Interpolation in weighted spaces of entire functions in $\mathbb{C}^2$

Bjarte Rom

*Department of Mathematical Sciences, Norwegian University of Science and Technology, 7491 Trondheim, Norway

Received 25 May 2005
Available online 29 December 2006
Submitted by G. Komatsu

Abstract

The purpose of this article is to construct complete interpolating sequences for special spaces of entire functions of two variables. The origin for the work is a result of Yu. Lyubarskii and A. Rashkovskii on sampling and interpolation for two-dimensional Fourier transforms. We also prove a theorem of Paley–Wiener type.

© 2007 Elsevier Inc. All rights reserved.

Keywords: Complete interpolating sequences; Paley–Wiener spaces

1. Introduction

Let $M$ be the algebraic sum of a finite number of nonzero vectors in $\mathbb{C}^2$ (for the definition, see Section 3), and let

$$H_M(z) = \sup_{\lambda \in M} \text{Re}\langle z, \lambda \rangle$$

be its support function. $\langle \cdot, \cdot \rangle$ denotes the $\mathbb{C}^2$-scalar product: $\langle z, \lambda \rangle = z_1 \bar{\lambda}_1 + z_2 \bar{\lambda}_2$ for $z = (z_1, z_2)$, $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$. The geometrical interpretation of $H_M(z)$ is that if $|z| = 1$, then $H_M(z)$ is the length of the set $M$ projected on the ray with direction $z$. In this article we study interpolation
problems in the space \( S^p_M \), \( 1 < p < \infty \), consisting of entire functions of exponential type\(^1\) with indicator not exceeding \( H_M(z) \), and with finite norm
\[
\| f \|_{S^p_M} = \sup_{\gamma \in \mathbb{C}^2} \left( \int_{\mathbb{C}^2} |f(z)|^p e^{-pH_M(z)} (dd^c H_M(z + \gamma))^2 \right)^{1/p} < \infty.
\] (1.1)

The form \((dd^c H_M(z))^2\) is the Monge–Ampère measure of \( H_M(z) \), and its definition is given in Section 3.

We say that a sequence \( \Omega = \{\omega\} \subset \mathbb{C}^2 \) is an interpolating sequence for \( S^p_M \) if for each \( \{a_\omega e^{-H_M(\omega)}\}_{\omega \in \Omega} \in l^p(\Omega) \), there exists \( f \in S^p_M \) solving the interpolating problem
\[
f(\omega) = a_\omega, \quad \omega \in \Omega.
\] (1.2)

The factor \( e^{-H_M(\omega)} \) in \( \{a_\omega e^{-H_M(\omega)}\}_{\omega \in \Omega} \) is introduced in order to compensate the exponential growth of \( f \). If the solution to this problem is always unique we say that \( \Omega \) is a complete interpolating sequence for \( S^p_M \). It will be shown that for each interpolating sequence \( \Omega \), the operator
\[
f \mapsto \{f(\omega)e^{-H_M(\omega)}\}_{\omega \in \Omega}
\]
is bounded from \( S^p_M \) onto \( l^p \). So by the Banach inverse operator theorem each complete interpolating sequence \( \Omega \) is also a sampling sequence, i.e.
\[
A \| \{f(\omega)e^{-H_M(\omega)}\}_{\omega \in \Omega} \|_{l^p} \leq \| f \|_{S^p_M} \leq B \| \{f(\omega)e^{-H_M(\omega)}\}_{\omega \in \Omega} \|_{l^p}, \quad f \in S^p_M,
\]
for some \( A, B > 0 \) independent of \( f \).

The purpose of this article is to construct complete interpolating sequences for the space \( S^p_M \). In contrast to the one-dimensional case where a full description of complete interpolating sequences is obtained for a number of spaces of entire functions (see e.g. \([13,14]\) and \([11]\)), the corresponding multi-dimensional problem is very far from being solved. This is the case because the machinery of generating functions, the main tool in the one-dimensional case, cannot be applied in several dimensions. Even the existence of complete interpolating sequences is not obvious, so at the present stage it is natural to just look for examples of complete interpolating sequences in a hope to get a hint about more general situations.

Except natural constructions related to products of one-dimensional interpolating sequences, the only examples of complete interpolating sequences in several variables known to the author concern the spaces
\[
P W_M = \left\{ f(z): f(z) = \int_M e^{i(z,\xi)} \phi(\xi) \, dm(\xi) \in L^2(M) \right\}
\]
endowed with the \( L^2(\mathbb{R}^2) \)-norm. Here \( M \subset \mathbb{R}^2 \) is a convex polygon, \( \mathbb{R}^2 \) is considered as the real plane in \( \mathbb{C}^2 \) and \( dm \) stands for the plane Lebesgue measure. This was studied in \([10]\), where it was shown that the collection of all pairwise intersections of the zero hyperplanes of a certain function forms a complete interpolating sequence if it is uniformly separated (i.e. the distance between each pair of distinct points is uniformly bounded off zero).

Following the construction in \([10]\) we construct entire functions with plane zeros in \( \mathbb{C}^2 \) which will generate complete interpolating sequences for \( S^p_M \). Being of complex dimension 1, the zero

---

\(^1\) We refer the reader to \([7]\) for main definitions and properties of entire functions of exponential type.
set $Z$ of an entire function in $\mathbb{C}^2$ cannot itself form such a sequence (in contrast to the one-dimensional case), however it can produce a discrete set $\Omega \subset Z$ which fits our needs, namely the collection of all pairwise intersections of the zero hyperplanes. In our construction $\Omega$ forms a complete interpolating sequence for $S_{M}^{p}$ if it is uniformly separated.

The article is organized as follows. In Section 2 we gather preliminary results. In Section 3 we construct complete interpolating sequences for $S_{M}^{p}$. In Section 4 we prove a uniqueness property in $S_{M}^{2}$, and in Section 5 we solve the interpolation problem stated in (1.2). In Section 6 we consider the special case when $M$ is a product domain. Finally, in Section 7 we give examples of functions in $S_{M}^{2}$. In particular, we prove a theorem of Paley–Wiener type for product domains.

2. Preliminaries

We need some information about interpolation in the one-dimensional Paley–Wiener spaces $L_{\sigma}^{p}$ consisting of entire functions of exponential type not larger than $\sigma$ whose $p$th powers are integrable over the real axis.

A sequence of points $\{\zeta_{k}\} \subset \{\zeta \in \mathbb{C}: |\text{Im} \ \zeta| < \text{Const}\}$ is called a complete interpolating sequence for $L_{\sigma}^{p}$ if the interpolation problem

$$f(\zeta_{k}) = a_{k}, \quad k \in \mathbb{Z},$$

has a unique solution $f \in L_{\sigma}^{p}$ for each $\{a_{k}\} \in l^{p}$. It follows from Banach inverse operator theorem that a complete interpolating sequence is a sampling set, i.e. there exists constant $K > 0$ such that

$$\frac{1}{K} \| f(\zeta_{k}) \|_{l^{p}} \leq \| f \|_{L_{\sigma}^{p}} \leq K \| f(\zeta_{k}) \|_{l^{p}} \quad (2.1)$$

for all $f \in L_{\sigma}^{p}$.

A special case of complete interpolating sequences for $L_{\sigma}^{p}$ is due to Levin (see [7, Lecture 22]).

**Definition 2.1.** An entire function $S(\zeta)$ is called a sine-type function of type $\sigma$ if all its zeros $\{\zeta_{k}\}$ are simple and lie in a horizontal strip, and also

(a) the zero set $\{\zeta_{k}\}$ is uniformly separated, i.e.

$$\inf_{k \neq l} |\zeta_{k} - \zeta_{l}| = \delta > 0, \quad (2.2)$$

(b) for every $\epsilon > 0$, we have

$$\frac{1}{C} < |S(\zeta)|e^{-\sigma|\text{Im} \ \zeta|} < C \quad \text{for dist}(\zeta, \{\zeta_{k}\}) > \epsilon. \quad (2.3)$$

We have the following theorem.

**Theorem 2.2.** (See e.g. [7, Lecture 22].) Let $S(\zeta)$ be a sine-type function of type $\sigma$ and $\{\zeta_{k}\}$ be its zero set. Then $\{\zeta_{k}\}$ is a complete interpolating sequence for $L_{\sigma}^{p}$, and the constant $K$ in (2.1) depends only on $\delta$ and $C$ in Definition 2.1.

Let $D$ be a closed convex $n$-gon in $\mathbb{C}$, and assume for definiteness that the origin belongs to $D$ (this can always be done by a suitable translation). From the origin we draw normals $N_{j}$,
\( j = 1, 2, \ldots, n \), to the sides of \( D \), and we number the angles \( \theta_j \) which the normals make with the positive direction of the real axis. Define the class \( L^p_D \) of entire functions of exponential type on \( \mathbb{C} \) with finite norm

\[
\|f\|_{p,D} = \sup_{\theta \in [0,2\pi]} \left\{ \int_0^\infty |f(re^{i\theta})|^p e^{-pH_D(re^{i\theta})} \right\} (1 < p < \infty).
\]

Here \( H_D(\zeta) = \sup_{\xi \in D} \text{Re} \frac{\xi}{\bar{\zeta}} \) is the support function of \( D \). An application of Pragmén–Lindelöf (see e.g. [8, Lemma 2.2]) gives the norm

\[
\|f\|_{p,D} = \max_{j=1,\ldots,n} \left\{ \int_0^\infty |f(re^{i\theta_j})|^p e^{-pH_D(re^{i\theta_j})} \right\} (1 < p < \infty). \tag{2.4}
\]

Note that if \( D \) degenerates into the segment \([-i\sigma, i\sigma]\) on the imaginary axis, \( L^p_D \) is just the Paley–Wiener space \( L^p_\sigma \) defined above.

Interpolation in \( L^p_D \) has been studied in [8]. Let \( \Sigma_D \) denote the class of functions \( S(\zeta) \) of exponential type satisfying

\[
|S(\zeta)|e^{-H_D(\zeta)} \asymp 1 \quad \text{for} \quad \text{dist}(\zeta, \Lambda) > \delta > 0, \quad \Lambda = \{ \lambda : S(\lambda) = 0 \}, \tag{2.5}
\]

where the constants in \( \asymp \) only depend on \( S \) and \( \delta \), and \( S(z) \neq 0 \) outside strips around the normals \( N_j, 1 \leq j \leq n \). The following interpolation result is a generalization of Theorem 2.2 (see [8, Theorem 2.3]).

**Theorem 2.3.** Let \( S(z) \in \Sigma_D, \) and let \( \Lambda = \{ \lambda_k \} \) be the sequence of its zeros, numbered in order of increasing modulus. Assume that all the zeros are simple and that the set \( \Lambda \) is uniformly separated (see (2.2)). Let \( \{a_k\} \) be a sequence such that \( \{a_k e^{-H_D(\lambda_k)}\} \in l^p, 1 < p < \infty \). Then

\[
f(z) = S(z) \sum_k \frac{a_k}{S'(\lambda_k)(z - \lambda_k)} \tag{2.6}
\]

solves the interpolation problem \( f(\lambda_k) = a_k \) in the space \( L^p_D \). The series \( f(z) \) converges in the norm \( \|\cdot\|_{p,D} \) given by (2.4) and uniformly on compact sets in \( \mathbb{C} \).

**Remark.** Some analogues of the spaces \( L^p_D \) can be naturally defined also in the case when \( D \) is an arbitrary convex compact set in \( \mathbb{C} \). If the boundary of \( D \) contains a smooth arc with positive curvature, the corresponding space \( L^p_D \) does not admit both complete and interpolating sequences. This statement has been proved (for \( p = 2 \)) by V.I. Lutsenko and R.S. Yulmukhametov in [9] and independently by Yu. Lyubarskii and K. Seip in [12].

Let \( D \) be the complex conjugate of \( D \). The Smirnov space \( G^2(D) \) consists of functions \( \psi(z) \) holomorphic in the exterior of \( D \) with the property that there exists a sequence of rectifiable contours \( \{c_k\} \) in the exterior of \( D \) and approaching \( \partial D \) such that

\[
\|\psi\|^2_{G^2(D)} = \sup_k \left\{ \int_{|z|>1} \left| \psi(z) \right|^2 |dz| \right\} < \infty.
\]

In addition, \( \psi(z) \to 0 \) as \( |z| \to \infty \).

\[\]

---

\[\] Here and in the sequel the sign \( \asymp \) means that the ratio of the two sides lies between two positive constants.
Theorem 2.4. (See e.g. [6, Appendix I.3].) Let \( D \) be a convex polygon in \( \mathbb{C} \). Then \( f \in L^2_D \) if and only if
\[
f(\zeta) = \int_{\partial D} e^{\zeta} \psi(\zeta) d\zeta,
\]
where \( \psi \in G^2(\bar{D}) \). In addition, \( \| f \|_{p,D} \asymp \| \psi \|_{G^2(\bar{D})} \).

This theorem is a generalization of the classical Paley–Wiener theorem which corresponds to the case when \( D \) degenerates into a segment on the imaginary axis.

The Hardy space \( H^p(\mathbb{C}_+) \), \( 1 < p < \infty \), is the space of analytic functions in the half-plane \( \mathbb{C}_+ = \{ z : \Im z > 0 \} \) satisfying the condition
\[
\sup_{y > 0} \int_{-\infty}^{\infty} |f(x + iy)|^p dx < \infty,
\]
and endowed with the norm
\[
\| f \|_{H^p(\mathbb{C}_+)} = \left( \sup_{y > 0} \int_{-\infty}^{\infty} |f(x + iy)|^p dx \right)^{1/p} = \left( \int_{-\infty}^{\infty} |f(x)|^p dx \right)^{1/p}.
\]
The proof of the last equality can be found in e.g. [5].

Given \( a \in \mathbb{R} \) and \( l > 0 \), we denote \( Q(a, l) = \{ z = x + iy : |x - a| < 2l, \ 0 < y < 2l \} \). A positive measure \( \mu \) in \( \mathbb{C}_+ \) is called a Carleson measure for \( \mathbb{C}_+ \) if, for each \( a \in \mathbb{R} \) and \( l > 0 \),
\[
\mu(\{ z \in \mathbb{C}_+ : |x - a| < 2l, \ 0 < y < 2l \}) \leq \text{Const} \cdot l.
\]
The following property is a characterization of Carleson measures (see e.g. [5, Chapter VIII]).

Theorem 2.5. Let \( \mu \) be a positive measure in \( \mathbb{C}_+ \), and fix \( p \in (0, \infty) \). Then
\[
\left( \int_{\mathbb{C}_+} |f(\zeta)|^p d\mu(\zeta) \right)^{1/p} \leq \text{Const} \| f \|_{H^p(\mathbb{C}_+)} \quad \text{for all } f \in H^p(\mathbb{C}_+)
\]
if and only if \( \mu \) is a Carleson measure in \( \mathbb{C}_+ \).

We also need some facts about Hardy spaces in two variables. Let \( \mathbb{C}_+ \times \mathbb{C}_+ \) denote the bi-upper half-plane \( \{ z \in \mathbb{C}^2 : \Im z_1 > 0, \ \Im z_2 > 0 \} \). The space \( H^p(\mathbb{C}_+ \times \mathbb{C}_+) \) is the space of analytic functions in \( \mathbb{C}_+ \times \mathbb{C}_+ \) satisfying the condition (see e.g. [3])
\[
\sup_{y_1 > 0, y_2 > 0} \int_{\mathbb{R}^2} |f(x_1 + iy_1, x_2 + iy_2)|^p dx_1 dx_2 < \infty
\]
and endowed with the norm
\[
\| f \|_{H^p_+} = \sup_{y_1 > 0, y_2 > 0} \int_{\mathbb{R}^2} |f(x_1 + iy_1, x_2 + iy_2)|^p dx_1 dx_2 = \int_{\mathbb{R}^2} |f(x_1, x_2)|^p dx_1 dx_2.
\]

\(^3\) This norm equivalence was not stated explicitly in [6], but it follows from the proof given there.
Standard arguments show that any \( f \in H^2(\mathbb{C}_+ \times \mathbb{C}_+) \) has the representation

\[
f(z) = \int_{0}^{\infty} \int_{0}^{\infty} e^{i\langle z, t \rangle} \phi(t) \, dm(t), \quad \text{where} \quad \phi(t) \in L^2((0, \infty) \times (0, \infty)).
\]  

(2.9)

For each \( z \in \mathbb{C}_+ \), let \( I_z \) denote the interval \( \{ s : |s - x| < y \} \) on the real line \( \mathbb{R} \). For each connected open set \( U \subset \mathbb{R}^2 \), define

\[
A(U) = \{(z_1, z_2) \in \mathbb{C}_+ \times \mathbb{C}_+ : I_{z_1} \times I_{z_2} \subset U \}.
\]

The following result in [2] comes in handy when proving that the solution to the interpolation problem satisfies the integrability condition in (1.1).

**Theorem 2.6.** Let \( \mu \) be a positive measure in \( \mathbb{C}_+ \times \mathbb{C}_+ \) and suppose \( 0 < p < \infty \). Then

\[
\left( \int_{\mathbb{C}_+ \times \mathbb{C}_+} |f(z)|^p \, d\mu(z) \right)^{1/p} \leq C_p \| f \|_{H^p(\mathbb{C}_+ \times \mathbb{C}_+)} \quad \text{for all} \quad f \in H^p(\mathbb{C}_+ \times \mathbb{C}_+)
\]

if and only if

\[
\mu(A(U)) \leq C|U| \quad \text{for all connected open sets} \quad U \subset \mathbb{R}^2.
\]  

(2.10)

A positive measure satisfying (2.10) is called a Carleson measure for the bi-upper half-plane.

A function \( \phi : D \to B \), where \( D \) is a domain in \( \mathbb{C} \) and \( B \) is a Banach space, is called analytic if for all \( \lambda \in D \) the derivative

\[
\phi'(\lambda) = \lim_{h \to 0} \frac{\phi(\lambda + h) - \phi(\lambda)}{h}
\]

exists, where the limit is considered with respect to the norm in \( B \). For every linear functional \( f \in B^* \), the function \( f[\phi(\lambda)] \) is analytic, and this permits theorems on complex-valued analytic functions to be extended to \( B \)-valued analytic function. In the results, the modulus will be replaced by the Banach norm. The reader is referred to [7, Lecture 6.2] for more on \( B \)-valued analytic functions.

3. Construction of a complete interpolating sequence

First we need some geometrical constructions. Given \( b^1, \ldots, b^N \in \mathbb{C}_2 \setminus \{0\} \) such that \( b^i \neq \alpha b^j \), \( \alpha \in \mathbb{R} \), \( i \neq j \), our set \( M \) has by definition the following representation:

\[
M = \left\{ \sum_{j=1}^{N} t_j b^j : |t_j| \leq \pi, \ 1 \leq j \leq N \right\}.
\]  

(3.1)

The support function of \( M \) is now explicit

\[
H_M(z) = \pi \sum_{j=1}^{N} |\text{Re}\{z, b^j\}|.
\]
Lemma 3.1. Let $M$ have the representation (3.1) and $\gamma \in \mathbb{C}^2$. Then

$$
\int_{\mathbb{C}^2} \left| f(z) \right|^p e^{-pH_M(z)} \left( dd^c H_M(z + \gamma) \right)^2 = \sum_{r=1}^{N} \sum_{s=1}^{N} A_{r,s} \int_{\Pi_{r,s}(\gamma)} \left| f(z) \right|^p e^{-pH_M(z)} dm_{r,s}(\gamma),
$$

where

$$
\Pi_{r,s}(\gamma) = \{ z \in \mathbb{C}^2 : \text{Re}(z + \gamma, b^r) = \text{Re}(z + \gamma, b^s) = 0 \},
$$

$$dm_{r,s}(\gamma) = \text{the Lebesgue measure on } \Pi_{r,s}(\gamma), \text{ and}
$$

$$A_{r,s} = 8\pi \left( |b^r_1|^2 |b^s_2|^2 + |b^s_1|^2 |b^r_2|^2 - b^r_1 \bar{b}^s_2 \bar{b}^r_2 b^s_1 - \bar{b}^r_2 b^r_1 \bar{b}^s_1 \bar{b}^r_1 \bar{b}^s_2 \right).$$

$A_{r,s} = 0$ if and only if there exists $\alpha \in \mathbb{C}$ such that $b^r = \alpha b^s$. This is equivalent to $\Pi_{r,s}$ being an analytic plane (i.e. of the form $az_1 + b z_2 = c$).

Before we prove this lemma, we need some more information about the Monge–Ampère operator. We use the notation

$$
\partial = \frac{\partial}{\partial z_1} dz_1 + \frac{\partial}{\partial z_2} dz_2, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}_1} d\bar{z}_1 + \frac{\partial}{\partial \bar{z}_2} d\bar{z}_2, \quad d = \partial + \bar{\partial} \quad \text{and} \quad d^c = i(\bar{\partial} - \partial)
$$

so that $dd^c = 2i \partial \bar{\partial}$. Also recall that $\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right)$ and $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right)$, where $z_j = x_j + iy_j$, $j = 1, 2$.

An upper semi-continuous function $u$ in $\mathbb{C}^2$ is said to be plurisubharmonic if the function $\lambda \mapsto u(a + \lambda w)$ is subharmonic in $\mathbb{C}$ for each $a$ and $w$ in $\mathbb{C}^2$. This is the same as saying that $dd^c u$ is positive. By this we mean that (see e.g. [4, Section 3.3])

$$dd^c u = 2 \sum_{j,k=1}^{2} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} i dz_j \wedge d\bar{z}_k
$$

is positive. By this we mean that (see e.g. [4, Section 3.3])

$$dd^c u = 2 \sum_{j,k=1}^{2} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \geq 0
$$

for every $w \in \mathbb{C}^2$. If $u$ is not a $C^2$ function, the coefficients in $dd^c u$ will generally just be measures, and $dd^c u$ is a positive $(1, 1)$ current (dual to $(1, 1)$ forms; a $(1, 1)$ form with distribution coefficients).

Following [1], we can now define $(dd^c u)^2$ for a plurisubharmonic function $u$ in $\mathbb{C}^2$ if $u$ is locally bounded on $\mathbb{C}^2$ using the fact that $dd^c u$ is a positive $(1, 1)$ current with measure coefficients. Note that if $u$ were of class $C^2$, given $\phi$ a smooth function with compact support in $\mathbb{C}^2$, Stokes’ theorem yields

$$
\int_{\mathbb{C}^2} \phi(dd^c u)^2 = - \int_{\mathbb{C}^2} d\phi \wedge d^c u \wedge dd^c u = - \int_{\mathbb{C}^2} du \wedge d^c \phi \wedge dd^c u
$$

$$= \int_{\mathbb{C}^2} v dd^c \phi \wedge dd^c u
$$

since $\phi$ vanish at infinity. The applications of Stokes’ theorem are justified if $u$ is $C^2$; for arbitrary locally bounded plurisubharmonic functions $u$ in $\mathbb{C}^2$, these formal calculations serve as
motivation to define $(dd^c u)^2$ as a positive measure (precisely, a positive current of bidegree $(2, 2)$ and hence a positive measure) via

$$\int_{\mathbb{C}^2} \phi (dd^c u)^2 := \int_{\mathbb{C}^2} u dd^c \phi \wedge dd^c u.$$ 

This defines $(dd^c u)^2$ as a $(2, 2)$ current (acting on $(0, 0)$ forms; i.e. test functions) since $u dd^c u$ has measure coefficients. We refer the reader to [1] or [4, p. 113] for the verification of positivity of $(dd^c u)^2$.

**Proof of Lemma 3.1.** First note that $H_M(z)$ is a continuous plurisubharmonic function in $\mathbb{C}^2$. Let $u_r = \text{Re}(z + \gamma, b^r)$ such that $H_M(z + \gamma) = \pi \sum_{r=1}^N |u_r|$, and let $\mathcal{H}$ be the Heaviside function,

$$\mathcal{H}(t) = \begin{cases} 1 & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Its derivative in distributional sense is Dirac’s $\delta$. Calculations then give

$$\delta \partial |u_r| = \partial \left[ (2\mathcal{H}(u_r) - 1) \frac{1}{2} \bar{b}_1^r d\bar{z}_1 + (2\mathcal{H}(u_r) - 1) \frac{1}{2} \bar{b}_2^r d\bar{z}_2 \right]$$

$$= \frac{1}{2} \delta (u_r) \left[ |b_1^r|^2 d z_1 + d \bar{z}_1 + |b_2^r|^2 d z_2 + d \bar{z}_2 + \bar{b}_1^r b_2^r d z_1 \wedge d \bar{z}_2 + |b_2^r|^2 d z_2 \wedge d \bar{z}_2 \right].$$

Further (see e.g. [1, Proposition 2.7]),

$$(dd^c H_M(z + \gamma))^2 = 32\pi \left( \frac{\partial^2 H_M(z + \gamma)}{\partial z_1 \partial \bar{z}_1} \cdot \frac{\partial^2 H_M(z + \gamma)}{\partial z_2 \partial \bar{z}_2} - \frac{\partial^2 H_M(z + \gamma)}{\partial z_2 \partial z_1} \cdot \frac{\partial^2 H_M(z + \gamma)}{\partial \bar{z}_1 \partial \bar{z}_2} \right) dV$$

$$= 8\pi \left( \sum_{r=1}^N \delta (u_r) |b_1^r|^2 \right) \left( \sum_{r=1}^N \delta (u_r) |b_2^r|^2 \right) - \left( \sum_{r=1}^N \delta (u_r) b_1^r \bar{b}_2^r \right) \left( \sum_{r=1}^N \delta (u_r) \bar{b}_1^r b_2^r \right) dV$$

$$= 8\pi \sum_{j=1}^N \sum_{k=1 \neq j}^N \left( |b_1^j|^2 |b_2^k|^2 + |b_1^k|^2 |b_2^j|^2 - b_1^j b_2^k \bar{b}_1^k \bar{b}_2^j \bar{b}_1^j b_2^k \delta (u_j) \delta (u_k) dV, \right.$$

where $dV = -\frac{1}{3} d z_1 \wedge d \bar{z}_1 \wedge d z_2 \wedge d \bar{z}_2$. Here we have used the facts $d z_j \wedge d z_j = 0$ and $d \bar{z}_j \wedge d \bar{z}_j = 0$, $j = 1, 2$. Calculations show that

$$|b_1^j|^2 |b_2^k|^2 + |b_1^k|^2 |b_2^j|^2 - b_1^j b_2^k \bar{b}_1^k \bar{b}_2^j \bar{b}_1^j b_2^k = 0 \quad \text{if and only if} \quad b^k = \alpha b^j, \quad \alpha \in \mathbb{C}.$$ 

Finally, by the definition of the $\delta$ function,

$$\int_{\mathbb{R}^4} \phi (u_r, v_r, u_s, v_s) \delta (u_r) \delta (u_s) du_r dv_r du_s dv_s = \int_{\mathbb{R}^2} \phi (0, v_r, 0, v_s) dv_r dv_s,$$

where $\phi$ is a bounded, continuous test function on $\mathbb{R}^4$. The result now follows after suitable changes of variables.

In proving the last assertion, we may assume that $b^r = (1, 0)$ (this can be achieved by an appropriate affine transformation). If $b^s = (\alpha, 0)$, a simple calculation yields $\Pi_{r,s}(\gamma) = \{ z \in \mathbb{C}^2; \ z_1 = -\gamma_1 \}$ which is an analytic plane. Conversely, assume $\Pi_{r,s}(\gamma)$ is an analytic plane. After a
holomorphic change of variables, we may assume \( \Pi_{\tau, s}(y) = \{z_1 = 0\} \). Straightforward calculations now show that \( b_2' = b_2^s = 0 \), and we are done. \( \square \)

Put

\[
S(z) = \prod_{k=1}^{N} \sin[\pi(i[z, b^k] - \alpha_k)],
\]

where the complex constants \( \alpha_k \) will be specified later. This function will be used for construction of the desired complete interpolation sequence for \( S^P_M \). Here are some of the properties of \( S \).

**Lemma 3.2.** For given \( \delta > 0 \), the function \( S \) satisfies

\[
|S(z)| \asymp e^{HM(z)} \quad \text{for dist}(i[z, b^k] - \alpha_k, \mathbb{Z}) > \delta, \ 1 \leq k \leq N.
\]  

**Proof.** We need to show

\[
Ae^{HM(z)} \leq |S(z)| \leq Be^{HM(z)} \quad \text{for dist}(i[z, b^k] - \alpha_k, \mathbb{Z}) > \delta, \ 1 \leq k \leq N,
\]

where \( A \) and \( B \) are positive constants. This will be shown by estimation of each factor in \( S(z) \). We have

\[
|\sin[\pi(i[z, b^k] - \alpha_k)]| \leq \text{Const} \left( |e^{-\pi(z, b^k)}| + |e^{\pi(z, b^k)}| \right) \\
\leq \text{Const} e^{\pi |\text{Re}(z, b^k)|}.
\]

Now we need opposite inequality:

\[
|\sin[\pi(i[z, b^k] - \alpha_k)]| = \frac{1}{2} \left| e^{-u+iv} - e^{u+iv} \right| \\
= \frac{1}{2} \left| e^{-u-i} - e^{u+i} \right| \\
= \frac{1}{2} \left| e^{u} \cos(v) \left( 1 - e^{-2|u|} \right) - i \sin v \left( 1 + e^{-2|u|} \right) \right| \\
\geq \frac{1}{2} e^{|u|} |\sin v| \geq \frac{1}{2} e^{-\pi |\text{Re}(z, b^k)|} e^{\pi |\text{Re}(z, b^k)|} |\sin v|.
\]

In addition, we have

\[
|\sin[\pi(i[z, b^k] - \alpha_k)]| \geq \frac{1}{2} |e^{-u} - e^{u}| = \frac{1}{2} (1 - e^{-|u|}) e^{|u|}. \quad (3.6)
\]

Since dist\((z, b^k) - \alpha_k, \mathbb{Z}) > \delta' \),

\[
\left| \frac{iu - v}{\pi} - n \right|^2 = (v + \pi n)^2 + u^2 > (\pi \delta')^2 = \delta^2 \quad \forall n \in \mathbb{Z}.
\]

Suppose \( |u| \geq \delta/2 \). Let \( k_1 = 1/2(1 - e^{-\delta/2}) \). Then by (3.6)

\[
|\sin[\pi(i[z, b^k] - \alpha_k)]| \geq k_1 e^{|u|} \geq k_1 e^{-\pi |\text{Re}(\alpha_k)|} e^{\pi |\text{Re}(z, b^k)|}.
\]

Let now \( |u| < \delta/2 \). Then by (3.7), \( (v + \pi n)^2 > 3/4 \delta^2 \) for all \( n \in \mathbb{Z} \), that is

\[
v \in \bigcup_{n \in \mathbb{Z}} \left( \pi n + \frac{\sqrt{3} \delta}{2}, \pi (n + 1) - \frac{\sqrt{3} \delta}{2} \right).
\]
Here \(| \sin v | \geq k_2 > 0\), and by (3.5)
\[
\left| \sin \left( \pi \left( i \langle z, b^k \rangle - \alpha_k \right) \right) \right| \geq \frac{1}{2} e^{-\pi |\text{Re} i\alpha_k|} k_2 e^{\pi |\text{Re}(z, b^k)|}.
\]
Finally, by choosing \(k = \min\left(\frac{1}{2} e^{-\pi |\text{Re} i\alpha_k|} k_2, k_1 e^{-\pi |\text{Re} i\alpha_k|}\right)\), we get the opposite inequality. \(\square\)

The zero set \(Z\) of the function \(S\) is the union of hyperplanes
\[
P^{(k,n)} = \{ z \in \mathbb{C}^2 : i \langle z, b^k \rangle = n + \alpha_k \}, \quad n \in \mathbb{Z}, \ 1 \leq k \leq N.
\]

**Definition 3.3.** \(\Omega\) is the set of points which are pairwise intersections of the hyperplanes \(P^{(k_1,n_1)}\) and \(P^{(k_2,n_2)}\), \(n_1, n_2 \in \mathbb{Z}\).

We say that \(\omega \in \Omega\) is a multiple point if it lies in the intersection of three or more hyperplanes. **Our main result is that \(\Omega\) is a complete interpolating sequence for \(S_M\) if it is uniformly separated and does not contain multiple points (Theorem 5.1).**

The rest of this section will be used to investigate when these two conditions are fulfilled. Let
\[
b^{(k,n)} = -i (n + \alpha_k) \frac{b^k}{|b^k|^2}
\]
and choose a unit vector \(c^k \in \mathbb{C}^2\) such that \(\langle c^k, b^k \rangle = 0\). Then \(P^{(k,n)}\) has the following representation:
\[
P^{(k,n)} = \{ z = b^{(k,n)} + c^k \zeta : \zeta \in \mathbb{C} \}.
\] (3.8)

**Lemma 3.4.** There exists a set \(E_M \subset \mathbb{C}^N\) of zero Lebesgue measure such that \(\Omega\) consists of no multiple points for all \(\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{C}^N \setminus E_M\).

**Proof.** Assume \(b_1^{k_1} b_2^{k_2} \neq b_1^{k_2} b_2^{k_1}\). We have to check that the equation
\[
b^{(k_1,n_1)} + c^{k_1} \xi_1 = b^{(k_2,n_2)} + c^{k_2} \xi_2 = \omega^{(k_1,n_1)(k_2,n_2)},
\] (3.9)
with respect to \(\xi_1\) and \(\xi_2\), has exactly one solution. This gives us the following equations
\[
c_1^{k_1} \xi_1 - c_1^{k_2} \xi_2 = b_1^{(k_2,n_2)} - b_1^{(k_1,n_1)},
\]
\[
c_2^{k_1} \xi_1 - c_2^{k_2} \xi_2 = b_2^{(k_2,n_2)} - b_2^{(k_1,n_1)},
\]
which have a unique solution \((\xi_1, \xi_2)\) if and only if \(c_1^{k_1} c_2^{k_2} \neq c_1^{k_2} c_2^{k_1}\), and this is satisfied if and only if \(b_1^{k_1} b_2^{k_2} \neq b_1^{k_2} b_2^{k_1}\).

Assume now that \(b_1^{k_1} b_2^{k_2} = b_1^{k_2} b_2^{k_1}\). The above equations only have infinitely many solutions whenever there is a certain linear relation between \(\alpha_{k_1}\) and \(\alpha_{k_2}\). This linear relation defines a subspace in the parameter space \(\mathbb{C}^N_{(\alpha)}\). Let \(E'_M\) be the countable union of such subspaces for all \((k_1, n_1)\) and \((k_2, n_2)\).

Denote the solution of (3.9) by
\[
\lambda_1^{(k_1,n_1)} = \xi_1, \quad \lambda_2^{(k_2,n_2)} = \xi_2
\]
and let
\[ X_k^{(k_1,n_1)} = \left\{ x_{(k,n)}^{(k_1,n_1)} : n \in \mathbb{Z} \right\}, \]
\[ X^{(k_1,n_1)} = \bigcup_{k \neq k_1} X_k^{(k_1,n_1)}, \]
\[ \Omega_k^{(k_1,n_1)} = \left\{ \omega^{(k_1,n_1)(n)} : n \in \mathbb{Z} \right\}, \]
\[ \Omega^{(k_1,n_1)} = \bigcup_{k \neq k_1} \Omega_k^{(k_1,n_1)}. \]

Then
\[ \Omega = \bigcup_{n_1 \in \mathbb{Z}} \bigcup_{1 \leq k_1 \leq N} \Omega^{(k_1,n_1)}. \]

Suppose \( \omega^{(k_1,n_1)(k_2,n_2)} = \omega^{(k_1,n_1)(k_3,n_3)} \) where \( (k_2, n_2) \neq (k_3, n_3) \). This gives us 6 equations and 3 unknown, and a solution is only possible if there exists a certain linear relation between \( \omega^{(k_1,n_1)} \). Suppose \( \omega^{(k_1,n_1)} \) is an arithmetic progression of the step length \( b^{(k_1,n_1)} = -i(n + \alpha_k) \frac{b_k}{|b_k|^2} \). This linear relation defines a hyperspace in the parameter space \( \mathbb{C}_0^N \). The countable union of such hyperplanes for all \( (k_1, n_1), (k_2, n_2) \) and \( (k_3, n_3) \) forms a set \( E_M \) of zero Lebesgue measure. Finally, we have \( E_M = E_M' \cup E_M'' \).

**Lemma 3.5.** Let \( s_j^{(k)} = 1/|\langle b, c^k \rangle| \). Then \( \Omega \) is uniformly separated if and only if \( s_l^{(k)} / s_m^{(k)} \in \mathbb{Q} \) for all \( k, l, m \in \{1, \ldots, N\} \).

**Proof.** A direct calculation shows that \( |x_{(k,n+1)}^{(k_1,n_1)} - x_{(k,n)}^{(k_1,n_1)}| = 1/(|b^k, c^k|) \), so each sequence \( X_{(k_1,n_1)} \) is an arithmetic progression of the step length \( s_k^{(k_1)} \). Therefore, the points of \( X_l^{(k_1,n_1)} \cup X_m^{(k_1,n_1)} \), \( l \neq m \), are uniformly separated if and only if the quotient of the steps \( s_l^{(k_1)} \) and \( s_m^{(k_1)} \) is rational. \( \square \)

4. Uniqueness

In this section we will show that if \( \Omega \) does not contain multiple points, then the only function \( f \in S_M^p \) satisfying \( f(\omega) = 0 \) for all \( \omega \in \Omega \) is \( f \equiv 0 \). This will be done by first showing that \( f \) vanishes on all the hyperplanes \( P^{(k,n)} \), and then showing that \( \Phi(z) = f(z)/S(z) \equiv 0 \).

First we want to find an entire function of one variable whose zero set is \( X^{(k_1,n_1)} \). Let \( U_{k_1} = \{1 \leq k \leq N : b_1^{k_1} b_2^{k_1} \neq b_1 b_2^{k_1}\} \) and \( \rho_k^{(k_1,n_1)} = -i(b^{(k_1,n_1)}, b^k) + \alpha_k \). Then the function
\[ L_k^{(k_1,n_1)}(\zeta) = \sin \left[ \pi \left( i(c^{(k_1),b^k}), \zeta - \rho_k^{(k_1,n_1)} \right) \right] \]
has \( X_k^{(k_1,n_1)} \) as zero set, and \( X^{(k_1,n_1)} \) is the zero set of the product
\[ L^{(k_1,n_1)}(\zeta) = \prod_{k \in U_{k_1}} L_k^{(k_1,n_1)}(\zeta), \quad \zeta \in \mathbb{C}. \tag{4.1} \]

**Lemma 4.1.** The function \( L^{(k_1,n_1)}(\zeta) \) satisfies
\[ |L^{(k_1,n_1)}(\zeta)| \geq e^{-\sum_{k \in U_{k_1}} |\text{Re}(c^{(k_1),b^k})|} = e^{H_{b_1^{k_1}}(\zeta)} \quad \text{for } \text{dist}(\zeta, X^{(k_1,n_1)}) > \delta > 0, \]
where $H_{D^{k_1}}(\zeta)$ is the support function of the convex set
\[ D^{k_1} = \left\{ \sum_{k \in U_{k_1}} t_k \langle c^{k_1}, b^k \rangle : |t_k| \leq \pi \right\}. \]

**Proof.** The proof is similar to the proof of Lemma 3.2. \(\square\)

**Lemma 4.2.** $X^{(k_1,n_1)}$ is a set of uniqueness in $L^p_{D^{k_1}}$.

**Proof.** We need to show $f \in L^p_{D^{k_1}}$ and $f|_{X^{(k_1,n_1)}} \equiv 0 \Rightarrow f \equiv 0$.

Define $\phi(\zeta) = f(\zeta)/L^{(k_1,n_1)}(\zeta)$. Then $\phi(\zeta)$ is an entire function in $\mathbb{C}$ and we have the estimate
\[ |\phi(\zeta)| \leq \text{Const} e^{-H_{D^{k_1}}(\zeta)} |f(\zeta)| \text{ for dist}(\zeta, X^{(k_1,n_1)}) > \delta > 0. \]

By the maximum principle, this estimate holds for every $\zeta \in \mathbb{C}$. [8, Lemma 2.8] gives us
\[ \lim_{|\zeta| \to \infty} \left| f(\zeta) \right| e^{-H_{D^{k_1}}(\zeta)} = 0, \]
and the Liouville Theorem yields $\phi \equiv 0$, and hence $f \equiv 0$. \(\square\)

**Lemma 4.3.** If $X^{(k_1,n_1)}$ is uniformly separated, then it is a complete interpolating sequence for $L^p_{D^{k_1}}$.

**Proof.** The sides of $D^{k_1}$ will be parallel to the vectors $B^k = \langle c^{k_1}, b^k \rangle$, $k \in U_{k_1}$. The points $x^{(k_1,n_1)}_{(k,n)}$ satisfy
\[ x^{(k_1,n_1)}_{(k,n)} \cdot B^k = -i(n + \alpha_k - i\langle b^{(k_1,n_1)}, b^k \rangle), \]
so they all lie on a line parallel to the line $\text{Im}(B^k)y = -\text{Re}(B^k)x$ and therefore parallel to the normal to the side $B^k$ of $D^{k_1}$.

The result now follows from Theorem 2.3. \(\square\)

Finally, we need a uniqueness result for the space $S^p_M$.

**Proposition 4.4.** Assume $\Omega$ does not contain multiple points, $f \in S^p_M$, and $f|_{\Omega} \equiv 0$, then $f \equiv 0$.

**Proof.** Assume $f \in S^p_M$ and $f|_{\Omega} \equiv 0$. First we will show that the zero set of $f$ contains the zero set of $S^{(k,n)}$. Fix $p^{(k,n)}$. We have
\[ \left| f(b^{k,n} + c^k \lambda) \right| e^{-H_{(b^{k,n}+c^k \lambda)}} \preceq \left| f(b^{k,n} + c^k \lambda) \right| e^{-H_{D^{k_1}}(\lambda)}. \]
(4.2)

As above, let $U_k = \{1 \leq j \leq N : b^j_1 b^j_2 \neq b^k_1 b^k_2 \}$. The vectors $i\langle c^k, b^j \rangle$, $j \in U_k$, are the normals to the sides of $D^k$.

Let $f_{k,n}(\lambda) = f(b^{k,n} + c^k \lambda)$ and $L^{(k,n)}(\lambda)$ be as in (4.1). Then $L^{(k,n)} \in \Sigma_{D^k}$ (see (2.5)). Due to the assumption $f|_{\Omega} \equiv 0$, we have $f_{k,n}|_{X^{(k,n)}} \equiv 0$, and $X^{(k,n)}$ is the zero set of $L^{(k,n)}$. From the proof of Lemma 4.3 we know that $X^{(k,n)} = \bigcup_{m \in \mathbb{Z}} \{x_{j,m} : m \in \mathbb{Z} \}$, where $\{x_{j,m} \}_{m \in \mathbb{Z}}$ lies on a line, say $L_j$, parallel to the normal $i\langle c^k, b^j \rangle$ of $D^k$.
Let
\[ w(r_j) = b^{(k,n)} + (r_j i (c^k, b^k) + \zeta_j) c^k, \quad \zeta_j \in \mathbb{C}. \]

Since
\[ \text{Re}\langle w(r_j), b^k \rangle = \text{Im}\alpha_k \quad \text{and} \quad \text{Re}\langle w(r_j), b^j \rangle = \text{Re}\left(\langle b^{(k,n)}, b^j \rangle + \zeta_j c^k, b^j \rangle \right), \]
w(r_j) lies in a shift of the plane \( \Pi_{k,j} \). Choose \( \zeta_j \) such that \( w(r_j) \) is uniformly separated from \( L_j \).

Then we know that \( |L^{(k,n)}(\lambda)| \asymp e^{H_{pk}(\lambda)} \) along \( w(r_j) \) as \( r_j \to \infty \). By (4.2) and the fact that \( |f(z)|^p e^{-pH_M(z)} \) is integrable along any shift of the plane \( \Pi_{k,j} \),
\[ f_{k,n}(\lambda) e^{-H_{pk}(\lambda)} \to 0 \quad \text{along } w(r_j) \text{ as } r_j \to \infty. \]

Since the zero set of \( f_{k,n} \) contains the zero set of \( L^{(k,n)} \), we have that \( \phi(\lambda) = f_{k,n}(\lambda)/L^{(k,n)}(\lambda) \) is an entire function. The above shows that \( |\phi(\lambda)| \) tends to zero along \( w(r_j) \) as \( r_j \to \infty, j \in U_k \).

Using the Pragmén–Lindelöf theorem, we can conclude that \( \Phi \) is bounded in the whole complex plane, hence it is a constant. According to the above limits, the constant has to be zero. Therefore, \( f_{k,n} \equiv 0 \). Since the multiplicity of zeros of \( S \) (which is defined on the set of regular points of \( Z(S) \), i.e. on \( Z(S) \setminus \Omega \)) equals 1, \( S \) divides \( f \) (see e.g. [16]), so
\[ \Phi(z) = \frac{f(z)}{S(z)}, \quad z \in \mathbb{C}^2, \]
is an entire function. Since
\[ |S(z)| \asymp e^{H_M(z)} \quad \text{for } \text{dist}(i\langle z, b^k \rangle - \alpha_k, \mathbb{Z}) > \delta, \ 1 \leq k \leq N, \]
\[ |\Phi(z)| \leq \text{Const} \text{ on the set } \text{dist}(i\langle z, b^k \rangle - \alpha_k, \mathbb{Z}) > \delta, \ 1 \leq k \leq N. \]
Plurisubharmonic arguments for \( \ln|\Phi(z)| \) extends this estimate to the whole \( \mathbb{C}^2 \), and from Liouville’s theorem it follows that \( \Phi \) is constant.

Fix \( \gamma \in \mathbb{C}^2 \) and pick one of planes \( \Pi_{r,s}(\gamma) \). Since it is non-analytic, it does not coincide with any of the analytic planes \( P^{(k,n)} \). It is possible to pick a sequence of points \( \{z_j\} \) going to infinity along \( \Pi_{r,s}(\gamma) \) such that \( |S(z_j)| \asymp e^{H_M(z_j)}. \) Since \( |f(z)|^p e^{-pH_M(z)} \) is integrable along \( \Pi_{r,s} \), we can conclude that
\[ \lim_{j \to \infty} |f(z_j)| e^{-H_M(z_j)} = 0. \]

This establishes \( \Phi \equiv 0 \) and hence \( f \equiv 0. \quad \square \)

5. Solution to the interpolation problem

Now we are able to solve the interpolation problem (1.2) under the assumption that \( \Omega \) is uniformly separated and does not contain multiple points.

For each \( \omega = \omega^{(k_1,n_1)}(k_2,n_2) \in \Omega \), define
\[ \phi_{\omega}(z) = \prod_{k \neq k_1,k_2} \sin\pi\left(i\langle z, b^k \rangle - \alpha_k\right) r_{k_1,n_1}(z) r_{k_2,n_2}(z), \quad (5.1) \]
where
\[ r_{k,n}(z) = \frac{\sin[\pi(i\langle z, b^k \rangle - \alpha_k)]}{i\langle z, b^k \rangle - n - \alpha_k}. \]
The function $\phi_\omega$ is zero on $\Omega \setminus \{\omega\}$. Let $\omega = \omega^{(k_1,n_1)(k_2,n_2)}$. Since $r_{k_j,n_j}(\omega) = \pi$, and $\text{Re}\langle\omega, b^{k_j}\rangle = \text{Im}\alpha_{k_j}$, $j = 1, 2$, it follows from the assumption that $\Omega$ is uniformly separated and the estimates in Lemma 3.2 that
\[ |\phi_\omega(\omega)| \asymp e^{H_M(\omega)}. \] (5.2)

**Theorem 5.1.** Assume that $\Omega$ is uniformly separated and does not contain multiple points. Given a sequence $a = \{a_\omega e^{-H_M(\omega)}\} \in l^p(\Omega)$, the solution $f = f_a \in S_M^p$ to the interpolation problem
\[ f(\omega) = a_\omega, \quad \omega \in \Omega, \]
exists and has the form
\[ f_a(z) = \sum_{\omega \in \Omega} \frac{a_\omega}{c_\omega} \phi_\omega(z), \quad z \in \mathbb{C}^2, \] (5.3)
where $c_\omega = \phi_\omega(\omega)$. The series converges in the norm $\|\cdot\|_{S_M^p}$ and also uniformly on compact sets in $\mathbb{C}^2$.

**Proof.** It is enough to assume $\alpha_\omega \neq 0$ for a finite number of $\omega$’s only. Then the series $f_a$ in (5.3) converges and obviously solves the interpolation problem. According to Proposition 4.4, $f_a$ is the unique solution to the interpolation problem. The general case will then follow as a limit case once the inequality
\[ \|f\|_{S_M^p} \leq \text{Const} \left\|\left\{a_\omega e^{-H_M(\omega)}\right\}\right\|_{l^p(\Omega)} \] (5.4)
has been established.

Further we may fix some $k_1, k_2 \in \{1, \ldots, N\}$, $k_1 \neq k_2$, and assume that all nonzero $a_\omega$ corresponds to points $\omega$ of the form $\omega = \omega^{(k_1,n_1)(k_2,n_2)}$, $n_1, n_2 \in \mathbb{Z}$. This can be done because each $a$ can be represented as a union of at most $\frac{1}{2} N(N + 1)$ such sequences with pairwise disjoint supports.

Fix $k_1$ and $k_2$, and set
\[ \omega_{n_1,n_2} = \omega^{(k_1,n_1)(k_2,n_2)}, \quad a_{n_1,n_2} = a_{\omega_{n_1,n_2}}, \quad c_{n_1,n_2} = c_{\omega_{n_1,n_2}}. \]

Using (5.1), we get
\[ f_a(z) = \left\{ \prod_{k \neq k_1,k_2} \sin[\pi(i\langle z,b^k\rangle - \alpha_k)] \right\} g_a(z), \]
where
\[ g_a(z) = \sum_{n_1} r_{k_1,n_1}(z) \left\{ \sum_{n_2} \frac{a_{n_1,n_2}}{c_{n_1,n_2}} r_{k_2,n_2}(z) \right\} = \sum_{n_1} V_{n_1}(\langle z,b^{k_2}\rangle) r_{k_1,n_1}(z) \]
and
\[ V_{n_1}(\xi) = \sum_{n_2} \frac{a_{n_1,n_2}}{c_{n_1,n_2}} \frac{\sin[\pi(i\xi - \alpha_k)]}{i\xi - n_2 - \alpha_k}. \]

Set $d_{(n_1)} = \{a_{n_1,n_2}/c_{n_1,n_2}\}_{n_2 \in \mathbb{Z}}$. Due to (5.2), we have $d_{(n_1)} \in l^p$, and $\sum_{n_1} \|d_{(n_1)}\|_p^p \asymp \|\{a_\omega e^{-H_M(\omega)}\}\|_{l^p(\Omega)}^p$. Now $V_{n_1} \in L^p_{i\pi}$, where $L^p_{i\pi}$ is the space $L^p_D$ when $D = [-\pi, \pi]$, so by Theorem 2.2,
\[ \|V_{n_1}\|_{L^p_{i\pi}} \leq \text{Const} \|d_{(n_1)}\|_{l^p(\Omega)}. \]
In addition,
\[
\sum_{n_1} \| V_{n_1} \|_{L^p_{\nu_n}}^p \leq \text{Const} \left\| \left\{ a_\omega e^{-H_M(\omega)} \right\} \right\|_{L^p(\Omega)}^p.
\]

Recall \( \Pi_{r,s}(\gamma) = \{ z \in \mathbb{C}^2 : \text{Re}(z + \gamma, b^r) = \text{Re}(z + \gamma, b^s) = 0 \} \). Establishing (5.4) amounts to showing
\[
\int_{\Pi_{r,s}(\gamma)} \left| f_\alpha(z) \right|^p e^{-pH_M(\cdot)} d\mu_{r,s}(\gamma) \leq \text{Const} \left\| \left\{ a_\omega e^{-H_M(\omega)} \right\} \right\|_{L^p(\Omega)}^p
\]
for all \( r, s \) with \( b_1^r b_2^s \neq b_2^r b_1^s \), and all \( \gamma \in \mathbb{C}^2 \). Here \( d\mu_{r,s}(\gamma) \) is the (area) Lebesgue measure on \( \Pi_{r,s}(\gamma) \). We will use Theorem 2.6 to prove this.

Fix \( r \) and \( s \) with \( b_1^r b_2^s \neq b_2^r b_1^s \), and for each \( \gamma \in \mathbb{C}^2 \) define the measure \( \mu_\gamma \) on \( \mathbb{C}_+ \times \mathbb{C}_+ \) as
\[
\mu_\gamma(B) = \text{mes}\{ \Pi \cap \text{cl} B \}
\]
for any set \( B \subseteq \mathbb{C}_+ \times \mathbb{C}_+ \). Here \( \text{mes} \) denotes Lebesgue area measure and \( \text{cl} B \) is the closure of \( B \) in \( \mathbb{C}^2 \).

**Lemma 5.2.** \( \mu_\gamma \) is a Carleson measure for the bi-upper half-plane. Moreover, the constant in (2.10) can be taken independent of \( \gamma \in \mathbb{C}^2 \).

For the moment, let us assume this lemma holds. We have
\[
\left| f_\alpha(z) \right| e^{-H_M(\cdot)} \leq \text{Const} \sum_{n_1, n_2} d_{n_1, n_2} \frac{\sin[\pi(i(z, b_1^k) - \alpha_k)] \sin[\pi(i(z, b_2^k) - \alpha_k)]}{i(z, b_1^k) - n_1 - \alpha_k} \frac{i(z, b_2^k) - n_2 - \alpha_k}{e^{\pi i(z, b_2^k)}}.
\]

Find complex constants \( r_1, r_2, s_1 \) and \( s_2 \) such that \( b^r = r_1 b_{k_1} + r_2 b_{k_2} \) and \( b^s = s_1 b_{k_1} + s_2 b_{k_2} \).

Let \( c_1 = -r_1 \text{Re}(\gamma, b_{k_1}) - r_2 \text{Re}(\gamma, b_{k_2}) \) and \( c_2 = -s_1 \text{Re}(\gamma, b_{k_1}) - s_2 \text{Re}(\gamma, b_{k_2}) \). Then, with \( \lambda_j = i(z, b_j^k) \), \( j = 1, 2, \)
\[
\Pi_{r,s}(\gamma) = \{ z \in \mathbb{C}^2 : \text{Re}(z + \gamma, b^r) = \text{Re}(z + \gamma, b^s) = 0 \}
= \{ \lambda : \text{Im}(\lambda_1 r_1 + \lambda_2 r_2) = c_1, \text{Im}(\lambda_1 s_1 + \lambda_2 s_2) = c_2 \}.
\]

Let
\[
\Pi_{++}(\gamma) = \Pi_{r,s}(\gamma) \cap \{ z : \text{Re}(z, b_{k_1}) \geq 0, \text{Re}(z, b_{k_2}) \geq 0 \},
\Pi_{+-}(\gamma) = \Pi_{r,s}(\gamma) \cap \{ z : \text{Re}(z, b_{k_1}) \geq 0, \text{Re}(z, b_{k_2}) < 0 \},
\Pi_{-+}(\gamma) = \Pi_{r,s}(\gamma) \cap \{ z : \text{Re}(z, b_{k_1}) < 0, \text{Re}(z, b_{k_2}) \geq 0 \},
\Pi_{--}(\gamma) = \Pi_{r,s}(\gamma) \cap \{ z : \text{Re}(z, b_{k_1}) < 0, \text{Re}(z, b_{k_2}) < 0 \}.
\]

Then
\[
\Pi_{r,s}(\gamma) = \Pi_{++}(\gamma) \cup \Pi_{+-}(\gamma) \cup \Pi_{-+}(\gamma) \cup \Pi_{--}(\gamma),
\]
and we will first consider \( \Pi_{++}(\gamma) \).

Define
\[
h(\lambda_1, \lambda_2) = \sum_{n_1, n_2} d_{n_1, n_2} \frac{\sin[\pi(\lambda_1 - \alpha_k) \sin[\pi(\lambda_2 - \alpha_k)]]}{\lambda_1 - n_1 - \alpha_k} \frac{\lambda_2 - n_2 - \alpha_k}{e^{\pi(\lambda_1 + \lambda_2)}}.
\]
Then \( h \in H^p(\mathbb{C}_+ \times \mathbb{C}_+) \), and Theorem 2.6 with \( \mu_\gamma \) as in Lemma 5.2, gives us

\[
\int_{\Pi^{++}(\gamma)} |f_a(z)|^p e^{-pH_M(z)} \, dm \leq \text{Const} \int_{\Pi_{r,s}(\gamma) \cap \mathbb{C}_+ \times \mathbb{C}_+} |h(\lambda_1, \lambda_2)|^p \, dm
\]

\[
= \text{Const} \int_{\mathbb{C}_+ \times \mathbb{C}_+} |h(\lambda_1, \lambda_2)|^p \, d\mu_\gamma
\]

\[
\leq \text{Const} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h(x_1, x_2)|^p \, dx_1 \, dx_2 = I,
\]

where \( \text{Re} \lambda_1 = x_1 \) and \( \text{Re} \lambda_2 = x_2 \), and, according to Lemma 5.2, the last constant is independent of \( \gamma \).

Define

\[
\phi(\zeta) = \sum_{n_1} V_{n_1}(-ix_2) \sin[\pi(\zeta - \alpha_{k_1})]/(\zeta - n_1 - \alpha_{k_1}) \in L^p_\pi.
\]

Since \( \phi(n_1 + \alpha_{k_1}) = \pi V_{n_1}(-ix_2) \), Theorem 2.2 can be applied to the function \( \phi(\zeta) \) and the sequence \( \{n_1 + \alpha_{k_1}\}_{n_1} \) to give

\[
I = \text{Const} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \sum_{n_1} V_{n_1}(-ix_2) \frac{\sin[\pi(x_1 - \alpha_{k_1})]}{x_1 - n_1 - \alpha_{k_1}} \right|^p \, dx_1 \, dx_2
\]

\[
\leq \text{Const} \int_{-\infty}^{\infty} \|V_{n_1}(-ix_2)\|^p_{L^p} \, dx_2
\]

\[
\leq \text{Const} \sum_{n_1} \|V_{n_1}\|^p_{L^p} \leq \text{Const} \|\{a_\omega e^{-H_M(\omega)}\}\|_{L^p(\Omega)}.
\]

Theorem 2.6 remains valid for the bi half-spaces \( \mathbb{C}_+ \times \mathbb{C}_-, \mathbb{C}_- \times \mathbb{C}_+ \) and \( \mathbb{C}_- \times \mathbb{C}_- \) as well, so the cases \( \Pi^{+-}(\gamma), \Pi^{-+}(\gamma) \) and \( \Pi^{--}(\gamma) \) can be dealt with similarly as the case \( \Pi^{++}(\gamma) \). All in all, this yields

\[
\int_{\Pi_{r,s}(\gamma)} |f_a(z)|^p e^{-pH_M(z)} \, dm_{r,s}(\gamma) \leq \text{Const} \|\{a_\omega e^{-H_M(\omega)}\}\|_{L^p(\Omega)}
\]

with constant independent of \( \gamma \).

Finally, that \( f_a \) converges uniformly on compact sets in \( \mathbb{C}^2 \) follows from an application of Hölder’s inequality, and the proof is done. \( \square \)

**Proof of Lemma 5.2.** Fix \( r \) and \( s \) with \( b_1^r b_2^s \neq b_2^r b_1^s \), and consider \( \Pi_{r,s}(\gamma), \gamma \in \mathbb{C}^2 \). Then (see Theorem 2.6 for notation)

\[
\Pi_{r,s}(\gamma) \cap \text{cl} A(U) \subseteq \{(z_1, z_2) \in \Pi_{r,s}(\gamma), (x_1, x_2) \in \text{cl} U\} := D_\gamma.
\]

Assume first that \( \Pi_{r,s}(\gamma) \) can be parameterized with \( x_1 \) and \( x_2 \) as parameters, i.e.

\[
\Pi_{r,s}(\gamma) = \{(x_1 + iy_1, x_2 + iy_2): y_1 = a_1 x_1 + a_2 x_2 + c_1(\gamma),
\]

\[
y_2 = b_1 x_1 + b_2 x_2 + c_2(\gamma), \ x_1, x_2, a_j, b_j, c_1(\gamma), c_2(\gamma) \in \mathbb{R}\}.
\]
Define the linear transformations $T : D_\gamma \to \text{cl} U$ as $T(z_1, z_2) = (x_1, x_2)$. It is easily checked that its inverse

$$T^{-1}(x_1, x_2) = \left( x_1 + i (a_1 x_1 + a_2 x_2 + c_1(\gamma)), x_2 + i (b_1 x_1 + b_2 x_2 + c_2(\gamma)) \right)$$

is well defined if and only if $\Pi_{r,s}(\gamma)$ is not an analytic plane. Now we have

$$\mu_\gamma(A(U)) \leq \int_{D_\gamma} |dz_1 dz_2| = \left| \det \begin{pmatrix} 1 + ia_1 & ia_2 \\ ib_1 & 1 + ib_2 \end{pmatrix} \right| \int_{\text{cl} U} dx_1 dx_2$$

which is what we are looking for.

The cases when $\Pi_{r,s}(\gamma)$ cannot be parameterized as above can be checked to satisfy $\mu_\gamma(A(U)) \leq |U|$, and the lemma is proved. \qed

6. Product domains

Let $b^{1,j} \in \mathbb{C} \times \{0\} \subset \mathbb{C}^2$ and $b^{2,k} \in \{0\} \times \mathbb{C} \subset \mathbb{C}^2$, and define the product domain

$$M = M_1 \times M_2,$$

where

$$M_1 = \left\{ \sum_{j=1}^{N_1} t_j b^{1,j} : |t_j| \leq \pi, \ 1 \leq j \leq N_1 \right\}$$

and

$$M_2 = \left\{ \sum_{k=1}^{N_2} t_k b^{2,k} : |t_k| \leq \pi, \ 1 \leq k \leq N_2 \right\}.$$

In this setting, $\Omega$ has the following representation:

$$\Omega = \bigcup_{\substack{j=1 \ k=1 \ j \leq N_1 \ k \leq N_2}} \left\{ -i(n_{1,j} + \alpha_{1,j})b^{1,j}_1, -i(m_{2,k} + \alpha_{2,k})b^{2,k}_2 : \frac{|n_{1,j}|^2}{|b^{1,j}_1|^2}, \frac{|m_{2,k}|^2}{|b^{2,k}_2|^2} \right\}_{n_{1,j}, m_{2,k} \in \mathbb{Z}}.$$

We have the following estimates when $M$ is a product domain.

**Proposition 6.1.** Let $M = M_1 \times M_2$ be a product domain, $\gamma \in \mathbb{C}^2$ and

$$\Pi_{r,s}(\gamma) = \left\{ z \in \mathbb{C}^2 : \text{Re}[z + \gamma, b^{1,r}] = \text{Re}[z + \gamma, b^{2,s}] = 0 \right\}.$$

For $f \in S^p_M$, $1 < p < \infty$, and every $1 \leq r \leq N_1$, $1 \leq s \leq N_2$,

$$\int_{\Pi_{r,s}(\gamma)} |f(z)|^p e^{-pH_M(z)} dm_{r,s} \leq \text{Const} \int_{\Pi_{r,s}(0)} |f(z)|^p e^{-pH_M(z)} dm_{r,s},$$

where the constant only depends on $p$. 
Proof. Let $D_\beta$ be the cone in $\mathbb{C}^2$ where
\[
e^{-HM(z)} = \left| e^{-\pi \sum_{j=1}^{N_1} \beta(1,j)(z,b^{1,j}) + \sum_{k=1}^{N_2} \beta(2,k)(z,b^{2,k})} \right|.
\]
Here $\beta(1,j) = \pm 1$ and $\beta(2,k) = \pm 1$ are chosen such that $\beta(1,j) \text{Re}(z,b^{1,j}) \geq 0$ and $\beta(2,k)(z,b^{2,k}) \geq 0$. $\mathbb{C}^2$ is divided into finitely many such cones.

Fix one of the cones $D_\beta$, and let us for definiteness assume $\beta(1,j) = 1$ and $\beta(2,k) = 1$ for every $j$ and $k$. Also fix $1 \leq s \leq N_2$. For $r_2$ fix, define
\[
\phi_{r_2}(z_1) = f(z_1, ir_2 b^{2,s}) e^{-\pi \sum_{j=1}^{N_1} z_1 b^{1,j} + \pi \sum_{k=1}^{N_2} b^{2,s}_k b^{2,k}_k}.
\]
Fix $R > 0$. Then $\phi_{r_2}(z_1)$ is an analytic function (with respect to $z_1$) with values in $L^p(0, R)$ (with respect to $r_2$), and
\[
\|\phi_{r_2}(z_1)\|_{L^p(0,R)} \leq C_R \quad \text{for all } z_1 \in D_\beta \cap \mathbb{C} \times \{0\}.
\]

Assume the vectors $\{b_1^{1,j}\}$ are enumerated with increasing angles formed with the positive real axis in the $z_1$-plane.

Let $\Gamma \times \{0\} = D_\beta \cap \mathbb{C} \times \{0\}$. Then $\Gamma$ is the smallest angle between the rays $\{\rho i b_1^{1,r}: \rho \geq 0\}$ and $\{\rho i b_1^{1,r+1}: \rho \geq 0\}$ for a suitable $1 \leq r \leq N_1$. We let $b_1^{1,r+1} = b_1^{1,1}$. Fix this $r$, and put $\Gamma_r = \Gamma$.

Note that $\text{Re} ir_1 b_1^{1,r} b_1^{1,j} \geq 0$ and $\text{Re} ir_1 b_1^{1,r+1} b_1^{1,j} \geq 0$ for $r_1 > 0$ and $1 \leq j \leq N_1$.

Integrability of $|f(z)|^p e^{-pHM(z)}$ along $\Pi_{r,s}$ yields
\[
\int_0^R \int_0^R |\phi_{r_2}(ir_1 b_1^{1,r})|^p dr_2 dr_1 < \text{Const},
\]
where the constant is independent of $R$. Similarly, we have that integrability of $|f(z)|^p e^{-pHM(z)}$ along $\Pi_{r+1,s}$ yields
\[
\int_0^R \int_0^R |\phi_{r_2}(ir_1 b_1^{1,r+1})|^p dr_2 dr_1 < \text{Const},
\]
where again the constant is independent of $R$. An application of Pragmén–Lindelöf, much as in [8, Lemma 2.2], we get that $\phi_{r_2} \in H^p(\Gamma_r, L^p(0, R))$. Here $H^p(\Gamma_r, L^p(0, R))$ denotes the space of analytic functions $g_1(\xi)$ in $\Gamma_r$ with values in $L^p(0, R)$ such that
\[
\|g_1(\xi)\|_{L^p(0,R)(t)}^p
\]
is integrable (with respect to $\xi$) on every ray from the origin and contained in $\Gamma_r$ (see e.g. [8] for more on these spaces).

As $R \to \infty$, we get $\phi_{r_2} \in H^p(\Gamma_r, L^p(0, \infty))$. Letting $\theta_r = \text{arg} i b_1^{1,r}$ and $\theta_{r+1} = \text{arg} i b_1^{1,r+1}$, an application of Cauchy’s formula (see e.g. [8, Lemma 2.3] for details) yields that $\phi_{r_2}$ admits the representation
\[
\phi_{r_2}(z_1) = \frac{e^{i\theta_r}}{2\pi i} \int_0^\infty \phi_{r_2}(\rho e^{i\theta_r}) e^{-\rho z_1} d\rho - \frac{e^{i\theta_{r+1}}}{2\pi i} \int_0^\infty \phi_{r_2}(\rho e^{i\theta_{r+1}}) e^{-\rho z_1} d\rho = \phi_{r_2}^r(z_1) + \phi_{r_2}^{r+1}(z_1),
\]
where $\Phi^r_{r_2}(\xi) = \phi_{r_2}^r(\xi e^{i\theta_r}) \in H^p(\mathbb{C}_+)$ and $\Phi^{r+1}_{r_2}(\xi) = \phi_{r_2}^{r+1}(\xi e^{i\theta_{r+1}}) \in H^p(\mathbb{C}_-)$. 


Fix $\gamma > 0$, and let $L'_\gamma = \{te^{i\theta} + iy e^{i\theta} : t \geq 0\}$. Define the measure $\mu_+$ on $\mathbb{C}_+$ as

$$
\mu(A) = \text{length}\left((e^{-i\theta} L'_\gamma \cap \Gamma_r) \cap A\right), \quad A \subset \mathbb{C}_+.
$$

Similarly, we define the measure $\mu_-$ on $\mathbb{C}_-$ as

$$
\mu(B) = \text{length}\left((e^{-i\theta+1} L'_\gamma \cap \Gamma_r) \cap B\right), \quad B \subset \mathbb{C}_-.
$$

It is easy to see that the measures $\mu_+$ and $\mu_-$ are Carleson measures in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively, and that the constant in (2.8) is 1. Applying Theorem 2.5, we get

$$
\int_0^\infty \int_0^\infty \left| \phi_{r_1}(r_1 e^{i\theta} + iy e^{i\theta}) + \gamma e^{i\theta} \right|^p dr_2 dr_1 < \text{Const} \int_{\Pi_{r,s}(0)} |f(z)|^p e^{-pH_M(z)} dm_{r,s}.
$$

Note that the constant is independent of $\gamma$. Similar arguments for the angle $-\Gamma_r$ show that $|f(z)|^p e^{-pH_M(z)}$ is integrable along the plane $\{z \in \mathbb{C}^2 : \Re\{z, b_{1,r}\} = -\gamma |b^{1,r}|, \Re\{z, b_{2,s}\} = 0\}$ for any $\gamma > 0$. Similar arguments for the angles $\Gamma_r-1$ and $-\Gamma_r-1$ gives us that $|f(z)|^p e^{-pH_M(z)}$ is integrable along the plane $\{z \in \mathbb{C}^2 : \Re\{z, b_{1,r}\} = c, \Re\{z, b_{2,s}\} = 0\}$ for any $c \in \mathbb{R}$. Reversing the roles of the variables $z_1$ and $z_2$ and applying the same procedure as above, we obtain that $|f(z)|^p e^{-pH_M(z)}$ is integrable along the plane $\{z \in \mathbb{C}^2 : \Re\{z, b_{1,r}\} = c_1, \Re\{z, b_{2,s}\} = c_2\}$ for any $c_1, c_2 \in \mathbb{R}$, and

$$
\int_{\Pi_{r,s}(\gamma)} |f(z)|^p e^{-pH_M(z)} dm_{r,s} \leq \text{Const} \int_{\Pi_{r,s}(0)} |f(z)|^p e^{-pH_M(z)} dm_{r,s}.
$$

\[\square\]

7. About $S^2_M$

In this section we want to describe a class of functions which belongs to $S^2_M$. If $M$ is a product domain, we show that this class constitute the whole space $S^2_M$.

Define the Paley–Wiener space

$$
P W_M = \left\{ f : f(z) = \int_{\partial_0 M^*} \phi(s) ds, \phi \in L^2(\partial_0 M^*) \right\},
$$

where $\partial_0 M^*$ is the union of parallelograms of the form

$$
P_{r,s} = \left\{ t_1 b^r + t_2 b^s + \pi \sum_{j=1}^N \beta(j)b^j : |t_1|, |t_2| \leq \pi, \beta(j) = \pm 1, b^r \neq \alpha b^s, \alpha \in \mathbb{C} \right\}.
$$

The signs $\beta(j)$ have to be chosen such that $P_{r,s} \subset \partial M$, and the inequality $b^r \neq \alpha b^s$ means that $P_{r,s}$ is not contained in any analytic plane (see Lemma 3.1). If $M$ is a product domain, $\partial_0 M^* = \partial_0 M = \partial M_1 \times \partial M_2$. $\partial_0 M$ is called the distinguished (or essential) boundary of $M$, and it is strictly smaller than the topological boundary $\partial M$. In many situations the distinguished boundary of a product domain plays the role of the boundary of a domain in one complex variables (e.g. in the Cauchy Integral Formula on polydiscs [15, Chapter 1.3]). In the general case, $\partial_0 M^*$ is strictly smaller than $\partial M$ and it can be considered as a generalization of the distinguished boundary $\partial_0 M$ of a product domain $M = M_1 \times M_2$. 
Endow $PW_M$ with the norm
\[ \|f\|_{PW_M}^2 = \int_{C^2} |f(z)|^2 e^{-2HM(z)} (dd^c H_M(z))^2. \]

**Proposition 7.1.** $PW_M \subset S^2_M$.

**Proof.** Let $f \in PW_M$. The inequality $|f(z)| \leq \text{Const} e^{H_M(z)}$ is straightforward.

Let $D_\beta$ be the cone in $C^2$ where $e^{-H_M(z)} = |e^{-\pi \sum_{j=1}^N \beta(j) \langle z, b^j \rangle}|$. Here $\beta(j) = \pm 1$ is chosen such that $\beta(j) \text{Re} \langle z, b^j \rangle > 0$. $C^2$ is divided into finitely many such cones. Let us for definiteness assume $\beta(j) = 1$ for every $j$.

Let $f \in PW_M$. Then we have the representation (see (7.2))
\[ f(z) \sum \int_{P_{r,s}} e^{\langle z, t \rangle} \phi(s) dm_{P_{r,s}} = \sum e^{\pi \sum_{j=1}^N \alpha(j) \langle z, b^j \rangle} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{\langle z, t_1 b^r + t_2 b^s \rangle} \phi(t_1, t_2) dt_1 dt_2, \]
where the sums are taken over $r$ and $s$ such that $P_{r,s} \subset \partial M$. Then, for $z \in D_\beta$,
\[ |e^{-H_M(z)} f(z)| \leq \sum \left| \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{\langle z, t_1 b^r + t_2 b^s \rangle} \phi(t_1, t_2) dt_1 dt_2 \right| \]
\[ = \sum \left| \int_{-2\pi}^{0} \int_{-2\pi}^{0} e^{\langle z, \tau_1 b^r + \tau_2 b^s \rangle} \phi(\tau_1, \tau_2) d\tau_1 d\tau_2 \right| = \sum |f_{r,s}(z)|. \]

Introducing the change of variables $\langle z, b^r \rangle = i\lambda_1^r$ and $\langle z, b^s \rangle = i\lambda_2^s$, we see that (see (2.9)) $f_{r,s}(\lambda_1^r, \lambda_2^s) \in H^2(\mathbb{C}_+ \times \mathbb{C}_+)$. The result now follows after repeating the arguments in the proof of Theorem 5.1 used for showing that the solution to interpolation problem lies in $S^p_M$. \[ \square \]

If $M = M_1 \times M_2$ is a product domain, we have the opposite inclusion as well.

**Theorem 7.2.** Let $M$ be a product domain. Then $f \in PW_M$ if and only if $f \in S^2_M$.

**Proof.** According to Proposition 7.1, we only need to show $S^2_M \subset PW_M$ when $M$ is a product domain. This amounts to showing that $f \in S^2_M$ admits the representation (7.1) with $\partial_0 M^* = \partial_0 M$.

Let $\Phi_{z_1}(z_2) = f(z_1, z_2)$. It follows from the proof of Proposition 6.1 that for $z_1 \in \mathbb{C}$ fix, $|\Phi_{z_1}(z_2)|^2 e^{-2H_M(z_1, z_2)}$ is integrable along the lines $\{\rho ib_2^s : \rho \in \mathbb{R}\}$ for $1 \leq s \leq N_2$. Hence $\Phi_{z_1}(z_2) \in L^2_M$, and according to Theorem 2.4, $\Phi_{z_1}(z_2)$ has the following representation
\[ \Phi_{z_1}(z_2) = \int_{\partial M_2} e^{\xi z_1} \psi_{z_1}(\xi) d\xi, \quad (7.3) \]
where $\psi_{z_1}(\xi) \in G^2(\overline{M}_2)$ and
\[
\|\Phi_{z_1}\|_{L^2_{M_2}} = \|\psi_{z_1}\|_{G^2(\overline{M}_2)} = \|\psi_{z_1}\|_{L^2(\partial M_2)}.
\]

Recall that $\overline{M}$ denotes the complex conjugate of $M$.

Let $\theta_s = \arg ib_{2,s}^2$. In the half-plane $\{\xi : \Re \xi > H_{M_2}(\rho e^{i\theta_s})\}$, $\psi_{z_1}(\xi)$ has the representation
\[
\psi_{z_1}(\xi) = \int_0^\infty e^{-\rho e^{i\theta_s} \xi} f_{z_1}(\rho e^{i\theta_s}) d\rho.
\]

From this formula it follows that $\psi_{z_1}(\xi)$ is analytic with respect to $z_1$ and of exponential type with indicator not exceeding $H_{M_1}(z_1)$ for $\xi$ fixed. Let $\theta_r = \arg ib_{1,r}^1$. Due to (7.4),
\[
\int_{-\infty}^\infty \|\psi_{e^{i\theta_r}}\|_{L^2(\partial M_2)} e^{-2H_{M_1}(\rho e^{i\theta_r})} d\rho < \sum_{k=1}^{N_2} \int_{H_{r,k}} |f(z)|^2 e^{-2H_{M}(z)} dm_{r,s} < \infty.
\]

Hence, $\psi_{z_1}(\xi) \in L^2_{M_1(z_1)}(G^2(\overline{M}_2(\xi)))$. Again, according to Theorem 2.4, there exists a function $\Psi(\xi)(w) \in G^2(\overline{M}_1(w), G^2(\overline{M}_2(\xi)))$ such that
\[
\psi_{z_1}(\xi) = \int_{\partial \overline{M}_1} e^{z_1 w} \Psi(\xi)(w) dw.
\]

Combining (7.3) and (7.5) we arrive at the representation for $\Phi_{z_1}(z_2) = f(z_1, z_2)$,
\[
f(z) = \int_{\partial \overline{M}_2} e^{z_2 \zeta} \int_{\partial \overline{M}_3} e^{z_1 \xi} \Psi(\xi)(w) dw d\xi = \int_{\partial_0 M} e^{(z, \lambda)} \Psi(\lambda) d\lambda_1 d\lambda_2,
\]
where $\lambda_1 = \bar{\zeta}$, $\lambda_2 = \bar{w}$, and $f \in PW_M$. \hfill $\square$

**Remark.** It is believed that $S^2_M = PW_M$ for the general case as well.

**References**