A two-species competition model on $\mathbb{Z}^d$

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Abstract

We consider a two-type stochastic competition model on the integer lattice $\mathbb{Z}^d$. The model describes the space evolution of two "species" competing for territory along their boundaries. Each site of the space may contain only one representative (also referred to as a particle) of either type. The spread mechanism for both species is the same: each particle produces offspring independently of other particles and can place them only at the neighboring sites that are either unoccupied, or occupied by particles of the opposite type. In the second case, the old particle is killed by the newborn. The rate of birth for each particle is equal to the number of neighboring sites available for expansion. The main problem we address concerns the possibility of the long-term coexistence of the two species. We have shown that if we start the process with finitely many representatives of each type, then, under the assumption that the limit set in the corresponding first passage percolation model is uniformly curved, there is positive probability of coexistence.
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1. Introduction

The *voter model* is an interacting particle system in which individuals (particles) of two species, *Red* and *Blue*, compete for "territory" on a (locally finite) graph.

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At each time $t \geqslant 0$, every vertex (site) of the graph is occupied by a single particle, either Red or Blue. At any time, a particle of color i at a vertex x may spontaneously die, at rate equal to the degree of x, and be replaced by a clone of a randomly chosen neighbor. Thus, a vertex of color i spontaneously flips to the opposite color j at rate equal to the number of neighboring vertices of color j. See [7] for a formal construction of this process and an exposition of its basic properties.

The *Richardson model* [8] was introduced as a model for the spatial spread of a population in a favorable environment. The environment is once again a locally finite graph. At any time a vertex may be occupied by *at most* one particle (some vertices may be unoccupied); all particles are of the same species. Once occupied, a vertex remains occupied forever. Each unoccupied vertex is spontaneously occupied at instantaneous rate equal to the number of occupied neighbors.

In this paper, we study a hybrid of the voter and Richardson models on the integer lattice $\mathbb{Z}^d$, which we dub the *two-species competition model*, or simply the *competition model*. The dynamics are as in the voter model, but unlike the voter model, vertices may be unoccupied. An unoccupied vertex is colonized at rate equal to the number of occupied neighbors, as in the Richardson model; at the instant of first colonization, the vertex flips to the color of a randomly chosen occupied neighbor. Once occupied, a vertex remains occupied forever, but its color may flip, as in the voter model: the flip rate is equal to the number of neighbors occupied by particles of the opposite color. The state of the system at any time t is given by the pair $(R(t), B(t))$, where $R(t)$ and $B(t)$ denote the set of sites occupied by Red and Blue particles, respectively. Note that the set $R(t) \cup B(t)$ of occupied sites evolves precisely as in the Richardson model, and so the growth of this set is governed by the same *Shape Theorem* (see Section 3.3 below) as is the Richardson model.

Our primary interest is in the possibility of long-term coexistence of the two species, given initial conditions in which only finitely many vertices are occupied (with at least one vertex of each color). It is clear that at least one of the two species must survive, and that for any nondegenerate finite initial configuration of colors there is positive probability that Red survives and positive probability that Blue survives. However, it is not at all obvious (except perhaps in the case where the ambient graph on which the competition takes place is the integer lattice $\mathbb{Z}$—see Section 2 below) that the event of mutual survival has positive probability. Our main result concerns the competition model on the graph $\mathbb{Z}^d$. Say that a compact, convex set $\mathscr{S}$ with boundary $\partial\mathscr{S}$ is *uniformly curved* if there exists $\varrho < \infty$ such that for every point $z \in \partial\mathscr{S}$ there is a ball of radius ϱ with z on its surface that contains $\mathscr{S}$.

Theorem 1. *If the limit shape $\mathscr{S}$ for the Richardson model is uniformly curved, then for any nondegenerate initial finite configuration the event of mutual survival of the two species has positive probability.*

The proof will be carried out in Sections 3 and 4 below. Theorem 1 is by no means a complete solution to the coexistence problem, because it remains unknown whether the limit shape $\mathscr{S}$ for the Richardson model is uniformly curved, or even if its boundary $\partial\mathscr{S}$ is strictly convex. Nevertheless, simulations give every indication that

it is, and Lalley [6] suggests a possible explanation of what lies behind the strict convexity of $\partial \mathcal{S}$.

The two-species complete model is superficially similar to the *two-type Richardson model* studied by Haggstrom and Pemantle [4], but differs in that it allows displacement of colors on occupied sites: in the two-type Richardson model, once a vertex is occupied by a particle (either red or blue) it remains occupied by that color forever. The main result of [4] is similar to Theorem 1, but requires no hypothesis about the Richardson shape: it states that mutual unbounded growth has positive probability. Because no displacements are allowed, the behavior of the two-type Richardson model is very closely tied up with the first-passage percolation process with exponential edge passage times. The two-species competition model is also closely related to first-passage percolation, but the connection is less direct, because the possibility of displacements implies that not only the first passages across edges play a role in the evolution.

We have run simulations of the competition model with initial configuration $R(0) = \{(0, 0)\}$ and $B(0) = \{(1, 0)\}$. Fig. 1 shows two snap shots of the same realization of the process taken at the times when the region occupied by both types, $R(t) \cup B(t)$, hits the boundary of the rectangles $[-300, 300] \times [-300, 300]$ and $[-400, 400] \times [-400, 400]$, respectively. Observe that the overall shape of the red and the blue clusters did not change significantly. We believe that the shape of the regions occupied by the red and blue types stabilizes as times goes to infinity.

For any subset $Z \subset \mathbb{R}^d$, define

$$\hat{Z} = \{x \in \mathbb{R}^d : \text{dist}\,(x, Z) \leqslant \tfrac{1}{2}\},$$

where dist denotes distance in the L^∞-norm on $\mathbb{R}^d$. For any subset $Z \subset \mathbb{R}^d$ and any scalar $s > 0$, let $Z/s = \{y/s : y \in Z\}$.
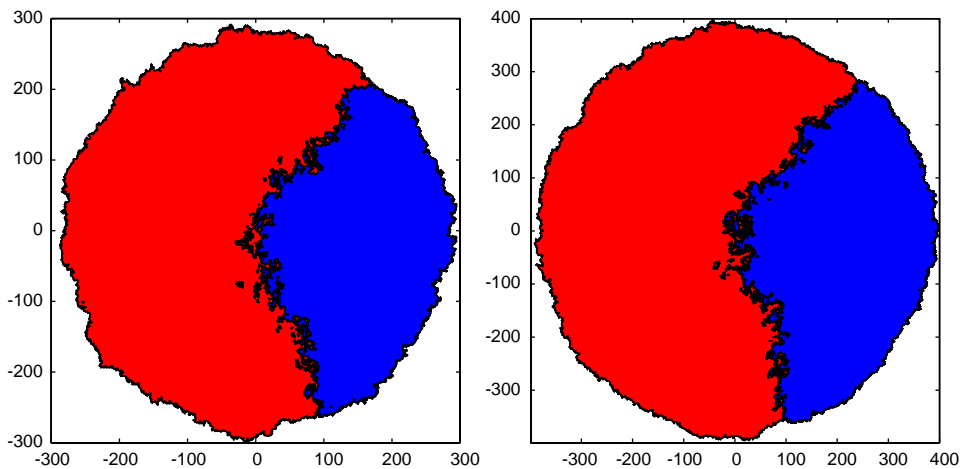

Fig. 1. Competition model with $R(0) = \{(0, 0)\}$, $B(0) = \{(1, 0)\}$.

Conjecture 1. *There exist random sets $\tilde{R}$ and $\tilde{B}$ such that with probability one*

$$\lim_{t\to\infty} \hat{R}(t)/t = \tilde{R}, \tag{1}$$

$$\lim_{t\to\infty} \hat{B}(t)/t = \tilde{B}, \tag{2}$$

and

$$\tilde{R} \cup \tilde{B} = \mathscr{S}. \tag{3}$$

If this is true, we expect that the limit sets $\tilde{R}$ and $\tilde{B}$ will be finite unions of angular sectors, as the simulation results shown in Fig. 1 suggest. The sizes and directions of these angular sectors (and even their number) will, we expect, be random, with distributions depending on the initial configuration. This is illustrated by simulation
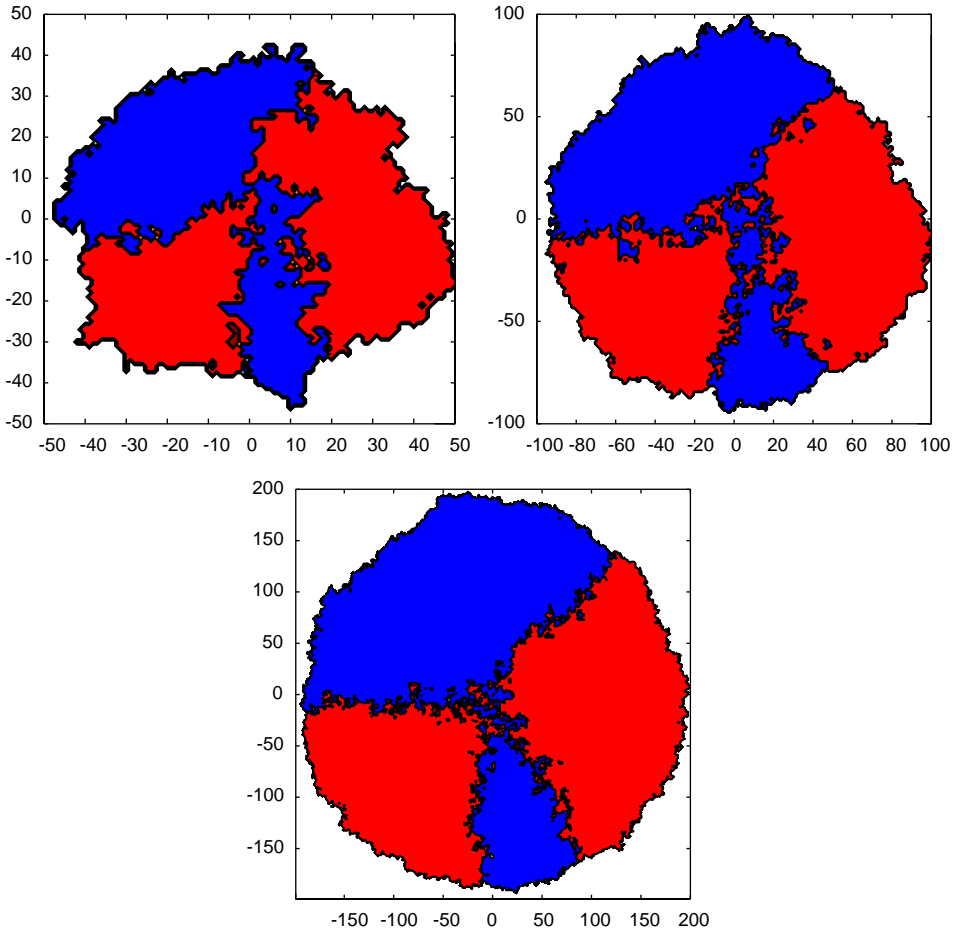

Fig. 2. Competition model with $R(0) = \{(-2,-2),\ (2,2)\}$ and $B(0) = \{(-2,2),\ (2,-2)\}$.

results summarized in Fig. 2 with initial configuration

$$R(0) = \{(-2, -2), \ (2, 2)\},$$

$$B(0) = \{(-2, 2), \ (2, -2)\}.$$

Three time progressive snap shots of the process were taken. The plots in Fig. 2 suggest that stabilization of the shape was taking place on the considered time interval.

2. The competition model on $\mathbb{Z}^1$

The coexistence problem for the Competition Model in one dimension is considerably simpler than in higher dimensions. Since the limit shape of the Richardson model in one dimension is an interval, no auxiliary hypothesis is needed.

Proposition 1. *For any nondegenerate finite initial configuration on $\mathbb{Z}$, the event of mutual survival in the two-species Competition Model has positive probability.*

Proof. Without loss of generality, we may assume that the initial configuration consists of a finite interval of red sites with rightmost point -1 and a finite interval of blue sites with leftmost point 0, since [a translate of] such a configuration may be reached in finite time, with positive probability, from *any* nondegenerate initial configuration. Let X_t and Y_t be the left- and right-most occupied sites (of either color) at time t, and let Z_t be the leftmost blue site. Note that as long as $X_t < Z_t < Y_t$, there will be both red and blue sites: all sites to the left of Z_t are red, and all sites to the right are blue. Each of the processes X_t and Y_t is a pure jump process, with jumps of size 1 occurring at rate 1; hence, with probability one, as $t \to \infty$,

$$X_t/t \longrightarrow -1$$

and

$$Y_t/t \longrightarrow 1.$$

The process Z_t behaves, up to the time of first exit from (X_t, Y_t), as a continuous-time simple nearest-neighbor random walk on the integers. Consequently, there is positive probability that Z_t never exits the interval (X_t, Y_t). But on this event, both species survive. □

This simple argument clearly shows what the difficulty in higher dimensions will be: In one dimension, the interface between (connected) red and blue clusters is just a point; but in higher dimensions, it will in general be a hypersurface, whose time evolution will necessarily be somewhat complicated.

3. Preliminaries

3.1. Graphical constructions

The Richardson model, the voter model, and the two-species competition model all admit *graphical contructions* using *percolation structures*. Such constructions make certain comparison arguments and duality relations transparent. We briefly review the construction here, primarily to emphasize that the same percolation structure can be used to simultaneously build versions of all three processes with all possible initial configurations. See, for instance, [3] for further details in the case of the Richardson model and the voter model.

The *percolation structure* Π is an assignment of independent, rate-1 Poisson processes to the directed edges xy of the lattice $\mathbb{Z}^d$. (For each pair $\{x, y\}$ of neighboring vertices, there are two directed edges xy and yx.) Above each vertex x is drawn a timeline, on which are placed marks at the occurrence times T_i^{xy} of the Poisson processes attached to directed edges emanating from x; at each such mark, an arrow is drawn from x to y. A *directed path* through the percolation structure Π may travel upward, at speed 1, along any timeline, and may (but does not have to) jump across any outward-pointing arrow that it encounters. A *reverse path* is a directed path run backward in time: thus, it moves downward along timelines and jumps across inward-pointing arrows. A *voter-admissible* path is a directed path that does not pass any inward-pointing arrows. Observe that for each vertex y and each time $t > 0$ there is a unique voter-admissible path beginning at time 0 and terminating at (y, t): its reverse path is gotten by traveling downward along timelines, starting at (y, t), jumping across all inward-pointing arrows encountered along the way.

Richardson model: A version $Z(t)$ of the Richardson model with initial configuration $Z(0) = \zeta$ is obtained by setting $Z(t)$ to be the set of all vertices y such that there is a directed path in the percolation structure Π that begins at $(x, 0)$ for some $x \in \zeta$ and ends at (y, t).

Voter model: A version $(R(t), B(t))$ of the voter model with initial configuration $R(0) = \beta$, $B(0) = \beta^c$ is gotten by defining $B(t) = R(t)^c$ and $R(t)$ to be the set of all vertices y such that the unique voter-admissible path terminating at (y, t) begins at $(x, 0)$ for some $x \in \beta$.

Two-species competition model: Fix an initial configuration $R(0) = \xi$, $B(0) = \zeta$. Erase all arrows that lie *only* on paths that begin at points $(x, 0)$ such that $x \notin \xi \cup \zeta$; denote the resulting sub-percolation structure $\Pi_{\xi,\zeta}$. Define $R(t)$ (respectively, $B(t)$) to be the set of all vertices y such that there is a voter-admissible path relative to $\Pi_{\xi,\zeta}$ that ends at (y, t) and starts at $(x, 0)$ with $x \in \xi$ (respectively, $x \in \zeta$).

The graphical construction yields as by-products comparison principles for the Richardson, voter, and competition models. First, the set $R(t) \cup B(t)$ of vertices occupied by either Red or Blue particles at time t in the competition model coincides with the set $Z(t)$ of occupied vertices in the Richardson model when $Z(0) = R(0) \cup B(0)$. Second, if $R^{(1)}(t), B^{(1)}(t)$ is the voter model with initial configuration $R^{(1)}(0) = \xi^{(1)}$, and $R(t), B(t)$ is the competition model with initial

configuration $R(0) = \xi \supset \xi^{(1)}$, $B(0) = \zeta \subset \xi^c$, then for all $t > 0$,

$$R(t) \supset R^{(1)}(t).$$

3.2. Voter model: invasion times

How long does it take for a Red vertex to be overrun by Blue? Clearly, for either the voter model or the competition model the answer will depend, at least in part, on how far away the nearest Blue vertices are. The comparison principle implies that, for any given value ρ of the distance to the nearest blue vertex, the worst case (for either model) is the voter model with initial configuration $R(0) = D(0, \rho)$ and $B(0) = D(0, \rho)^c$, where $D(x, \rho)$ denotes the disk of radius ρ centered at x (more precisely, its intersection with the lattice $\mathbb{Z}^d$).

Lemma 1. *Fix* $\beta \in (\frac{1}{2}, 1)$, *and denote by* $(R(t), B(t))$ *the state of the voter model at time* t. *There exist constants* $c_1, c_2 > 0$ *(depending on* β*) such that for all* $\rho \geqslant 1$ *and all* $t \in [0, \rho]$, *if* $R(0)$ *contains the disk of radius* ρ^β *centered at the vertex* x, *then*

$$P\{x \notin R(t)\} \leqslant c_1 \exp\{-c_2 \rho^{\beta - 1/2}\}.$$

Remark. This holds for *any* norm on $\mathbb{R}^d$, not just the Euclidean norm: in particular, it holds for the *Richardson norm* defined below. The constant $c_2 > 0$ may, of course, depend on the norm.

Proof of Lemma 1. The dual process of the voter model is the coalescing random walk (see [3] or [7]). Thus, the probability that the vertex x is blue at time t coincides with the probability that a continuous-time simple random walker started at x at time 0 will land in the set $B(0)$ at time t. (This is not difficult to deduce directly from the graphical construction above: the event $x \in B(t)$ occurs if and only if the reverse voter-admissible path started at (x, t) will terminate at $(y, 0)$ for some $y \in B(0)$; but the reverse voter-admissible path is a simple random walk.) Hence, if $R(0) \supset D(x, \rho^\beta)$ then this probability is dominated by the probability that the continuous-time simple random walk exits the ball $D(x, \rho^\beta)$ by time ρ.   □

3.3. Richardson model: shape theorem

The first-order asymptotic behavior of the Richardson model on the integer lattice $\mathbb{Z}^d$ is described by the *Shape Theorem* [2]. Denote by $Z(t)$ the set of vertices of $\mathbb{Z}^d$ that are occupied at time t, and by P_ζ the probability measure describing the law of the process given the initial condition $Z(0) = \zeta$. For any subset $Z \subset \mathbb{R}^d$, define

$$\hat{Z} = \left\{ x \in \mathbb{R}^d : \text{dist}(x, Z) \leqslant \tfrac{1}{2} \right\},$$

where dist denotes distance in the L^∞-norm on $\mathbb{R}^d$. For any subset $Z \subset \mathbb{R}^d$ and any scalar $s > 0$, let $Z/s = \{y/s : y \in Z\}$.

Theorem 2 (*The Shape Theorem*). *There exists a nonrandom compact convex set* $\mathscr{S} \subset \mathbb{R}^d$, *invariant under permutation of and reflection in the coordinate hyperplanes, and*

with non-empty interior, such that for any finite initial configuration $Z(0) = \zeta$ *and any* $\varepsilon > 0$, *with* P_ζ-*probability one, eventually (i.e., for all sufficiently large s)*

$$(1 - \varepsilon)\mathscr{S} \subseteq \frac{\hat{Z}(s)}{s} \subseteq (1 + \varepsilon)\mathscr{S}.$$

The exact shape of the limiting set $\mathscr{S}$ remains unknown. A simple argument shows that $\mathscr{S}$ is convex, but nobody has succeeded in proving that it is *strictly* convex. Let $|\cdot|$ be the norm on $\mathbb{R}^d$ associated with the shape set $\mathscr{S}$, that is, for $x \in \mathbb{R}^d$, $|x| = \inf\{t : x \in \mathscr{S} \cdot t\}$. That this is in fact a norm follows from the convexity of $\mathscr{S}$. The Shape Theorem is equivalent to the statement that the set of occupied sites grows at speed one, relative to the norm $|\cdot|$, in every direction.

The Richardson model admits a description as a first passage percolation model, as follows. To each edge of the lattice $\mathbb{Z}^d$, attach a mean one exponential random variable, the "passage time", in such a way that the passage times of distinct edges are mutually independent. For any self-avoiding path γ, define the traversal time $\tau(\gamma)$ to be the sum of the passage times of the edges in γ. For any finite set ζ of vertices and any vertex x, define the passage time $T(\zeta, x)$ from ζ to x to be the infimum of the traversal times $\tau(\gamma)$ of all self-avoiding paths connecting x to ζ. A version of the Richardson model $Z(t)$ with initial configuration $Z(0) = \zeta$ is given by

$$Z(t) := \{x \in \mathbb{Z}^d : T(\zeta, x) \leqslant t\}.$$

The first-passage percolation representation gives simultaneous realizations of Richardson evolutions for all initial configurations. Since traversal times of paths are the same backwards and forwards, the following *duality property* is immediate: For any finite subsets $F, G \subset \mathbb{Z}^d$ and any $t > 0$,

$$P_F\{Z(t) \cap G = \emptyset\} = P_G\{Z(t) \cap F = \emptyset\}. \tag{4}$$

Kesten [5] and Alexander [1] have established large deviation results for the passage times in first passage percolation that specialize to the Richardson model as follows.

Theorem 3. *There exist constants* c_1 *and* $c_2 > 0$ *such that for any* $\varepsilon > 0$, $y \in \mathbb{Z}^d$ *and* $|y|^{1/2+\varepsilon} < t < |y|^{3/2-\varepsilon}$,

$$P\{|T(\mathbf{0}, y) - |y|\| \leqslant t\} \geqslant 1 - c_1 \exp\{-c_2 t / \sqrt{|y|}\}.$$

3.4. A triangle inequality

The hypothesis of Theorem 1 is that the Richardson shape $\mathscr{S}$ is *uniformly curved*, that is, that there exists $\varrho > 0$ such that for each $x \in \partial\mathscr{S}$ there is a (Euclidean) ball $D^*(x)$ of radius ϱ containing $\mathscr{S}$ with x on its surface. Denote by $\pi : \mathbb{R}^d \backslash \{0\} \to \partial\mathscr{S}$ the natural projection onto the boundary of the Richardson shape, that is, for any $x \in \mathbb{R}^d$ such that $x \neq 0$,

$$\pi x = x / |x|.$$

Lemma 2. *Suppose that $\mathscr{S}$ is uniformly curved, then there exists a constant $c > 0$ such that, for all $x \notin \mathscr{S}$ and $y \in \mathscr{S}$*

$$|x - y| \geqslant |x - \pi x| + c|\pi x - y|^2$$

Proof. Let $\| \cdot \|_2$ be the Euclidean norm (L_2-norm) on $\mathbb{R}^d$. Any two norms on $\mathbb{R}^d$ are equivalent, and so the Euclidean norm is equivalent to the Richardson norm: in particular, there is a constant $\delta > 1$ such that, for any $x \in \mathbb{R}^d$,

$$\frac{1}{\delta} \|x\|_2 \leqslant |x| \leqslant \delta \|x\|_2.$$

Let l be the tangent hyperplane to $D^*(x)$ at πx. For any $y \in \mathscr{S}$, denote by y_p the (orthogonal) projection of y on l. Since $\mathscr{S}$ is uniformly curved (and hence strictly convex),

$$|x - y| \geqslant |x - \pi x| + |y - y_p|.$$

Elementary trigonometric observations imply that, since $y \in D^*(x)$ and y_p is on the tangent line l, there exists a constant c_0 that depends only on ϱ and such that for all such y's

$$\|y - y_p\|_2 > c_0 \|\pi x - y\|_2^2.$$

Hence,

$$|y - y_p| > \frac{c_0}{\delta^3} |\pi x - y|^2$$

which proves the inequality of the lemma. $\square$

4. Proof of Theorem 1

4.1. Strategy

We begin by showing that it suffices to restrict attention to a special class of initial configurations, which we dub *sliced Richardson shapes*. These are obtained as follows: Run the Richardson model (starting from the initial configuration with two adjacent occupied sites, one at the origin) for a (large) time T, and let Z_T be the occupied set. Let ξ_T be the subset of Z_T consisting of all points with positive first coordinates, and let $\zeta_T = Z_T \backslash \xi$. Observe that, starting from *any* nondegenerate finite initial configuration the competition model can evolve to a sliced Richardson shape in finite time, with positive probability. (This will occur if, following the first time that there are adjacent Red and Blue sites, only these sites reproduce, and only on their sides of the hyperplane separating them.) Thus, it suffices to prove that for all sufficiently large T, with positive probability the sliced Richardson shape ξ_T, ζ_T is

such that

$$P_{\xi,\zeta}\left\{\bigcap_{t\geqslant 0} R(t) = \emptyset \text{ or } \bigcap_{t\geqslant 0} B(t) = \emptyset\right\} < 4\varepsilon. \tag{5}$$

Here and in the sequel $P_{\xi,\zeta}$ will denote the probability measure governing the evolution of the competition model under the initial condition $R(0) = \xi$, $B(0) = \zeta$. By the Bonferroni inequality, it suffices to prove that

$$P_{\xi,\zeta}\left\{\bigcap_{t\geqslant 0} R(t) = \emptyset\right\} < 2\varepsilon. \tag{6}$$

The idea behind the proof of (6) is this: If the initial condition ξ, ζ is such that ξ and ζ are, approximately, the intersections of $k\mathscr{S}$ with complementary angular sectors $\mathscr{A}, \mathscr{A}^c$ in $\mathbb{R}^d$ based at the origin, for large k, then at time $t = \delta k$ the sets $R(t), B(t)$ should, with high probability, be approximately the intersections of $k(1 + \delta)\mathscr{S}$ with the same angular sectors $\mathscr{A}, \mathscr{A}^c$. This is because (1) the Shape Theorem for the Richardson model implies that $R(t) \cup B(t)$ should be close to $(1 + \delta)k\mathscr{S}$; (2) the uniform curvature of $\mathscr{S}$ implies that the *first* occupations of vertices in $\mathscr{A} \cap ((t + k)\mathscr{S}\backslash k\mathscr{S})$ and $\mathscr{A}^c \cap ((t + k)\mathscr{S}\backslash k\mathscr{S})$ should (except for those near the boundaries) be by Red and Blue, respectively; and (3) Lemma 1 implies that, once a region is totally occupied by Red, it must remain so (except near its boundary) for a substantial amount of time afterward.

4.2. Stabilization estimate

The key step is to show that once one of the species (say Red) has occupied an angular sector in the Richardson shape, it is very unlikely for the opposite species (Blue) to make a large incursion into this sector for some time afterward. Henceforth, let d be the metric associated with the Richardson norm. For any set ζ and any vertex x, define the distance $d(\zeta, y)$ between ζ and x to be the infimum of the distances $d(y, x)$ for all vertices in $y \in \zeta$. For any point $x \in \mathbb{R}^d$ and any $r > 0$, denote by $D(x; r)$ the disk of radius r centered at x relative to the metric d. (We shall not attempt to distinguish between open and closed disks, as this distinction will not matter in any of the estimates.) For $r_1 < r_2$ denote by $D(x; r_1, r_2)$ the annular region $D(x; r_2)\backslash D(x; r_1)$. For each $z \in \partial\mathscr{S}$ and any $\varrho > 0$, define the *angular sector* $\mathscr{A}(z; \varrho)$ of aperture ϱ centered at z by

$$\mathscr{A}(z; \varrho) := \{y \in \mathbb{R}^d\backslash\{0\} : d(\pi y, z) < \varrho\}.$$

Fix $r > 0$, $n \geqslant 1$, $\beta \in (1/2, 1)$ and $\alpha \in (1/2, 1)$ such that $(\beta + 1)/2 < \alpha$, and let $A_1 \subset A_2$ be angular sectors with common center z and apertures $r < r + n^{\alpha-1}$, respectively. Fix $\delta \in (0, 1)$, and define

$$\mathscr{R}_0 = \mathscr{R}_0^n = D(0; n/(1 + \delta), n - n^\beta) \cap A_2 \cap \mathbb{Z}^d,$$

$$\mathscr{B}_0 = \mathscr{B}_0^n = (D(0; n/(1 + \delta)) \cup (D(0; n/(1 + \delta), n + n^\beta) \cap A_2^c)) \cap \mathbb{Z}^d,$$

$$\mathscr{B}_1 = \mathscr{B}_1^n = (D(0; n) \cup A_1^c) \cap \mathbb{Z}^d,$$

$$\mathscr{R}_1 = \mathscr{R}_1^n = (D(0; n, n(1 + \delta) - (n + \delta n)^\beta) \cap A_1) \cap \mathbb{Z}^d.$$

Lemma 3. *There exist constants $c_1, c_2 > 0$ such that the following is true, for any $n \geqslant 1$. If the initial configuration ξ, ζ is such that $\xi \supset \mathscr{R}_0^n$ and $\zeta \subset \mathscr{B}_0^n$, then*

$$1 - P_{\xi,\zeta}\{B(\delta n) \subset \mathscr{B}_1^n\} \leqslant c_1 n^{3d} \exp\{-c_2 (\delta n)^{\beta - 1/2}\}. \tag{7}$$

Proof. To prove (7) we find exponential upper bounds on

$$P_{\mathscr{R}_0}, \mathscr{B}_0\{B(\delta n) \cap \mathscr{R}_1 \neq \emptyset\}$$

and

$$P_{\mathscr{R}_0}, \mathscr{B}_0\{B(\delta n) \cap (\mathscr{R}_1 \cup \mathscr{B}_1)^c \neq \emptyset\}.$$

Claim 1. *For all sufficiently large n and for all $x \in \mathscr{R}_1$*

$$d(x, \mathscr{R}_0) + 4(\delta n)^\beta < d(x, \mathscr{B}_0).$$

Proof. First, observe that by Lemma 2, for every $x \in \mathscr{R}_1$ such that $|x| > n + n^\beta$, we have

$$d(x, \mathscr{B}_0) > |x - \pi x(n + n^\beta)| + cn^{2\alpha - 1}.$$

Also,

$$d(x, \mathscr{R}_0) \leqslant |x - \pi x(n + n^\beta)| + 2n^\beta.$$

These inequalities imply Claim 1 for all $x \in \mathscr{R}_1 \cap D(0; n + n^\beta)^c$ and all sufficiently large n.

Next, if $x \in \mathscr{R}_1$ is such that $n < |x| < n + n^\beta$, then

$$d(x, \mathscr{R}_0) \leqslant 2n^\beta.$$

Also, by Lemma 2, for all $y \in \mathscr{B}_0$,

$$d(y, x) > c(\delta n)^{2\alpha - 1}.$$

These two inequalities imply Claim 1 for all $x \in \mathscr{R}_1 \cap D(0; n, n + n^\beta)$ and all sufficiently large n. $\square$

Claim 2. *With probability $\to 1$ as $n \to \infty$, for every $x \in \mathscr{R}_1$, every site of the ball $D(x; (\delta n)^\beta)$ will be colonized by time $\tau_x = d(x, \mathscr{R}_0) + 2(\delta n)^\beta$. In particular, there exist constants $c_1 > 0$ and $c_2 > 0$ (not depending on x) such that for every $x \in \mathscr{R}_1$*

$$P_{\mathscr{R}_0, \mathscr{B}_0}\{D(x, (\delta n)^\beta) \not\subset R(\tau_x) \cup B(\tau_x)\} \leqslant c_1 (\delta n)^{d\beta} \exp\{-c_2 (\delta n)^{\beta - 1/2}\}.$$

Proof. Fix $x \in \mathscr{R}_1$ and let $\tau_x = d(x, \mathscr{R}_0) + 2(\delta n)^\beta$. For every $z \in D(x; (\delta n)^\beta)$,

$$d(z, \mathscr{R}_0) + (\delta n)^\beta \leqslant \tau_x. \tag{8}$$

Notice that the number of sites in $D(x; (\delta n)^{\beta})$ is of order $(\delta n)^{d\beta}$. By Theorem 3 and (8), it follows that for some $c_1 > 0$ and $c_2 > 0$ (not depending on x)

$$P_{\mathscr{R}_0, \mathscr{B}_0}\{D(x; (\delta n)^{\beta}) \not\subset R(\tau_x) \cup B(\tau_x)\} = P_{\mathscr{R}_0 \cup \mathscr{B}_0}\{D(x; (\delta n)^{\beta}) \not\subset Z(\tau_x)\}$$
$$\leqslant c_1(\delta n)^{d\beta} \exp\{-c_2(\delta n)^{\beta - 1/2}\}.$$

Hence, with probability $\to 1$ as $n \to \infty$, for every $x \in \mathscr{R}_1$, every site of the ball $D(x; (\delta n)^{\beta})$ will be colonized at time τ_x. $\square$

Define the boundary $\partial \mathscr{A}$ of a set $\mathscr{A} \subset \mathbb{Z}^d$ as the set of all $z \in \mathscr{A}$ that have at least one nearest neighbor that is not in $\mathscr{A}$. Next, for a set $\mathscr{A} \subset \mathbb{Z}^d$ let $\tau(\mathscr{A})$ be the first time at which the blue species reaches $\mathscr{A}$.

Claim 3. *With probability $\to 1$ as $n \to \infty$, for every $x \in \mathscr{R}_1$, the blue species will not reach the ball $D(x; (\delta n)^{\beta})$ by time τ_x. In particular, there exist constants $c_1 > 0$ and $c_2 > 0$ such that for every $x \in \mathscr{R}_1$*

$$P_{\mathscr{R}_0, \mathscr{B}_0}\{\tau(D(x; (\delta n)^{\beta})) < \tau_x\} \leqslant c_1(\delta n)^{(d-1)(\beta+1)} \exp\{-c_2(\delta n)^{\beta - 1/2}\}.$$

Proof. Notice that

$$P_{\mathscr{R}_0, \mathscr{B}_0}\{\tau(D(x; (\delta n)^{\beta})) \leqslant \tau_x\} \leqslant P_{\mathscr{B}_0}\{Z(\tau_x) \cap D(x; (\delta n)^{\beta}) \neq \emptyset\}$$
$$= P_{D(x;(\delta n)^{\beta})}\{Z(\tau_x) \cap \mathscr{B}_0 \neq \emptyset\}.$$

By Claim 1, for large n we have $d(x, \mathscr{B}_0) > \tau_x + 2(\delta n)^{\beta}$. Hence,

$$P_{D(x;(\delta n)^{\beta})}\{Z(\tau_x) \cap \mathscr{B}_0 \neq \emptyset\} \leqslant P_{D(x;(\delta n)^{\beta})}\{Z(\tau_x) \not\subset D(x; \tau_x + 2(\delta n)^{\beta})\}.$$

Obviously, for every $z \in \partial D(x; (\delta n)^{\beta})$ and $y \in \partial D(x; \tau_x + 2(\delta n)^{\beta})$, we have

$$d(z, y) \geqslant \tau_x + (\delta n)^{\beta}/2.$$

Apply Theorem 3 to each pair of such vertices to get

$$P\{T(z, y) < \tau_x\} < c_1 \exp\{-c_2(\delta n)^{\beta - 1/2}\}.$$

The number of vertices in $\partial D(x, (\delta n)^{\beta})$ is of order $(\delta n)^{(d-1)\beta}$, and the number of vertices in $\partial D(x, \tau_x + 2(\delta n)^{\beta})$ is of order at most $(\delta n)^{(d-1)}$. Hence,

$$P_{D(x;(\delta n)^{\beta})}\{Z(\tau_x) \not\subset D(x; \tau_x + 2(\delta n)^{\beta})\} \leqslant c_1(\delta n)^{(d-1)(\beta+1)} \exp\{-c_2(\delta n)^{\beta - 1/2}\}.$$

This finishes the proof of Claim 3. $\square$

Claim 4. *With probability $\to 1$ as $n \to \infty$, there are no Blue particles in $\mathscr{R}_1$ at time δn. In particular, there exist constants $c_1 > 0$ and $c_2 > 0$ such that*

$$P_{\mathscr{R}_0, \mathscr{B}_0}\{\mathscr{R}_1 \cap B(\delta n) \neq \emptyset\} \leqslant c_1(\delta n)^{(d-1)(\beta+1)+d} \exp\{-c_2(\delta n)^{\beta - 1/2}\}.$$

Proof. By Claim 3, for all $x \in \mathscr{R}_1$ with $\tau_x \geqslant \delta n$,

$$P_{\mathscr{R}_0, \mathscr{B}_0}\{x \in B(\delta n)\} \leqslant c_1 (\delta n)^{(d-1)(\beta+1)} \exp\{-c_2 (\delta n)^{\beta-1/2}\}. \qquad (9)$$

Next, for all $x \in \mathscr{R}_1$ with $\tau_x < \delta n$, Claims 2 and 3 imply that for some $c_1 > 0$ and $c_2 > 0$

$$P_{\mathscr{R}_0, \mathscr{B}_0}\{D(x, (\delta n)^\beta) \not\subset R(\tau_x)\} \leqslant c_1 (\delta n)^{(d-1)(\beta+1)} \exp\{-c_2 (\delta n)^{\beta-1/2}\}.$$

Hence, by Lemma 1, there exist constants c_1 and c_2 such that for every such x we have

$$P_{\mathscr{R}_0, \mathscr{B}_0}\{x \notin R(\delta n)\} \leqslant c_1 (\delta n)^{(d-1)(\beta+1)} \exp\{-c_2 (\delta n)^{\beta-1/2}\}. \qquad (10)$$

The number of vertices in $\mathscr{R}_1$ is of order $(\delta n)^d$. Thus, by combining (9) and (10), we get

$$P_{\mathscr{R}_0, \mathscr{B}_0}\{B(\delta n) \cap \mathscr{R}_1 \neq \emptyset\} \leqslant c_1 (\delta n)^{(d-1)(\beta+1)+d} \exp\{-c_2 (\delta n)^{\beta-1/2}\}. \quad \square$$

Claim 5. *With probability $\to 1$ as $n \to \infty$, the blue species will not reach the set* $(\mathscr{R}_1 \cup \mathscr{B}_1)^c$ *by time δn. In particular, there exist constants $c_1 > 0$ and $c_2 > 0$ such that*

$$P_{\mathscr{R}_0, \mathscr{B}_0}\{\tau((\mathscr{R}_1 \cup \mathscr{B}_1)^c) \leqslant \delta n\} \leqslant c_1 n^{2(d-1)} \exp\{-c_2 (\delta n)^{\beta-1/2}\}.$$

Proof. For large n the distance between the sets $\mathscr{B}_0$ and $(\mathscr{R}_1 \cup \mathscr{B}_1)^c$ is greater than $\delta n + (\delta n)^\beta$. The number of vertices on the boundary of $\mathscr{B}_0$ is of order n^{d-1}. Using the same line of argument as in the proof of Claim 3 we get

$$\begin{aligned} P_{\mathscr{R}_0, \mathscr{B}_0}\{\tau((\mathscr{R}_1 \cup \mathscr{B}_1)^c) \leqslant \delta n\} &\leqslant P_{\mathscr{B}_0}\{Z(\delta n) \not\subset \mathscr{R}_1 \cup \mathscr{B}_1\} \\ &\leqslant c_1 n^{2(d-1)} \exp\{-c_2 (\delta n)^{\beta-1/2}\}. \quad \square \end{aligned}$$

Now, Claims 4 and 5 imply (7) and finish the proof of Lemma 3:

$$P_{\mathscr{R}_0, \mathscr{B}_0}\{B(\delta n) \not\subset \mathscr{B}_1\} \leqslant c_1 n^{3d} \exp\{-c_2 (\delta n)^{\beta-1/2}\}. \quad \square$$

4.3. Proof of (6)

Let Z_t be the set of sites occupied by the Richardson evolution (started from the default initial configuration) at time t. Fix $T = T_0 \geqslant 1$ and $\delta > 0$, and set

$$T_n = T(1+\delta)^n,$$

$$t_n = T\delta(1+\delta)^{n-1},$$

and

$$\tau_n = T_n - T = \sum_{j=1}^{n} t_n.$$

Fix $\beta \in (1/2, 1)$, and for each $n = 0, 1, \ldots$ define events

$$F_n := \{(1 - T_n^{\beta-1})\mathscr{S} \subseteq \hat{Z}(T_n)/T_n \subseteq (1 + T_n^{\beta-1})\mathscr{S}\},$$

$$G_n := \{(1 - T_n^{\beta-1})\mathscr{S} \subseteq \hat{Z}(\tau_n)/T_n \subseteq (1 + T_n^{\beta-1})\mathscr{S}\}.$$

By the Kesten–Alexander large deviation theorems (Theorem 3),

$$\lim_{T \to \infty} \sum_{n=0}^{\infty} P(F_n^c) = 0$$

and so, for sufficiently large T, the probability is nearly 1 that the configuration Z_T will be such that

$$\sum_{n=0}^{\infty} P_{Z_T}(G_n^c) < \varepsilon. \tag{11}$$

Fix an initial configuration Z so that the preceding estimate holds for $Z_T = Z$, and use this to construct the split Richardson shape $\xi = \xi_T, \zeta = \zeta_T$ as in Section 4.1 above: ξ and ζ are the subsets of Z_T with positive and nonpositive first coordinates, respectively. Since the union of the Red and Blue sites in the competition model evolves as the Richardson model, it follows from (11) that if the initial configuration is $R(0) = \xi$, $B(0) = \zeta$ then with probability in excess of $1 - \varepsilon$,

$$(1 - T_n^{\beta-1})\mathscr{S} \subseteq \frac{\hat{\xi}(\tau_n) \cup \hat{\zeta}(\tau_n)}{T_n} \subseteq (1 + T_n^{\beta-1})\mathscr{S} \tag{12}$$

for all $n = 0, 1, 2, \ldots$. Denote by G_* the event that (12) holds for all $n \geqslant 0$.

On the event G_*, the union of the Red and Blue regions will, at each time $T_n - T$, fill a region close enough to a Richardson shape that estimate (7) will be applicable whenever the Red and Blue populations are restricted (at least approximately) to angular sectors. Thus, define sequences $A_1^n \subset A_2^n$ of concentric angular sectors with apertures $r_1(n) < r_2(n)$ such that

$$r_2(n) - r_1(n) = T_n^{\alpha-1}$$

and

$$r_2(n + 1) = r_1(n)$$

and with $r_2(0)$ chosen so that A_2^0 is the halfspace consisting of all points in $\mathbb{R}^d$ with positive first coordinates. Here $\frac{1}{2} < \beta < (\beta + 1)/2 < \alpha < 1$ as in Section 4.2 above. Note that the second equality guarantees that $A_2^{n+1} = A_1^n$. This in turn, together with the fact that the sequence T_n is increasing, implies that the angular sectors are nested: $A_i^{n+1} \subset A_i^n$. Moreover, because T_n is an exponentially growing sequence and $\alpha < 1$,

$$\lim_{n \to \infty} r_1(n) := r_\infty > 0$$

provided $T = T_0$ is sufficiently large. Therefore, the intersection

$$A^\infty = \bigcap_{n=1}^\infty A_1^n$$

is an angular sector with nonempty interior.

Finally, for each $n \geqslant 0$ define H_n to be the event that at time τ_n there are no blue sites in A_2^n outside the (Richardson norm) disk of radius T_{n-1}. (For $n = 0$, set $T_{-1} = T_0/(1 + \delta)$.) On the event $G_n \cap H_n$, the set of all occupied sites is close to $T_n \cdot \mathscr{S}$, and the red sites fill at least the outer layer of this set in the sector A_2^n. We claim that for all sufficiently large T,

$$P_{\xi_T, \zeta_T}\left(\bigcap_{n=0}^\infty (G_n \cap H_n)\right) \geqslant 1 - 2\varepsilon. \tag{13}$$

To see this, let v be the smallest index n such that $(G_n^c \cup H_n^c)$ occurs. Since $v = n$ can only occur on $G_{n-1} \cap H_{n-1}$,

$$P_{\xi_T, \zeta_T}\{v = n\} \leqslant P_{\xi_T, \zeta_T}(G_n^c) + P_{\xi_T, \zeta_T}(H_n^c \,|\, G_{n-1} \cap H_{n-1}).$$

Inequality (11) provides a bound on the sum of the first of these terms, and Lemma 3 bounds the second. Thus, for T sufficiently large,

$$\sum_{n=0}^\infty P_{\xi_T, \zeta_T}\{v = n\} < 2\varepsilon,$$

this proves (13).

On the event $G_n \cap H_n$, the Red species must at time τ_n occupy at least the outer layer of the occupied set in the angular sector A^∞. Consequently, on the event $\bigcap_{n \geqslant 1}(G_n \cap H_n)$, Red survives! This proves (6).   □

5. Concluding remark

The preceding argument, in addition to proving that the event that mutual survival has positive probability, also goes part of the way towards proving Conjecture 1: If at a large time T one of the colors (say Red) occupies the outer layer of an angular sector, then with conditional probability approaching 1 as $T \to \infty$ it will occupy a slightly smaller angular sector forever after. Since the same is true for the other species, it follows that in at least some evolutions Red and Blue will each occupy angular sectors.

Unfortunately, it remains unclear what happens near the interface at large times. Although the preceding arguments show that neither Red nor Blue can make too deep an incursion into the other species' sector(s), it may be possible for one to repeatedly make small incursions across the interface that engender more (and necessarily thinner) angular sectors in its zone of occupation. Thus, it may be that the limit shapes exist, but consist of countably many angular sectors.

Finally, it remains unclear if stabilization must eventually occur on the event of mutual survival, that is, if it is necessarily the case that at large times T the outer layer of the occupied region must segregate into well-defined Red and Blue zones. Since local coalescence occurs in the voter model, one naturally expects that the same will be true in the competition model; thus far, we have been unable to prove this.

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