KG(n, k))\) to be exactly \(n - 2k + 2\). In response to Füredi’s problem of computing \(\chi(KG^2(n, k))\), Kim and Nakprasit [3] showed that \(\chi(KG^2(2k + 1, k)) \leq 4k\) when \(k\) is odd and \(\chi(KG^2(2k + 1, k)) \leq 4k + 2\) when \(k\) is even. They also established the following lower bounds for \(\chi(KG^2(2k + 1, k))\): \(k + 4\) when \(k \equiv 0 \pmod{6}\); \(k + 3\) when \(k \equiv 1, 2, 4 \pmod{6}\); \(k + 2\) when \(k \equiv 3, 5 \pmod{6}\). We will establish improved upper bounds as follows: \(\chi(KG^2(2k + 1, k)) \leq 3k + 2\) when \(k \geq 3\) and \(\chi(KG^2(9, 4)) \leq 12\).

The following type of labeling of a graph was motivated by the frequency assignment problem in communication networks. The reader is referred to Yeh [7] for a recent survey on relevant concepts and results. An \(L(2, 1)\)-labeling of a graph \(G\) is a function \(\varphi\) from \(V(G)\) into the set \(\{0, 1, \ldots, m\}\) for some natural number \(m\) such that

\[|\varphi(u) - \varphi(v)| \geq \begin{cases} 2 & \text{if } d_C(u, v) = 1, \\ 1 & \text{if } d_C(u, v) = 2. \end{cases}\]

The span \(\lambda(G)\) of \(G\) is the smallest \(m\) such that an \(L(2, 1)\)-labeling into \(\{0, 1, \ldots, m\}\) exists. Note that \(\chi(G^2) \leq \lambda(G) + 1\). Kang [2] showed that \(\lambda(KG(2k + 1, k)) \leq 4k + 2\). Actually, the upper bound can be reduced to \(4k + 1\) by a similar argument.

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A subset $S$ of $[n]$ is said to be 2-stable if $2 < |x - y| < n - 2$ for any two distinct elements $x$ and $y$, or equivalently, $S$ does not contain two consecutive numbers in the cyclic ordering of $[n]$. The Schrijver graph $SG(n, k)$ is the subgraph of $KG(n, k)$ induced by all the vertices that are 2-stable subsets. Schrijver [5] proved that $\chi(SG(n, k)) = \chi(KG(n, k))$ and the deletion of any vertex from $SG(n, k)$ will produce a graph of smaller chromatic number. The Schrijver graph $SG(2k + 1, k)$ is an odd cycle $C_{2k+1}$. It is known that $\chi(C_{2k+1}) \leq 5$ and Griggs and Yeh [1] proved $\lambda(C_{2k+1}) = 4$. In this paper, we show that $\chi(SG^2(2k + 2, k)) = \lambda(SG(2k + 2, k)) = 2k + 2$ for $k \geq 4$, $\chi(SG^2(8, 3)) = 8$, $\lambda(SG(8, 3)) = 9$, $\chi(SG^2(6, 2)) = 9$, and $\lambda(SG(6, 2)) = 8$.

2. Kneser graph $KG(2k + 1, k)$

Throughout this section, let $G$ denote the Kneser graph $KG(2k + 1, k)$, $k > 1$, also known as the Odd graph. We use $n$ to denote $2k + 1$ for short. Note that $G$ is a triangle-free graph. From general results proved in Valencia-Pabon and Vera [6], we know that the diameter of $G$ is $k$ and the following holds.

**Lemma 1.** Let $A$ and $B$ be vertices of $G$.

$$|A \cap B| = s \quad \text{if and only if} \quad d_G(A, B) = \begin{cases} 2(k - s) & \text{when } 2s \geq k, \\ 2s + 1 & \text{when } 2s < k. \end{cases}$$

We use the notation $|a|_n$ to denote the number $s$ such that $0 \leq s < n$ and $s \equiv a \pmod{n}$. We pre-color the vertices of $G$ by the mapping $f$ from $V(G)$ to $[n]$ such that $f(A) = |\sum_{i \in A} f(i)|_n$. Let the color classes be $S_i = \{A \in V(G) \mid f(A) = i\}$ for $i \in [n]$. The color classes may not be independent sets, nevertheless vertices at distance two in $G$ receive distinct colors under $f$. Let $H[S]$ denote the subgraph of a graph $H$ induced by the set $S$ of vertices. The following three Lemmas were essentially proved in Kang [2].

**Lemma 2.** Each $G^2[S_i] = G[S_i]$ is a union of independent edges and isolated vertices, hence $\chi(G^2[S_i]) \leq 2$.

**Lemma 3.** Let $AB$ be an edge of $G$ and let $\{z\} = [n] \setminus (A \cup B)$. If $A \in S_i$ and $B \in S_j$, then $z = |j - i|_n$.

**Lemma 4.** Let $AB$ be an edge in $S_i$. Then $A$ and $B$ cannot both have neighbors in $S_{i+1}$, nor can they both have neighbors in $S_{i-1}$.

**Lemma 3** implies that, if $|j - i|_n \in A$ and $A \in S_i$, then there is no edge in $G$ between $A$ and any vertex of $S_j$ if $|j - i|_n \notin A$ and $A \in S_i$, then there is exactly one edge $AB$ in $G$ between $A$ and the vertex $B = [n] \setminus (A \cup \{|j - i|_n\})$ of $S_j$. These are valid even when $i = j$. By applying the above Lemmas, we have the following result, in which $\overline{m} \in A$ denotes $m \notin A$.

**Lemma 5.** The induced subgraph $G[S_0 \cup S_1]$ is a disjoint union of subgraphs, each of which belongs to one of the following five types.

1. A path $P = (A, B, C, D)$ of length three, $A, B \in S_0$ and $C, D \in S_1$, such that $n - 2, n - 1, \overline{0} \in A$, $n - 2, n - 1, \overline{0} \in B$, $n - 2, n - 1, \overline{0} \in D$.
2. A path $P = (A, B, C)$ of length two, $A, B \in S_0$ and $C \in S_1$, such that $n - 2, n - 1, \overline{0} \in A$, $n - 2, n - 1, \overline{0} \in B$, $n - 2, n - 1, \overline{0} \in C$.
3. A path $P = (A, B, C)$ of length two, $A, C \in S_0$ and $B \in S_1$, such that $n - 2, n - 1, \overline{0} \in A$, $n - 2, n - 1, \overline{0} \in B$, $n - 2, n - 1, \overline{0} \in C$.
4. An edge $AB, A \in S_0$ and $B \in S_1$, such that $n - 2, n - 1, \overline{0} \in A$ and $n - 2, n - 1, \overline{0} \in B$.
5. An isolated vertex $A$ such that $n - 1, \overline{0} \in A$ if $A \in S_0$, and $n - 2, n - 1, A \in S_1$ if $A \in S_1$.

**Lemma 6.** Let $A \in S_0$, $B \in S_1$ and $d_G(A, B) = 2$. Then the unique a in $A \setminus B$ and the unique b in $B \setminus A$ satisfy $b = |a + 1|_n$.

**Proof.** The existence and uniqueness of $a$ and $b$ are easy consequences of Lemma 1. Then $|a + \sum_{t \in A \setminus B} f(t)|_n = 0$ and $|b + \sum_{t \in A \setminus B} f(t)|_n = 1$ imply $|b - a|_n = 1$. Since $a, b \in [n]$, it follows that $b = |a + 1|_n$. □

**Lemma 7.** For $k \geq 3$, $\chi(G^2[S_0 \cup S_1]) \leq 3$.

**Proof.** By Lemma 5, each component of $G[S_0 \cup S_1]$ is one of the five types of subgraphs. Let $\sigma$ be a finite sequence of $m$‘s or $\overline{m}$‘s, where each $m \in [n]$. Let $S_0(\sigma)$ denote the set $\{A \in S_i \mid A$ contains every entry of $\sigma\}$. In order to find a proper $3$-coloring of $G[S_0 \cup S_1]$, we partition $S_0$ into 14 sets and $S_1$ into 13 sets with appropriate adjacencies as depicted in Fig. 1. It is routine to check that those 14 (respectively 13) subsets do constitute a partition of $S_0$ (respectively $S_1$). The circled numbers in Fig. 1 give colors to vertices of the indicated type. It is evident that no vertices of the same color are adjacent in $G$.

**Lemma 2** implies $d_G(A, B) \neq 2$ for all $A, B \in S_i$. Therefore, if $A, B \in S_0 \cup S_1$ and $d_G(A, B) = 2$, then one of $A$ and $B$ is in $S_0$ and the other is in $S_1$. The next three Claims will prove our lemma.
**Claim 1**. The union of $S_0(n-2, n-1, \bar{0}, \bar{1}, \bar{T}, S_0(n-3, n-2, n-1, \bar{0}, \bar{1}, 2), S_0(n-3, n-2, n-1, \bar{0}, \bar{1}, 2), S_0(n-3, n-2, n-1, \bar{0}, \bar{1}, 2), S_0(n-2, n-1, 0), S_0(n-2, n-1, 0), S_0(n-2, n-1, 0), S_0(n-2, n-1, 0), S_0(n-2, n-1, 0), S_0(n-2, n-1, 0)$ is an independent set in $G[S_0 \cup S_1]$. 

**Proof of Claim 1.** From the way we chose the subsets of $S_0$, we know their union is an independent set in $G[S_0 \cup S_1]$. Suppose $A \in S_0(\bar{T}), B \in S_1(1)$, and $d_G(A, B) = 2$. Lemma 6 implies $A \in S_0(0, \bar{T})$. Therefore, the union of $S_0(n-2, n-1, \bar{0}, \bar{1}, \bar{T}), S_0(n-3, n-2, n-1, \bar{0}, \bar{1}, 2), S_0(n-3, n-2, n-1, \bar{0}, \bar{1}, 2), S_0(n-3, n-2, n-1, \bar{0}, \bar{1}, 2)$, and $S_1(n-2, n-1, 0, 1)$ is an independent set in $G[S_0 \cup S_1]$. Similarly, the union of $S_0(n-2, n-1, 0), S_0(n-2, n-1, 0), S_0(n-2, n-1, 0), S_0(n-2, n-1, 0, 1), S_1(n-2, n-1, 0, 1)$ is an independent set in $G[S_0 \cup S_1]$. If $A \in S_0(n-2, n-1, 0)$ and $B \in S_1(n-2, n-1, 0, 1)$, then $|A \setminus B| \geq 2$. Lemma 1 implies $d_G(A, B) \neq 2$. Suppose $A \in S_0(0), B \in S_1(0)$, and $d_G(A, B) = 2$. Lemma 6 implies $A \in S_0(0, 1)$. Hence $S_0(n-1, 0, 1) \cup S_1(n-2, n-1, 0, 1)$ is an independent set in $G[S_0 \cup S_1]$ and we are done.

**Claim 2.** The union of $S_0(n-2, n-1, 0, 1), S_0(n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2)$ is an independent set in $G[S_0 \cup S_1]$. 

**Proof of Claim 2.** The reasons why the distance in $G$ between any two vertices in the above sets is different from two is listed in Table 1.

**Claim 3.** The union of $S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2), S_0(n-3, n-2, n-1, 0, 1, 2)$ is an independent set in $G[S_0 \cup S_1]$. 

![Fig. 1. The structure of $G[S_0 \cup S_1]$ and three independent sets of $G^2[S_0 \cup S_1]$.](image-url)
Table 1
Proof for Claim 2

<table>
<thead>
<tr>
<th>( A )</th>
<th>( B )</th>
<th>Why ( d_c(A, B) \not= 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_0(n-2, \overline{n-1}, \overline{0}, 1) )</td>
<td>( S_1(n-2, n-1, 0, 0, \overline{1}, \overline{2}) )</td>
<td>(</td>
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<tr>
<td>( S_0(n-2, n-1, 0, 1, 2) )</td>
<td>( S_1(n-2, n-1, 0, 0, \overline{1}, \overline{2}) )</td>
<td>(</td>
</tr>
<tr>
<td>( S_0(n-3, \overline{n-2}, n-1, \overline{0}, \overline{1}, \overline{2}) )</td>
<td>( S_1(n-2, n-1, 0, 0, \overline{1}, \overline{2}) )</td>
<td>( 1 \in A \setminus B, 2 \not\in B \setminus A )</td>
</tr>
<tr>
<td>( S_0(n-2, n-1, \overline{0}) )</td>
<td>( S_1(n-2, n-1, 0, 0, \overline{1}, \overline{2}) )</td>
<td>( n-2 \in A \setminus B, n-1 \not\in B \setminus A )</td>
</tr>
<tr>
<td>( S_0(n-2, \overline{n-1}, \overline{0}, 1) )</td>
<td>( S_1(n-3, n-2, n-1, 0, 1, \overline{2}) )</td>
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<td>( S_0(n-2, n-1, 0, 1, 2) )</td>
<td>( S_1(n-3, n-2, n-1, 0, 1, \overline{2}) )</td>
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<tr>
<td>( S_0(n-2, n-1, 0, 1, 2) )</td>
<td>( S_1(n-3, n-2, n-1, 0, 1, \overline{2}) )</td>
<td>( 1 \in A \setminus B, 2 \not\in B \setminus A )</td>
</tr>
<tr>
<td>( S_0(n-2, n, 0, 1) )</td>
<td>( S_1(n-3, n-2, n-1, 0, 1, \overline{2}) )</td>
<td>( n-2 \in A \setminus B, n-1 \not\in B \setminus A )</td>
</tr>
<tr>
<td>( S_0(n-2, n, 0, 1) )</td>
<td>( S_1(n-3, n-2, n-1, 0, 1, \overline{2}) )</td>
<td>( 0 \in A \setminus B, 1 \not\in B \setminus A )</td>
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Table 2
Proof for Claim 3

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<th>( A )</th>
<th>( B )</th>
<th>Why ( d_c(A, B) \not= 2 )</th>
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<tbody>
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<td>( S_0(n-3, \overline{n-2}, n-1, \overline{0}, 1, \overline{2}) )</td>
<td>( S_1(n-2, \overline{n-1}, \overline{0}, \overline{1}, 1) )</td>
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<tr>
<td>( S_0(n-3, \overline{n-2}, n-1, \overline{0}, 1, 2) )</td>
<td>( S_1(n-2, \overline{n-1}, \overline{0}, \overline{1}, 1) )</td>
<td>( n-3 \in A \setminus B, n-2 \not\in B \setminus A )</td>
</tr>
<tr>
<td>( S_0(n-2, n-1, \overline{0}) )</td>
<td>( S_1(n-3, n-2, n-1, 0, 1, \overline{2}) )</td>
<td>( n-3 \in A \setminus B, n-2 \not\in B \setminus A )</td>
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<td>( n-3 \in A \setminus B, n-2 \not\in B \setminus A )</td>
</tr>
</tbody>
</table>

Proof of Claim 3. Suppose \( A \in S_0(n-3, \overline{n-2}, n-1, \overline{0}, 1, \overline{2}) \). If \( |B^{\setminus} \setminus \{n-3, n-2, 0, 2\}| \geq 2 \), then \( |B \setminus A| \geq 2 \). Hence the union of \( S_0(n-3, \overline{n-2}, n-1, \overline{0}, 1, \overline{2}) \), \( S_1(n-2, n-1, 0, 1, 2) \), \( S_1(n-3, \overline{n-2}, n-1, 0, 1, \overline{2}) \), and \( S_1(n-3, n-2, n-1, 0, 1, \overline{2}) \) is an independent set in \( G^2(S_0 \cup S_1) \). The reasons that the remaining union is an independent set is listed in Table 2.

The proof of our Lemma is thus complete.

Lemma 8. The induced subgraphs \( G[S_i \cup S_{i+1} \cup \cdots \cup S_{i+1}] \) and \( G[S_i \cup S_{i+1} \cup \cdots \cup S_{i+1}] \) in \( G \) are isomorphic. Moreover, the induced subgraphs \( G^2[S_i \cup S_{i+1} \cup \cdots \cup S_{i+1}] \) and \( G^2[S_i \cup S_{i+1} \cup \cdots \cup S_{i+1}] \) in \( G^2 \) are isomorphic.

Proof. Define a mapping \( g \) on the vertex set of \( G \) by \( g([a_1, a_2, \ldots, a_l]) = [a_1-2(j-i)l, a_2-2(j-i)l, \ldots, a_k-2(j-i)l] \). Clearly, \( g \) is a bijection. Recall that \( f(A) = |\{j \in A | j \in [n]\} \). This implies that \( f(g(A)) = |\{f(A) - 2k(j-i)|A| = |f(A)| + |A| - \sum_{j \in A} j \} \).

Hence, \( A \in S_{i+m} \) if and only if \( g(A) \in S_{j+m} \) for \( m = 0, 1, \ldots, l \). Therefore, the mapping \( g \) is a bijection between \( S_i \cup S_{i+1} \cup \cdots \cup S_{i+1} \) and \( S_{i+1} \cup S_{i+1} \cup \cdots \cup S_{i+1} \). It is easy to check that \( |A \cap B| = t \) if and only if \( |g(A) \cap g(B)| = t \). It follows from Lemma 1 that \( d_c(A, B) = d_c(g(A), g(B)) \). Therefore, \( g \) is an isomorphism between \( G[S_i \cup S_{i+1} \cup \cdots \cup S_{i+1}] \) and \( G[S_i \cup S_{i+1} \cup \cdots \cup S_{i+1}] \). Moreover, \( g \) induces an isomorphism between \( G^2[S_i \cup S_{i+1} \cup \cdots \cup S_{i+1}] \) and \( G^2[S_i \cup S_{i+1} \cup \cdots \cup S_{i+1}] \).

Theorem 9. For \( k \geq 3 \), \( \chi(K^2(2k+1, k)) \leq 3k+2 \).

Proof. Again, let \( G \) denote \( K^2(2k+1, k) \). By Lemma 8, \( G^2[S_0 \cup S_1], G^2[S_2 \cup S_3], \ldots, G^2[S_{2k-2} \cup S_{2k-1}] \) are all isomorphic. By Lemma 2, \( \chi(G^2[S_{2k}]) \leq 2 \). By Lemma 7, \( \chi(G^2[S_0 \cup S_1]) \leq 3 \) for \( k \geq 3 \). Therefore, \( \chi(G^2) \leq 3k+2 \) for \( k \geq 3 \).

Note that \( K^5(5, 2) \) is the Petersen graph whose square is a complete graph on 10 vertices. Kim and Nakprasit [3] proved that \( \chi(K^2(7, 3)) = 6 \) and \( 11 \leq \chi(K^2(9, 4)) \leq 18 \).

Theorem 10. \( \chi(K^2(9, 4)) \leq 12 \).

Proof. For \( G = K^2(9, 4) \), our pre-coloring gives us the following classes.

\( S_1 = \{0127, 0136, 0145, 0235, 0478, 0568, 1234, 1378, 1468, 1567, 2368, 2458, 2467, 3457 \} \),
\( S_2 = \{0128, 0137, 0146, 0236, 0245, 0578, 1235, 1478, 1568, 2368, 2478, 2658, 2567, 3458, 3467 \} \).
Let $S$ be a 2-stable $k$-subset of $[n]$ containing a fixed number $x$. Starting from $x$, we enumerate the distances between consecutive elements of $S$ in the cyclic order. The sequence of “gaps” $x_j \geq 2, 1 \leq k \leq k$, satisfy the following equation.

$$x_1 + x_2 + \cdots + x_k = n.$$  \hfill (2)

Conversely, any such sequence of positive integers determine a 2-stable $k$-subset of $[n]$ containing $x$. Therefore, the vertex set of the Schrijver graph $SG(n, k)$ has cardinality $\binom{n}{k}$. In particular, $SG(2k + 2, k)$ has $(k + 1)^2$ vertices. We note the following facts. If $k$ is odd, then $(k + 1)^2 \equiv 0 \pmod{2k + 2}$. If $k$ is even, then $(k + 1)^2 \equiv k + 1 \pmod{2k + 2}$.

Throughout this section, we let $G = SG(2k + 2, k)$ and $m = \lfloor \frac{k}{2} \rfloor$. In this case, there are two types of gaps that satisfy Eq. (2). In the first type, all gaps equal to 2, except one gap equals to 4. In the second type, all gaps equal to 2, except two gaps equal to 3.

For $0 \leq i < 2k + 1$, let $v(0, i)$ represent the $k$-subset $(i, i + 2, \ldots, i + 2k - 2)$, in which each element $t$ is actually $|t|_{2k+2}$. We make the convention that all indices and elements are taken modulo $2k + 2$ in this section. We also regard $v(0, i)$ as a sequence with the elements ordered in the above manner.

For $1 \leq j \leq m$, let $E_j$ be the sequence whose last $j$ entries are all equal to 1 and the first $k - j$ entries are all equal to 0. When a $k$-set $A$, regarded as a sequence, and a sequence $B$ are added to get $A + B$, we just add the two sequences entry-wise to get a $k$-set if all the sums are distinct. Now for $0 \leq i \leq 2k + 1$ and $1 \leq j \leq m$, let $v(j, i)$ represent the $k$-subset $v(0, i) + E_j$.

We collect all $v(j, i)$ into the set $V_j$ which consists of all $k$-subsets that have a unique gap of length $4$. Then we collect all $v(j, i)$ into the set $V_j$ when $1 \leq j \leq m$.

Observation 1. For $0 \leq j \leq m - 1$, $|V_j| = 2k + 2$. If $k$ is odd, $|V_m| = 2k + 2$; if $k$ is even, $|V_m| = k + 1$.

The reason that $|V_m| = k + 1$ for even $k$ is as follows. For $0 \leq i \leq k$, we have $v(m, i) = v(0, i) + E_m = i, i + 2m - 2, (i + 2m) + 1, \ldots, (i + 4m - 2) + 1, i, i + 2m - 2, (i + 2m - 2) + k + 1, (i + 2m) + k + 2, \ldots, (i + 4m - 2) + k + 2 = v(0, i) + 2k + 1 + E_m = v(m, i + k + 1)$.

Observation 2. For $1 \leq j \leq m - 1$ and $0 \leq i \leq 2k + 1$, the vertex $v(j, i)$ has exactly four neighbors: $v(j, i - 1), v(j, i + 1), v(j - 1, i - 1), v(j + 1, i + 1)$. If $k$ is odd, the vertex $v(m, i)$ has exactly four neighbors: $v(m, i - 1), v(m, i + 1), v(m - 1, i - 1), v(m + 1, i + 1)$. Then $v(0, i)$ has exactly one neighbor outside $G[V_0]$ and it is $v(1, i + 1)$.

Observation 3. In the induced subgraph $G[V_0]$, the vertex $v(0, i)$ is adjacent to the vertex $v(0, j)$ if and only if $|i - j|$ is odd.

Lemma 11. We have $\chi(G^2) \geq 2k + 2$ and $\lambda(G) \geq 2k + 2$.

Proof. By Observation 3, $G[V_0]$ is a complete bipartite graph with partite sets $X = \{v(0, 0), v(0, 2), \ldots, v(0, 2k)\}$ and $Y = \{v(0, 1), v(0, 3), \ldots, v(0, 2k + 1)\}$. It follows that $\chi(G^2) \geq 2k + 2$. Since at least one buffer color is needed between $X$ and $Y$ to get an $L(2, 1)$-labeling of $G$, we have $\lambda(G) \geq 2k + 2$.

Theorem 12. For $k \geq 5$, $\chi(SG(2k + 2, k)) = \lambda(SG(2k + 2, k)) = 2k + 2$.

Proof. Let $\varphi$ be a mapping from $V_0 \cup V_1 \cup \cdots \cup V_m$ to $\{0, 1, \ldots, 2k + 2\}$ defined as follows.

$$\varphi(v(j, i)) = \begin{cases} \lfloor \frac{i}{2} \rfloor & \text{if } j = 0 \text{ and } i \text{ is even,} \\ (i - 1)/2 + k + 2 & \text{if } j = 0 \text{ and } i \text{ is odd,} \\ \varphi(v(j - 1, i + 3)) & \text{if } k \text{ is odd and } 1 \leq j \leq m, \\ \varphi(v(j, i + 1)) & \text{if } k \text{ is even and } 1 \leq j \leq m - 1, \\ \varphi(v(j, i + k + 1)) & \text{if } k \text{ is even, } j = m, \text{ and } 0 \leq i \leq k, \\ \varphi(v(j, i + k + 3)) & \text{if } k \text{ is even, } j = m, \text{ and } k < i. \end{cases} \hfill (3)$$

In order to verify that $\varphi$ is an $L(2, 1)$-labeling for $G = SG(2k + 2, k)$, we call $\{0, 1, \ldots, k\}$ the class of low labels and $\{k + 2, k + 3, \ldots, 2k + 2\}$ the class of high labels. The absolute difference between a low label and a high label is at least two. According to Eq. (3), the low labels and the high labels alternately appear in $V_j$ and each label appears exactly once.
First suppose that \( k \geq 5 \) is odd. Now for a fixed \( t \), \( 0 \leq t \leq 2k + 1 \), we have \( \varphi(v(j, j + t)) = \varphi(v(0, 4j + t)) \). Hence whether \( \varphi(v(j, j + t)) \) is a low or a high label depends on whether \( t \) is even or odd. We have \(|\varphi(v(j, j + 1 + t)) - \varphi(v(j, j + t))| = |\varphi(v(0, 4j + t + 4)) - \varphi(v(0, 4j + t))| \geq 2 \) and \( |\varphi(v(j + 2, j + 2 + t)) - \varphi(v(j, j + t))| = |\varphi(v(0, 4j + t + 8)) - \varphi(v(0, 4j + t))| \geq 1 \).

It follows from the adjacencies obtained in Observation 2 that if \( v(j, i) \), \( 2 \leq j \leq m - 2 \), satisfies Eq. (1).

From Observation 3, we see that the labels of neighbors of \( v(0, i) \) in \( V_0 \) belong to the opposite class of the label of \( v(0, i) \). Hence the labels of \( v(0, i) \) and \( v(1, i) \) satisfy Eq. (1).

We know that \( v(m, i) \) and \( v(m, i + k + 1) \) are adjacent. However, \( \varphi(v(m, i)) = \varphi(v(0, 3m + i)), \varphi(v(m - 1, i - 1)) = \varphi(v(0, 3m + i - 4)), \varphi(v(m, i + k + 1)) = \varphi(v(0, 3m + i + k + 1)) \), and \( \varphi(v(3m - 1, i + k)) = \varphi(v(0, 3m + i + k - 3)) \). It follows that labels of \( v(m, i) \) and \( v(m - 1, i - 1) \) satisfy Eq. (1) when \( k \geq 5 \).

Next we suppose that \( k \geq 6 \) is even. Similar to the previous case, when the fixed \( t \) satisfies \( 0 \leq t \leq k \), we have \( \varphi(v(j, j + t)) = \varphi(v(0, 4j + t)) \). Hence whether \( \varphi(v(j, j + t)) \) is a low or a high label depends on whether \( t \) is even or odd. The same argument used before also shows that the label of \( v(j, i) \), \( 0 \leq j \leq m - 2 \), satisfies Eq. (1).

The vertices \( v(m, i) \) and \( v(m, i + k + 1) \) are adjacent. However, \( \varphi(v(m, i)) = \varphi(v(m, i + k + 1)) = \varphi(v(0, 3m + i)), \varphi(v(m - 1, i - 1)) = \varphi(v(0, 3m + i - 4)), \varphi(v(m - 1, i + k)) = \varphi(v(0, 3m + i + k - 3)) \), and \( \varphi(v(m - 2, i + k - 1)) = \varphi(v(0, 3m + i + k - 7)) \). It follows that labels of \( v(m, i) \) and \( v(m - 1, i - 1) \) satisfy Eq. (1) when \( k \geq 6 \).

Since the “buffer” label \( k + 1 \) was not used in Eq. (3), if we substitute \( k + 1 \) for \( 0 \), then \( \chi(SG^2(2k + 2, k)) = 2k + 2 \) for \( k \geq 5 \).

Theorem 13. We have \( \chi(SG^2(6, 2)) = 9, \lambda(SG(6, 2)) = 8, \) and \( \chi(SG^2(10, 4)) = \lambda(SG(10, 4)) = 10 \).

Proof. We give \( L(2, 1) \)-labelings for \( SG(6, 2) \) and \( SG(10, 4) \) in Figs. 2 and 3. In each figure, antipodal vertices on the outermost circle are identified. In Fig. 3, we also omitted the edges of the complete bipartite graph \( G[V_0] \) except one cycle. Since \( SG^2(6, 2) \) is a complete graph, we have \( \chi(SG^2(6, 2)) = 9 \), and hence \( \lambda(SG(6, 2)) = 8 \). By Lemma 11, \( \lambda(SG(10, 4)) = 10 \) and \( \chi(SG^2(10, 4)) \geq 10 \). In Fig. 3, if we replace 5 by 2 and 0 by 5, then we see that \( \chi(SG^2(10, 4)) = 10 \).

Theorem 14. We have \( \chi(SG^2(8, 3)) = 8 \) and \( \lambda(SG(8, 3)) = 9 \).
We give an $L(2, 1)$-labeling for $SG(8, 3)$ in Fig. 4. Again, antipodal vertices on the outermost circle are joined by edges which have been omitted. By Lemma 11, we know $\lambda(SG(8, 3)) \geq 8$. Assume that there is an $L(2, 1)$-labeling for $SG(8, 3)$ using labels from 0 to 8. Let $A = \{v(0, 0), v(0, 2), v(0, 4), v(0, 6)\}$, $B = \{v(0, 1), v(0, 3), v(0, 5), v(0, 7)\}$, $C = \{v(1, 0), v(1, 2), v(1, 4), v(1, 6)\}$, and $D = \{v(1, 1), v(1, 3), v(1, 5), v(1, 7)\}$. Note that $A$ and $B$ are the partite sets of a complete bipartite graph. We may assume that each vertex of $A$ is labeled with an element in $\{0, 1, 2, 3\}$ and each vertex of $B$ is labeled with an element in $\{5, 6, 7, 8\}$. Since any vertex of $C$ is at distance two with any vertex of $A$, each vertex of $C$ is labeled with an element in $\{4, 5, 6, 7, 8\}$. Similarly, each vertex of $D$ is labeled with an element in $\{0, 1, 2, 3, 4\}$. Since any vertex of $C$ is within distance two to any vertex of $D$, 4 cannot appear simultaneously as labels of vertices of $C$ and $D$. Assume that none of the vertices of $D$ is labeled with 4. So each vertex of $D$ is labeled with an element in $\{0, 1, 2, 3\}$.

Observe that $P = v(0, 0)v(1, 1)v(1, 5)v(0, 4)$ is a path. Any $L(2, 1)$-labeling for $SG(8, 3)$ restricted to $P$ produces an $L(2, 1)$-labeling for $P$. Since each vertex of $P$ is labeled with an element in $\{0, 1, 2, 3\}$, the only possible labelings for $P$ are 1302 and its reverse. It follows that $v(1, 3)$ and $v(1, 7)$ are labeled with 1 or 2. Since $v(1, 3)$ is adjacent to $v(1, 7)$, the assumed $L(2, 1)$-labeling for $SG(8, 3)$ cannot exist. So $\lambda(SG(8, 3)) = 9$. In Fig. 4, if we substitute 4 for 2 and 7 for 6, then we see that $\chi(SG^2(8, 3)) = 8$.  

We conclude with a summary of our results as follows. We have $\chi(SG^2(2k + 2, k)) = \lambda(SG(2k + 2, k)) = 2k + 2$ for $k \geq 4$, $\chi(SG^2(8, 3)) = 8$, $\lambda(SG(8, 3)) = 9$, $\chi(SG^2(6, 2)) = 9$, and $\lambda(SG(6, 2)) = 8$.

References