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European Journal of Combinatorics 25 (2004) 1345–1376
European Journal of Combinatorics
www.elsevier.com/locate/ejc

A mixed hook-length formula for affine Hecke algebras

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Accepted 14 October 2003

Available online 15 January 2004

Dédié au Professeur Alain Lascoux pour son 60-ème anniversaire

Abstract

Let $\widehat{H}_l$ be the affine Hecke algebra corresponding to the group GL_l over a p-adic field with residue field of cardinality q. We will regard $\widehat{H}_l$ as an associative algebra over the field $\mathbb{C}(q)$. Consider the $\widehat{H}_{l+m}$-module W induced from the tensor product of the evaluation modules over the algebras $\widehat{H}_l$ and $\widehat{H}_m$. The module W depends on two partitions λ of l and μ of m, and on two non-zero elements of the field $\mathbb{C}(q)$. There is a canonical operator J acting on W; it corresponds to the trigonometric R-matrix. The algebra $\widehat{H}_{l+m}$ contains the finite dimensional Hecke algebra H_{l+m} as a subalgebra, and the operator J commutes with the action of this subalgebra on W. Under this action, W decomposes into irreducible subspaces according to the Littlewood–Richardson rule. We compute the eigenvalues of J, corresponding to certain multiplicity-free irreducible components of W. In particular, we give a formula for the ratio of two eigenvalues of J, corresponding to the "highest" and the "lowest" components. As an application, we derive the well known q-analogue of the hook-length formula for the number of standard tableaux of shape λ.

1. Introduction

In this article we will work with the affine Hecke algebra corresponding to the general linear group GL_l over a local non-Archimedean field. Let q be a formal parameter. Let H_l be the finite dimensional Hecke algebra over the field $\mathbb{C}(q)$ of rational functions in q, with

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doi:10.1016/j.ejc.2003.10.010

the generators $T_1, \dots, T_{l-1}$ and the relations

$$(T_i - q)(T_i + q^{-1}) = 0; \tag{1.1}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}; \tag{1.2}$$

$$T_i T_j = T_j T_i, \qquad j \neq i,\ i+1 \tag{1.3}$$

for all possible indices i and j. The *affine Hecke algebra* $\widehat{H}_l$ is the $\mathbb{C}(q)$-algebra generated by the elements $T_1, \dots, T_{n-1}$ and the pairwise commuting invertible elements $Y_1, \dots, Y_n$ subject to the relations (1.1)–(1.3) and

$$T_i Y_i T_i = Y_{i+1}; \tag{1.4}$$

$$T_i Y_j = Y_j T_i, \qquad j \neq i,\ i+1. \tag{1.5}$$

By definition, the affine algebra $\widehat{H}_l$ contains H_l as a subalgebra. There is also a homomorphism $\pi_1 : \widehat{H}_l \to H_l$ identical on the subalgebra $H_l \subset \widehat{H}_l$; it can be defined [1, Theorem 3.4] by setting $\pi_1(Y_1) = 1$. Indeed, then by (1.4) we have

$$\pi_1(Y_i) = T_{i-1} \dots T_1 T_1 \dots T_{i-1} \tag{1.6}$$

for every $i = 1, \dots, l$. Denote by X_i the right hand side of the equality (1.6). Using the relations (1.2) and (1.3), one can check that

$$T_i X_j = X_j T_i, \qquad j \neq i,\ i+1$$

and that the elements $X_1, \dots, X_l$ are pairwise commuting. These elements are invertible in H_l, because the generators $T_1, \dots, T_{l-1}$ are invertible: we have

$$T_i^{-1} = T_i - q + q^{-1} \tag{1.7}$$

due to (1.1). The elements $X_1, \dots, X_l$ are called the *Murphy elements* [14] of the Hecke algebra H_l; they play an important role in the present article.

More generally, for any non-zero $z \in \mathbb{C}(q)$, one can define a homomorphism $\pi_z : \widehat{H}_l \to H_l$, also identical on the subalgebra $H_l \subset \widehat{H}_l$, by setting $\pi_z(Y_1) = z$. It is called the *evaluation homomorphism* at z. By pulling any irreducible H_l-module V back through the homomorphism π_z we obtain a module over the algebra $\widehat{H}_l$, called an *evaluation module* at z and denoted by $V(z)$. By definition, the $\widehat{H}_l$-module $V(z)$ is irreducible.

Throughout this article l is a positive integer. For any index $i = 1, \dots, l-1$ let $\sigma_i = (i\,i+1)$ be the adjacent transposition in the symmetric group S_l. Take any element $\sigma \in S_l$ and choose a reduced decomposition $\sigma = \sigma_{i_1} \cdots \sigma_{i_L}$. As usual put $T_\sigma = T_{i_1} \cdots T_{i_L}$; this element of the algebra H_l does not depend on the choice of reduced decomposition of σ due to (1.2) and (1.3). The element of maximal length in S_l will be denoted by σ_0. We will write T_0 instead of T_{σ_0} for short. The elements T_σ form a basis of H_l as a vector space over the field $\mathbb{C}(q)$. We will also use the basis in H_l formed by the elements T_σ^{-1}.

The $\mathbb{C}(q)$-algebra H_l is semisimple; see [6, Section 4] for a short proof of this well known fact. The simple ideals of H_l are labelled by partitions λ of l, like the equivalence classes of irreducible representations of the symmetric group S_l. In Section 3 of the present article, for any partition λ of l we will construct a certain left ideal V_λ in the algebra H_l. Under the action of the algebra H_l via left multiplication, the subspace $V_\lambda \subset H_l$ is irreducible; see Corollary 3.5. The H_l-modules V_λ for different partitions λ are pairwise

non-equivalent; see Corollary 3.6. At $q = 1$, the algebra $H_n(q)$ specializes to the group ring $\mathbb{C}S_l$. The $H_n(q)$-module V_λ then specializes to the irreducible representation of S_l, corresponding [18] to the partition λ. Our construction of V_λ employs a certain limiting process called *fusion procedure* [1]; see Section 2 for details, cf. [7, 9].

Using this definition of the H_l-module V_λ, consider the evaluation module $V_\lambda(z)$ over the affine Hecke algebra $\widehat{H}_l$. Take a partition μ of a positive integer m and a non-zero element $w \in \mathbb{C}(q)$, then also consider the evaluation module $V_\mu(w)$ over the algebra $\widehat{H}_m$. The tensor product $\widehat{H}_l \otimes \widehat{H}_m$ is naturally identified with the subalgebra in $\widehat{H}_{l+m}$, generated by the elements

$$T_1, \dots, T_{l-1}, Y_1, \dots, Y_l \qquad \text{and} \qquad T_{l+1}, \dots, T_{l+m-1}, Y_{l+1}, \dots, Y_{l+m}.$$

Let us denote by W be the $\widehat{H}_{l+m}$-module induced from the module $V_\lambda(z) \otimes V_\mu(w)$ over the subalgebra $\widehat{H}_l \otimes \widehat{H}_m \subset \widehat{H}_{l+m}$. Identify the underlying vector space of the module W with the left ideal in H_{l+m} generated by $V_\lambda \otimes V_\mu \subset H_l \otimes H_m$, so that the subalgebra $H_{l+m} \subset \widehat{H}_{l+m}$ acts on W via left multiplication. Further, denote by W' be the $\widehat{H}_{l+m}$-module induced from the module $V_\mu(w) \otimes V_\lambda(z)$ over the subalgebra $\widehat{H}_m \otimes \widehat{H}_l \subset \widehat{H}_{l+m}$. The underlying vector space of W' is identified with the left ideal in H_{l+m} generated by $V_\mu \otimes V_\lambda \subset H_m \otimes H_l$. Note that then due to (4.5) and (4.6) we have the equality of left ideals $W' = WT_\tau$, where τ is the element of the symmetric group S_{l+m} permuting

$$(1, \dots, m, m+1, \dots, l+m) \mapsto (l+1, \dots, l+m, 1, \dots, l). \tag{1.8}$$

Suppose that $z^{-1}w \notin q^{2\mathbb{Z}}$. Then the $\widehat{H}_{l+m}$-modules W and W' are irreducible and equivalent; see for instance [20, Remark 8.7]. Hence there is a unique, up to a multiplier from $\mathbb{C}(q)$, non-zero intertwining operator of $\widehat{H}_{l+m}$-modules $I : W \to W'$. The existence of this operator does not depend on the choice of realization of the $\widehat{H}_{l+m}$-modules W and W'. For our choice of W and W', we will give an explicit formula for the operator I; see Proposition 4.2. This formula fixes the normalization of I, in particular.

Let $J : W \to W$ be the composition of the operator $I : W \to W'$, and the operator $W' \to W$ of multiplication by the element T_τ^{-1} on the right. Since the subalgebra $H_{l+m} \subset \widehat{H}_{l+m}$ acts on the left ideals W and W' in H_{l+m} via left multiplication, the operator J commutes with this action of H_{l+m}. Under this action, the vector space W splits into irreducible components according to the Littlewood–Richardson rule [13, Section I.9]. On every irreducible component appearing with multiplicity one, the operator J acts as multiplication by a certain element of $\mathbb{C}(q)$. In this article, we compute these elements of $\mathbb{C}(q)$ for certain multiplicity-free components of W; see Theorems 4.5 and 4.6. Note that without affecting the eigenvalues of the operator J, one can replace V_λ and V_μ in our definition of W by any left ideals in the algebras H_l and H_m respectively, equivalent to V_λ and V_μ as modules over these two algebras.

Let us give an example of applying our Theorems 4.5 and 4.6. Write

$$\lambda = (\lambda_1, \lambda_2, \dots) \qquad \text{and} \qquad \lambda = (\mu_1, \mu_2, \dots),$$

where the parts of λ and μ are as usual arranged in the non-increasing order. Consider also the conjugate partitions

$$\lambda^* = (\lambda_1^*, \lambda_2^*, \dots) \qquad \text{and} \qquad \mu^* = (\mu_1^*, \mu_2^*, \dots).$$

There are two distinguished irreducible components of the H_{l+m}-module W which are multiplicity free. They correspond to the two partitions of $l+m$

$$\lambda+\mu=(\lambda_1+\mu_1,\lambda_2+\mu_2,\ldots) \qquad \text{and} \qquad (\lambda^*+\mu^*)^*.$$

Let us denote by $h_{\lambda\mu}(z,w)$ the ratio of the corresponding two eigenvalues of the operator J; this ratio does not depend on the normalization of this operator.

Corollary 1.1. *We have*

$$h_{\lambda\mu}(z,w)=\prod_{a,b}\frac{z^{-1}w-q^{-2(\mu_a+\lambda_b^*-a-b+1)}}{z^{-1}w-q^{2(\lambda_a+\mu_b^*-a-b+1)}} \tag{1.9}$$

where the product is taken over all $a,b=1,2,\ldots$ such that $b\leqslant\lambda_a,\mu_a$.

We will derive this result from Theorems 4.5 and 4.6, using Proposition 4.7. Now consider the Young diagrams of λ and μ. For the partition λ, this is the set (2.5). The condition $b\leqslant\lambda_a,\mu_a$ in Corollary 1.1 means that the node (a,b) belongs to the intersection of the diagrams corresponding to λ and μ. Recall that the number $\lambda_a+\lambda_b^*-a-b+1$ is the *hook-length* corresponding to the node (a,b) of the Young diagram of λ. The numbers appearing in (1.9),

$$\lambda_a+\mu_b^*-a-b+1 \qquad \text{and} \qquad \mu_a+\lambda_b^*-a-b+1,$$

may be called the *mixed hook-lengths* of the first and second kind respectively. Both these numbers are positive for any node (a,b) in the intersection of the Young diagrams of λ and μ; hence there are no cancellations of factors in (1.9).

According to the famous formula from [3], the product of the hook-lengths of the Young diagram of λ is equal to the ratio $l!/\dim V_\lambda$. We call the equality (1.9) the mixed hook-length formula. Its counterpart for the degenerate, or graded affine Hecke algebras [2, 12] which does not involve the parameter q has appeared in [15]. The q-analogue of the hook-length formula [3] is also known; see for instance [13, Example I.3.1]. As another application of our Theorem 4.5 we give a new proof of this q-analogue; see the end of Section 4.

2. Fusion procedure for the algebra H_l

In this section, for any standard tableau Λ of shape λ we will construct a certain non-zero element $F_\Lambda\in H_l$. Under left multiplication by the elements of H_l, the left ideal $H_lF_\Lambda\subset H_l$ is an irreducible H_l-module. The irreducible H_l-modules corresponding to two standard tableaux are equivalent if and only if these tableaux have the same shape. The idea of this construction goes back to [1, Section 3] were no proofs were given however. The element F_Λ is related to the q-analogue of the Young symmetrizer in the group ring $\mathbb{C}S_l$ constructed in [5]; see the end of next section for details of this relation.

For each $i = 1, \dots, l-1$ introduce the H_l-valued rational function in two variables $x, y \in \mathbb{C}(q)$

$$F_i(x,y) = T_i + \frac{q-q^{-1}}{x^{-1}y-1}. \tag{2.1}$$

As a direct calculation using (1.1) and (1.2) shows, these functions satisfy

$$F_i(x,y)F_{i+1}(x,z)F_i(y,z) = F_{i+1}(y,z)F_i(x,z)F_{i+1}(x,y). \tag{2.2}$$

Due to (1.3) these rational functions also satisfy the relations

$$F_i(x,y)F_j(z,w) = F_j(z,w)F_i(x,y); \qquad j \neq i,\ i+1. \tag{2.3}$$

Using (1.1) once again, we obtain the relations

$$F_i(x,y)F_i(y,x) = 1 - \frac{(q-q^{-1})^2 xy}{(x-y)^2}. \tag{2.4}$$

Our construction of the element $F_\Lambda \in H_l$ is based on the following simple observation. Consider the rational function of x, y, z defined as the product on either side of (2.2). The factor $F_{i+1}(x,z)$ on the left hand side of (2.2), and the factor $F_i(x,z)$ on the right hand side have singularities at $x = z$. However,

Lemma 2.1. *Restriction of the rational function* (2.2) *to the set of* (x,y,z) *such that* $x = q^{\pm 2}y$, *is regular at* $x = z \neq 0$.

Proof. Let us expand the product on the left hand side of (2.2) in the factor $F_{i+1}(x,z)$. By the definition (2.1) we will get the sum

$$F_i(x,y)T_{i+1}F_i(y,z) + \frac{q-q^{-1}}{x^{-1}z-1}F_i(x,y)F_i(y,z).$$

Here the restriction to $x = q^{\pm 2}y$ of the first summand is evidently regular at $x = z$. After the substitution $y = q^{\mp 2}x$, the second summand takes the form

$$\frac{q-q^{-1}}{x^{-1}z-1}(T_i \mp q^{\pm 1})\left(T_i + \frac{q-q^{-1}}{q^{\pm 2}x^{-1}z-1}\right) = \frac{q-q^{-1}}{x^{-1}z-q^{\mp 2}}(q^{\pm 1} \mp T_i).$$

The rational function of x, z at the right hand side of the last displayed equality is also evidently regular at $x = z$. □

Let Λ be any standard tableau of shape λ. Here we refer to the *Young diagram*

$$\{(a,b) \in \mathbb{Z}^2 \mid 1 \leqslant a,\ 1 \leqslant b \leqslant \lambda_a\} \tag{2.5}$$

of the partition λ. Any bijective function on the set (2.5) with values $1, \dots, l$ is called a *tableau*. The values of this function are the *entries* of the tableau. The symmetric group S_l acts on the set of all tableaux of given shape by permutations of their entries. The tableau Λ is standard if $\Lambda(a,b) < \Lambda(a+1,b)$ and $\Lambda(a,b) < \Lambda(a,b+1)$ for all possible integers a and b. If $\Lambda(a,b) = i$ then put $c_i(\Lambda) = b - a$. The difference $b - a$ here is the *content* corresponding to the node (a,b) of the Young diagram (2.5).

Now introduce l variables $z_1, \dots, z_l \in \mathbb{C}(q)$. Equip the set of all pairs (i, j), where $1 \leqslant i < j \leqslant l$, with the following ordering. The pair (i, j) precedes another pair (i', j') if $j < j'$, or if $j = j'$ but $i < i'$. Take the ordered product

$$\prod_{(i,j)}^{\longrightarrow} F_{j-i}(q^{2c_i(\Lambda)} z_i, q^{2c_j(\Lambda)} z_j) \tag{2.6}$$

over this set. Consider the product (2.6) as a rational function taking values in H_l, of the variables $z_1, \dots, z_l$. Denote this function by $F_\Lambda(z_1, \dots, z_l)$. Let $\mathcal{Z}_\Lambda$ be the vector subspace in $\mathbb{C}(q)^{\times l}$ consisting of all tuples $(z_1, \dots, z_l)$ such that $z_i = z_j$ whenever the numbers i and j appear in the *same column* of the tableau Λ, that is whenever $i = \Lambda(a, b)$ and $j = \Lambda(c, b)$ for some a, b and c. Note that the point $(1, \dots, 1) \in \mathbb{C}(q)^{\times l}$ belongs to the subspace $\mathcal{Z}_\Lambda$.

Theorem 2.2. *The restriction of the rational function $F_\Lambda(z_1, \dots, z_l)$ to the subspace $\mathcal{Z}_\Lambda \subset \mathbb{C}(q)^{\times l}$ is regular at the point $(1, \dots, 1)$.*

Proof. Consider any standard tableau Λ' obtained from the tableau Λ by an adjacent transposition of its entries, say by $\sigma_k \in S_l$. Using the relations (2.2) and (2.3), we derive the equality of rational functions in the variables $z_1, \dots, z_l$

$$\begin{aligned} &F_\Lambda(z_1, \dots, z_l) F_{l-k}(q^{2c_{k+1}(\Lambda)} z_{k+1}, q^{2c_k(\Lambda)} z_k) \\ &\quad = F_k(q^{2c_k(\Lambda)} z_k, q^{2c_{k+l}(\Lambda)} z_{k+1}) F_{\Lambda'}(z'_1, \dots, z'_l), \end{aligned} \tag{2.7}$$

where the sequence of variables $(z'_1, \dots, z'_l)$ is obtained from the sequence $(z_1, \dots, z_l)$ by exchanging the terms z_k and z_{k+1}. Observe that

$$(z'_1, \dots, z'_l) \in \mathcal{Z}_{\Lambda'} \quad \Leftrightarrow \quad (z_1, \dots, z_l) \in \mathcal{Z}_\Lambda.$$

Also observe that here $|c_k(\Lambda) - c_{k+1}(\Lambda)| \geqslant 2$ because the tableaux Λ and Λ' are standard. Therefore the functions

$$F_k(q^{2c_k(\Lambda)} z_k, q^{2c_{k+l}(\Lambda)} z_{k+1}) \qquad \text{and} \qquad F_{l-k}(q^{2c_{k+1}(\Lambda)} z_{k+1}, q^{2c_k(\Lambda)} z_k)$$

appearing in the equality (2.7), are regular at $z_k = z_{k+1} = 1$. Moreover, their values at $z_k = z_{k+1} = 1$ are invertible in the algebra H_l; see the relation (2.4). Due to these two observations, the equality (2.7) shows that Theorem 2.2 is equivalent to its counterpart for the tableau Λ' instead of Λ.

Denote by Λ° the *column tableau* of shape λ. By definition, we have $\Lambda^\circ(a+1, b) = \Lambda^\circ(a, b) + 1$ for all possible nodes (a, b) of the Young diagram (2.5). There is a chain $\Lambda, \Lambda', \dots, \Lambda^\circ$ of standard tableaux of the same shape λ, such that each subsequent tableau in the chain is obtained from the previous one by an adjacent transposition of the entries. Due to the above argument, it now suffices to prove Theorem 2.2 only in the case $\Lambda = \Lambda^\circ$. Note that

$$(T_k - q)^2 = (-q - q^{-1})(T_k - q) \qquad \text{for } k = 1, \dots, l-1. \tag{2.8}$$

Consider the ordered product (2.6) when $\Lambda = \Lambda^\circ$. Suppose that the factor

$$F_{j-i}(q^{2c_i(\Lambda^\circ)} z_i, q^{2c_j(\Lambda^\circ)} z_j) \tag{2.9}$$

in that product has a singularity at $z_i = z_j = 1$. Then $c_i(\Lambda^\circ) = c_j(\Lambda^\circ)$. If here $i = \Lambda^\circ(a, b)$ then $i + 1 = \Lambda^\circ(a + 1, b) < j$. The next factor after (2.9) is

$$F_{j-i-1}(q^{2c_{i+1}(\Lambda^\circ)}z_{i+1}, q^{2c_j(\Lambda^\circ)}z_j) \tag{2.10}$$

where $c_{i+1}(\Lambda^\circ) = c_i(\Lambda^\circ) - 1$. Due to the relations (2.2) and (2.3), the product of all the factors before (2.9) is divisible on the right by

$$F_{j-i-1}(q^{2c_i(\Lambda^\circ)}z_i, q^{2c_{i+1}(\Lambda^\circ)}z_{i+1}). \tag{2.11}$$

Note that the restriction of (2.11) to $z_i = z_{i+1}$ equals $T_{j-i-1} - q$. Also note that the restriction to $z_i = z_{i+1}$ of the ordered product of three factors (2.9)–(2.11) is regular at $z_i = z_j = 1$ due to Lemma 2.1.

Now for every pair (i, j) such that (2.9) is singular at $z_i = z_j = 1$, insert the factor (2.11) divided by $(-q - q^{-1})$ immediately before the two adjacent factors (2.9) and (2.10) in the product (2.6) with $\Lambda = \Lambda^\circ$. These insertions do not alter the values of restriction of the entire product to $\mathcal{Z}_{\Lambda^\circ}$ due to (2.8). But with these insertions, restriction of the product to $\mathcal{Z}_{\Lambda^\circ}$ is evidently regular. □

Due to Theorem 2.2, an element $F_\Lambda \in H_l$ can now be defined as the value at the point $(1, \ldots, 1)$ of the restriction to $\mathcal{Z}_\Lambda$ of the function $F_\Lambda(z_1, \ldots, z_l)$. Note that for $l = 1$ we have $F_\Lambda = 1$. For any $l \geqslant 1$, take the expansion of the element $F_\Lambda \in H_l$ in the basis of the elements T_σ where σ is ranging over S_l.

Proposition 2.3. *The coefficient in $F_\Lambda \in H_l$ of the element T_0 is* 1.

Proof. Expand the product (2.6) as a sum of the elements T_σ with coefficients from the field of rational functions of $z_1, \ldots, z_l$; these functions take values in $\mathbb{C}(q)$. The decomposition in S_l with ordering of the pairs (i, j) as in (2.6)

$$\sigma_0 = \prod_{(i,j)}^{\longrightarrow} \sigma_{j-i}$$

is reduced, hence the coefficient at $T_0 = T_{\sigma_0}$ in the expansion of (2.6) is 1. By the definition of F_Λ, the coefficient of T_0 in F_Λ must also be 1. □

In particular, Proposition 2.3 shows that $F_\Lambda \neq 0$ for any standard tableau Λ. Denote by α_l the involutive antiautomorphism of the algebra H_l over the field $\mathbb{C}(q)$, defined by setting $\alpha_l(T_i) = T_i$ for every index $i = 1, \ldots, l - 1$. Note that each of the Murphy elements $X_1, \ldots, X_l$ of the algebra H_l is α_l-invariant.

Proposition 2.4. *The element $F_\Lambda T_0^{-1}$ is α_l-invariant.*

Proof. Any element of the algebra H_l of the form $F_i(x, y)$ is α_l-invariant. Hence applying the antiautomorphism α_l to an element of H_l the form (2.6) just reverses the ordering of the factors corresponding to the pairs (i, j). Using the relations (2.2) and (2.3), we can rewrite the reversed product as

$$\prod_{(i,j)}^{\longrightarrow} F_{l-j+i}(q^{2c_i(\Lambda)}z_i, q^{2c_j(\Lambda)}z_j)$$

where the pairs (i, j) are again ordered as in (2.6). But due to (1.2) and (1.3), we also have the identity in the algebra H_l

$$F_{l-i}(x, y)T_0 = T_0F_i(x, y).$$

This identity along with the equality $\alpha_l(T_0) = T_0$ implies that every value of the function $F_\Lambda(z_1, \dots, z_l)T_0^{-1}$ is α_l-invariant. So is the element $F_\Lambda T_0^{-1} \in H_l$. □

Proposition 2.5. *If $k = \Lambda(a, b)$ and $k + 1 = \Lambda(a + 1, b)$ then the element $F_\Lambda \in H_l$ is divisible on the left by $T_k - q$.*

Proof. Using the relations (2.2) and (2.3), one demonstrates that the product (2.6) is always divisible on the left by the function

$$F_k(q^{2c_k(\Lambda)}z_k, q^{2c_{k+l}(\Lambda)}z_{k+1}).$$

If here $k = \Lambda(a, b)$ and $k + 1 = \Lambda(a + 1, b)$ then restriction of this function to $z_i = z_{i+1}$ equals $T_k - q$. Hence the required property of the element $F_\Lambda \in H_l$ immediately follows from the definition of this element. □

Fix any standard tableau Λ of shape λ. Let $\rho \in S_l$ be the permutation such that $\Lambda = \rho \cdot \Lambda^\circ$, that is $\Lambda(a, b) = \rho(\Lambda^\circ(a, b))$ for all possible a and b. For any $j = 1, \dots, l$ take the subsequence of the sequence $\rho(1), \dots, \rho(l)$ consisting of all $i < j$ such that $\rho^{-1}(i) > \rho^{-1}(j)$. Denote by $\mathcal{A}_j$ the result of reversing this subsequence. Let $|\mathcal{A}_j|$ be the length of sequence $\mathcal{A}_j$. We have a reduced decomposition in the symmetric group S_l,

$$\rho = \overrightarrow{\prod_{j=1,\dots,l}} \left(\overrightarrow{\prod_{k=1,\dots,|\mathcal{A}_j|}} \sigma_{j-k} \right). \tag{2.12}$$

Let $\sigma_{i_L} \cdots \sigma_{i_1}$ be the product of adjacent transpositions at the right hand side of (2.12). For each tail $\sigma_{i_K} \cdots \sigma_{i_1}$ of this product, the image $\sigma_{i_K} \cdots \sigma_{i_1} \cdot \Lambda^\circ$ is a standard tableau. This can easily be proved by induction on the length $K = L, \dots, 1$ of the tail; see also the proof of Proposition 2.6 below. Note that for any $i \in \mathcal{A}_j$ and $k \in \{1, \dots, l-1\}$ the elements of the algebra H_l,

$$F_k(q^{2c_i(\Lambda)}, q^{2c_j(\Lambda)}) \qquad \text{and} \qquad F_k(q^{2c_j(\Lambda)}, q^{2c_i(\Lambda)}),$$

are well defined and invertible. Indeed, if $i = \Lambda(a, b)$ and $j = \Lambda(c, d)$ for some a, b and c, d then $a < c$ and $b > d$. So $c_i(\Lambda) - c_j(\Lambda) = b - a - d + c \geqslant 2$ here.

Proposition 2.6. *We have the equality in the algebra H_l*

$$F_\Lambda \cdot \overrightarrow{\prod_{j=1,\dots,l}} \left(\overrightarrow{\prod_{k=1,\dots,|\mathcal{A}_j|}} F_{l-j+k}(q^{2c_j(\Lambda)}, q^{2c_i(\Lambda)}) \right)$$
$$= \overrightarrow{\prod_{j=1,\dots,l}} \left(\overrightarrow{\prod_{k=1,\dots,|\mathcal{A}_j|}} F_{j-k}(q^{2c_i(\Lambda)}, q^{2c_j(\Lambda)}) \right) \cdot F_{\Lambda^\circ} \qquad \text{where } i = \mathcal{A}_j(k).$$

Proof. We will proceed by induction on the length $L = |\mathcal{A}_1| + \cdots + |\mathcal{A}_l|$ of the element $\rho \in S_l$. Let n be the minimal of the indices j such that the sequence $\mathcal{A}_j$ is not empty. Then we have $\mathcal{A}_n(1) = n-1$. Indeed, if $\mathcal{A}_n(1) < n-1$ then $\rho^{-1}(\mathcal{A}_n(1)) > \rho^{-1}(n-1)$. Then $\mathcal{A}_n(1) \in \mathcal{A}_{n-1}$, which would contradict the minimality of n. The tableau $\sigma_{n-1} \cdot \Lambda$ is standard; denote it by Λ'. In our proof of Theorem 2.2 we used the equality (2.7). Setting $k = n-1$ in that equality and then using Theorem 2.2 itself, we obtain the equality in H_l

$$F_\Lambda F_{l-n+1}(q^{2c_n(\Lambda)}, q^{2c_{n-1}(\Lambda)}) = F_{n-1}(q^{2c_{n-1}(\Lambda)}, q^{2c_n(\Lambda)}) F_{\Lambda'}. \tag{2.13}$$

For each index $j = 1, \ldots, l$ denote by $\mathcal{A}'_j$ the counterpart of the sequence $\mathcal{A}_j$ for the standard tableau Λ' instead of Λ. Each of the sequences $\mathcal{A}'_1, \ldots, \mathcal{A}'_{n-2}$ and $\mathcal{A}'_n$ is empty. The sequence $\mathcal{A}'_{n-1}$ is obtained from the sequence $\mathcal{A}_n$ by removing its first term $\mathcal{A}_n(1) = n-1$. By replacing the terms $n-1$ and n, whenever any of them occurs, respectively by n and $n-1$ in all the sequences $\mathcal{A}_{n+1}, \ldots, \mathcal{A}_l$ we obtain the sequences $\mathcal{A}'_{n+1}, \ldots, \mathcal{A}'_l$.

Assume that the Proposition 2.6 is true for Λ' instead of Λ. Write the product on the left hand side of the equality to be proved in Proposition 2.6 as

$$F_\Lambda F_{l-n+1}(q^{2c_n(\Lambda)}, q^{2c_{n-1}(\Lambda)}) \cdot \overrightarrow{\prod_{k=2,\ldots,|\mathcal{A}_n|}} F_{l-n+k}(q^{2c_n(\Lambda)}, q^{2c_i(\Lambda)})$$
$$\times \overrightarrow{\prod_{j=n+1,\ldots,l}} \left(\overrightarrow{\prod_{k=1,\ldots,|\mathcal{A}_j|}} F_{l-j+k}(q^{2c_j(\Lambda)}, q^{2c_i(\Lambda)}) \right)$$

where in the first line $i = \mathcal{A}_n(k)$, while in the second line $i = \mathcal{A}_j(k)$. Using the equality (2.13) and the description of the sequences $\mathcal{A}'_1, \ldots, \mathcal{A}'_l$ as given above, the latter product can be rewritten as

$$F_{n-1}\left(q^{2c_{n-1}(\Lambda)}, q^{2c_n(\Lambda)}\right) F_{\Lambda'}$$
$$\times \overrightarrow{\prod_{j=1,\ldots,l}} \left(\overrightarrow{\prod_{k=1,\ldots,|\mathcal{A}'_j|}} F_{l-j+k}(q^{2c_j(\Lambda')}, q^{2c_i(\Lambda')}) \right) \qquad \text{where } i = \mathcal{A}'_j(k).$$

By the inductive assumption, this product equals

$$F_{n-1}\left(q^{2c_{n-1}(\Lambda)}, q^{2c_n(\Lambda)}\right) \cdot \overrightarrow{\prod_{j=1,\ldots,l}} \left(\overrightarrow{\prod_{k=1,\ldots,|\mathcal{A}'_j|}} F_{j-k}(q^{2c_i(\Lambda')}, q^{2c_j(\Lambda')}) \right)$$

times F_{Λ°, where we keep to the notation $i = \mathcal{A}'_j(k)$. Using the description of the sequences $\mathcal{A}'_1, \ldots, \mathcal{A}'_l$ once again, the last product can be rewritten as on the right hand side of the equality to be proved in Proposition 2.6. □

Proposition 2.7. *If $k = \Lambda(a,b)$ and $k+1 = \Lambda(a,b+1)$ then the element $F_\Lambda \in H_l$ is divisible on the left by $T_k + q^{-1}$.*

Proof. Given a pair of indices (a, b) such that $\lambda_a > b$, it suffices to prove Proposition 2.7 for just one standard tableau Λ of shape λ. Indeed, let $\tilde{\Lambda}$ be another standard tableau of the same shape, such that $\tilde{\Lambda}(a, b) = \tilde{k}$ and $\tilde{\Lambda}(a, b+1) = \tilde{k}+1$ for some $\tilde{k} \in \{1, \dots, l-1\}$. Let σ be the permutation such that $\tilde{\Lambda} = \sigma \cdot \Lambda$. There is a decomposition $\sigma = \sigma_{i_N} \cdots \sigma_{i_1}$ such that for each $K = 1, \dots, N-1$ the tableau $\Lambda_K = \sigma_{i_K} \cdots \sigma_{i_1} \cdot \Lambda$ is standard. Note that this decomposition is not necessarily reduced. Using Theorem 2.2, we get

$$\overleftarrow{\prod_{K=1,\dots,N}} F_{i_K}(q^{2c_{i_K}(\Lambda_K)}, q^{2c_{i_K+1}(\Lambda_K)}) \cdot F_\Lambda = F_{\tilde{\Lambda}} \cdot \overleftarrow{\prod_{K=1,\dots,N}} F_{l-i_K}(q^{2c_{i_K+1}(\Lambda_K)}, q^{2c_{i_K}(\Lambda_K)}) \tag{2.14}$$

where $\Lambda_N = \tilde{\Lambda}$. Note that here for every $K = 1, \dots, N$ the factor

$$F_{l-i_K}(q^{2c_{i_K+1}(\Lambda_K)}, q^{2c_{i_K}(\Lambda_K)})$$

is invertible. Further, we have the equality $\sigma\sigma_k = \sigma_{\tilde{k}}\sigma$ by the definition of the permutation σ. Using the relations (2.2) and (2.3), we obtain the equality

$$\overleftarrow{\prod_{K=1,\dots,N}} F_{i_K}(q^{2c_{i_K}(\Lambda_K)}, q^{2c_{i_K+1}(\Lambda_K)}) \cdot F_k(q^{2c_k(\Lambda)}, q^{2c_{k+1}(\Lambda)}) = F_{\tilde{k}}(q^{2c_{\tilde{k}}(\tilde{\Lambda})}, q^{2c_{\tilde{k}+1}(\tilde{\Lambda})}) \cdot \overleftarrow{\prod_{K=1,\dots,N}} F_{l-i_K}(q^{2c_{i_K+1}(\Lambda_K)}, q^{2c_{i_K}(\Lambda_K)}).$$

The last equality along with the equality (2.14) shows, that Proposition 2.7 implies its counterpart for the tableau $\tilde{\Lambda}$ and the index $\tilde{k}$, instead of Λ and k respectively. Here we also use the equalities

$$F_k(q^{2c_k(\Lambda)}, q^{2c_{k+1}(\Lambda)}) = T_k + q^{-1},$$
$$F_{\tilde{k}}(q^{2c_{\tilde{k}}(\tilde{\Lambda})}, q^{2c_{\tilde{k}+1}(\tilde{\Lambda})}) = T_{\tilde{k}} + q^{-1}.$$

Let us consider the column tableau Λ° of shape λ. Put $m = \Lambda^\circ(a, b)$. Also put $n = \Lambda^\circ(\lambda_b^*, b)$; then $\Lambda^\circ(a, b+1) = n + a$. We will prove that the element $F_{\Lambda^\circ} \in H_l$ is divisible on the left by the product

$$\overleftarrow{\prod_{i=m,\dots,n}} \left(\overrightarrow{\prod_{j=n+1,\dots,n+a}} F_{i+j-n-1}(q^{2c_i(\Lambda^\circ)}, q^{2c_j(\Lambda^\circ)}) \right). \tag{2.15}$$

Then Proposition 2.7 will follow. Indeed, put $k = m + a - 1$; this is the value of the index $i + j - n - 1$ in (2.15) when $i = m$ and $n = n + a$. Let Λ be the tableau such that Λ° is obtained from the tableau $\sigma_k \cdot \Lambda$ by the permutation

$$\overleftarrow{\prod_{i=m,\dots,n}} \left(\overrightarrow{\prod_{j=n+1,\dots,n+a}} \sigma_{i+j-n-1} \right).$$

The tableau Λ is standard. Moreover, then $\Lambda(a,b)=k$ and $\Lambda(a,b+1)=k+1$. Note that the rightmost factor in the product (2.15), corresponding to $i=m$ and $n=n+a$, is

$$F_{m+a-1}(q^{2c_m(\Lambda^\circ)},q^{2c_{n+a}(\Lambda^\circ)})=T_k+q^{-1}.$$

Denote by F the product of all factors in (2.15) but the rightmost one. Further, denote by G the product obtained by replacing each factor in F

$$F_{i+j-n-1}(q^{2c_i(\Lambda^\circ)},q^{2c_j(\Lambda^\circ)})$$

respectively by

$$F_{l-i-j+n+1}(q^{2c_j(\Lambda^\circ)},q^{2c_i(\Lambda^\circ)}).$$

The element $F\in H_l$ is invertible, and we have $FF_\Lambda=F_{\Lambda^\circ}G$. Therefore the divisibility of the element F_{Λ° on the left by the product (2.15) will imply the divisibility of the element F_Λ on the left by T_k+q^{-1}.

Take the tableau obtained from Λ° by removing the entries $n+a+1,\dots,l$. This is the column tableau corresponding to a certain partition of $n+a$; let us denote this tableau by Υ°. The proof of Theorem 2.2 shows that the element $F_{\Lambda^\circ}\in H_l$ is divisible on the left by the element $F_{\Upsilon^\circ}\in H_{n+a}$. Here we use the standard embedding $H_{n+a}\to H_l$ where $T_i\mapsto T_i$ for each $i=1,\dots,n-a-1$. Hence it suffices to prove the divisibility of the element $F_{\Upsilon^\circ}\in H_{n+a}$ on the left by the product (2.15). Therefore it suffices to consider only the case where $n+a=l$. We will actually prove that F_{Λ° is divisible on the right by

$$\overrightarrow{\prod_{i=m,\dots,n}}\left(\overleftarrow{\prod_{j=n+1,\dots,l}}F_{l-i-j+n+1}(q^{2c_i(\Lambda^\circ)},q^{2c_j(\Lambda^\circ)})\right). \tag{2.16}$$

The divisibility of F_{Λ° on the left by the product (2.15) where $n+a=l$ will then follow by Proposition 2.4.

The element $F_{\Lambda^\circ}\in H_l$ is the value at the point $(1,\dots,1)$ of the restriction to the subspace $\mathcal{Z}_{\Lambda^\circ}\subset\mathbb{C}(q)^{\times l}$ of the rational function $F_{\Lambda^\circ}(z_1,\dots,z_l)$. This function has been defined as the ordered product (2.6) where $\Lambda=\Lambda^\circ$. Let us change the ordering of the pairs (i,j) in (2.6) to the *lexicographical* one, so that now the pair (i,j) precedes another pair (i',j') if $i<i'$, or if $i=i'$ but $j<j'$. This reordering does not alter any value of the function $F_{\Lambda^\circ}(z_1,\dots,z_l)$ due to the relations (2.3). Using the new ordering, we can once again prove that the restriction of $F_{\Lambda^\circ}(z_1,\dots,z_l)$ to the subspace $\mathcal{Z}_{\Lambda^\circ}$ is regular at the point $(1,\dots,1)$. Indeed, take any factor (2.9) in the product (2.6) such that $c_i(\Lambda^\circ)=c_j(\Lambda^\circ)$. If here $j=\Lambda^\circ(a,b)$ then $j-1=\Lambda^\circ(a-1,b)>i$. The factor in (2.6) immediately before (2.9) is now

$$F_{j-i-1}(q^{2c_i(\Lambda^\circ)}z_i,q^{2c_{j-1}(\Lambda^\circ)}z_{j-1}) \tag{2.17}$$

where $c_{j-1}(\Lambda^\circ)=c_j(\Lambda^\circ)+1$. Due to the relations (2.2) and (2.3), the product of all the factors after (2.9) is divisible on the left by

$$F_{j-i-1}(q^{2c_{j-1}(\Lambda^\circ)}z_{j-1},q^{2c_j(\Lambda^\circ)}z_j). \tag{2.18}$$

The restriction to $z_{j-1}=z_j$ of the ordered product of the three factors (2.9), (2.17) and (2.18) is regular at $z_i=z_j=1$; cf. Lemma 2.1.

With the new ordering, consider the product of all those factors in (2.6) where $i \geqslant m$. Any such factor is regular at $z_i = z_j = 1$, because we are considering only the case $n + a = l$. At the point $(z_1, \dots, z_l) = (1, \dots, 1)$, the product of these factors takes the value

$$\prod_{i=m,\dots,l-1}^{\rightarrow} \left(\prod_{j=i+1,\dots,l}^{\rightarrow} F_{j-i}(q^{2c_i(\Lambda^\circ)}, q^{2c_j(\Lambda^\circ)}) \right). \tag{2.19}$$

The argument given in the previous paragraph not only shows that the restriction of $F_{\Lambda^\circ}(z_1, \dots, z_l)$ to the subspace $\mathcal{Z}_{\Lambda^\circ}$ is regular at $(1, \dots, 1)$, it also shows that the element F_{Λ° is divisible on the right by the product (2.19). Using (2.2) and (2.3), the product (2.19) is equal to (2.16) multiplied on the left by

$$\prod_{i=m,\dots,n-1}^{\rightarrow} \left(\prod_{j=i+1,\dots,n}^{\rightarrow} F_{j-i}(q^{2c_i(\Lambda^\circ)}, q^{2c_j(\Lambda^\circ)}) \right) \times \prod_{i=n+1,\dots,l-1}^{\rightarrow} \left(\prod_{j=i+1,\dots,l}^{\rightarrow} F_{j-i+n-m+1}(q^{2c_i(\Lambda^\circ)}, q^{2c_j(\Lambda^\circ)}) \right). \quad \square$$

Let us now regard F_Λ as an element of the algebra H_{l+1}, by using the standard embedding $H_l \to H_{l+1}$ where $T_i \mapsto T_i$ for any $i = 1, \dots, l-1$.

Proposition 2.8. *We have equality of rational functions in z, valued in H_{l+1}*

$$\prod_{k=1,\dots,l}^{\rightarrow} F_k(z, q^{2c_k(\Lambda)}) \cdot F_\Lambda = \frac{T_1 \cdots T_l - z T_1^{-1} \cdots T_l^{-1}}{1 - z} \cdot F_\Lambda.$$

Proof. Denote by $F(z)$ the rational function with the values in H_{l+1}, defined as the product of the left hand side of the equality to be proved. Note that

$$F(0) = T_1 \cdots T_l F_\Lambda \qquad \text{and} \qquad F(\infty) = T_1^{-1} \cdots T_l^{-1} F_\Lambda$$

due to (1.7). It remains to show that $F(z)$ may have a pole only at $z = 1$ and that this pole is simple. Since $c_1(\Lambda) = 0$ the factor $F_1(z, q^{2c_1(\Lambda)})$ in the product defining $F(z)$ has a simple pole at $z = 1$. Take any $z_0 \in \mathbb{C}(q)$. Suppose there is an index $j \in \{2, \dots, l\}$ such that $z_0 = q^{2c_j(\Lambda)}$. The factor $F_j(z, q^{2c_j(\Lambda)})$ has a pole at $z = z_0$. We shall prove that when we estimate the order of the pole of $F(z)$ at $z = z_0$ from above, any factor with $j > 1$ does not count.

Let $i \in \{1, \dots, j-1\}$ be the maximal index such that $|c_i(\Lambda) - c_j(\Lambda)| = 1$. Note that $i = \Lambda(a, b)$; then either $j = \Lambda(a+1, b)$ or $j = \Lambda(a, b+1)$. Consider the sequence of tableaux of shape λ,

$$\Lambda' = \sigma_{j-1} \cdot \Lambda\,, \Lambda'' = \sigma_{j-2} \cdot \Lambda', \dots, \Lambda^{(j-i-1)} = \sigma_{i+1} \cdot \Lambda^{(j-i-2)}.$$

Each of these tableaux is standard. Using this sequence, we obtain the relation

$$F_\Lambda \cdot \prod_{k=i+1,\dots,j-1}^{\leftarrow} F_{l-k}(q^{2c_j(\Lambda)}, q^{2c_k(\Lambda)}) = \prod_{k=i+1,\dots,j-1}^{\leftarrow} F_k(q^{2c_k(\Lambda)}, q^{2c_j(\Lambda)}) \cdot F_{\Lambda^{(j-i-1)}} \tag{2.20}$$

in the algebra H_l; cf. the proof of Proposition 2.6. Each of the factors

$$F_{l-k}(q^{2c_j(\Lambda)}, q^{2c_k(\Lambda)})$$

in (2.20) is invertible. The entries i and $i+1$ of the tableau $\Lambda^{(j-i-1)}$ correspond to the same nodes of the Young diagram (2.5) as the entries i and j of the tableau Λ respectively. Using either Proposition 2.5 or 2.7, the element $F_{\Lambda^{(j-i-1)}}$ is divisible on the left by

$$F_i(q^{2c_i(\Lambda)}, q^{2c_j(\Lambda)}) = T_i \mp q^{\pm}.$$

The relation (2.20) now shows that the element F_Λ is divisible on the left by

$$\prod_{k=i,\dots,j-1}^{\leftarrow} F_k(q^{2c_k(\Lambda)}, q^{2c_j(\Lambda)}).$$

Using the relations (2.2) and (2.3), we obtain an equality in the algebra H_{l+1}

$$\begin{aligned} &\prod_{k=1,\dots,l}^{\rightarrow} F_k(z, q^{2c_k(\Lambda)}) \cdot \prod_{k=i,\dots,j-1}^{\leftarrow} F_k(q^{2c_k(\Lambda)}, q^{2c_j(\Lambda)}) \\ &\quad = \prod_{k=1,\dots,i-1}^{\rightarrow} F_k(z, q^{2c_k(\Lambda)}) \cdot \prod_{k=i+1,\dots,j-1}^{\leftarrow} F_{k+1}(q^{2c_k(\Lambda)}, q^{2c_j(\Lambda)}) \\ &\qquad \times F_i(z, q^{2c_i(\Lambda)}) F_{i+1}(z, q^{2c_j(\Lambda)}) F_i(q^{2c_i(\Lambda)}, q^{2c_j(\Lambda)}) \\ &\qquad \times \prod_{k=i+1,\dots,j-1}^{\rightarrow} F_{k+1}(z, q^{2c_k(\Lambda)}) \cdot \prod_{k=j+1,\dots,l}^{\rightarrow} F_k(z, q^{2c_k(\Lambda)}). \end{aligned} \tag{2.21}$$

The product in the line (2.21) above is regular at $z = q^{2c_j(\Lambda)}$; cf. Lemma 2.1.

Now take any other index $j' \neq j$ such that $c_j(\Lambda) = c_{j'}(\Lambda)$. We assume that $j' > 1$. Let $i' \in \{1, \dots, j'-1\}$ be the corresponding maximal index such that $|c_{i'}(\Lambda) - c_{j'}(\Lambda)| = 1$. If $j' > j$, then also $i' > j$ because the tableau Λ is standard. Thus the two sets of indices $\{i', \dots, j'\}$ and $\{i, \dots, j\}$ are always disjoint. Therefore we can apply the above argument to both factors $F_j(z, q^{2c_j(\Lambda)})$ and $F_{j'}(z, q^{2c_{j'}(\Lambda)})$ in the product defining $F(z)$ simultaneously, and so on. In this way we show that when estimating from above the order of the pole of the function $F(z)$ at $z = z_0$, all the factors $F_j(z, q^{2c_j(\Lambda)})$ where $z_0 = q^{2c_j(\Lambda)}$ but $j > 1$, do not count. □

Now denote by ι the embedding $H_l \to H_{l+1}$ defined by setting $\iota(T_i) = T_{i+1}$.

Proposition 2.9. *We have the equality*

$$\prod_{k=1,\dots,l}^{\rightarrow} F_k(z, q^{2c_k(\Lambda)}) \cdot F_\Lambda = \iota(F_\Lambda) \cdot \prod_{k=1,\dots,l}^{\leftarrow} F_{l-k+1}(z, q^{2c_k(\Lambda)}).$$

Proof. Take the variables $z_1, \dots, z_l \in \mathbb{C}(q)$. Using the relations (2.2) and (2.3) and the definition (2.6) of $F_\Lambda(z_1, \dots, z_l)$ we obtain the equality of rational functions in the variables $z, z_1, \dots, z_l$

$$\prod_{k=1,\dots,l}^{\rightarrow} F_k(z, q^{2c_k(\Lambda)} z_k) \cdot F_\Lambda(z_1, \dots, z_l)$$
$$= \iota(F_\Lambda(z_1, \dots, z_l)) \cdot \prod_{k=1,\dots,l}^{\leftarrow} F_{l-k+1}(z, q^{2c_k(\Lambda)} z_k).$$

Restricting, in the above displayed equality, the function $F_\Lambda(z_1, \dots, z_l)$ to $\mathcal{Z}_\Lambda$, and then evaluating the restriction at the point $(1, \dots, 1) \in \mathcal{Z}_\Lambda$, we derive Proposition 2.9 from Theorem 2.2. □

3. Young symmetrizers for the algebra H_l

For every standard tableau Λ of shape λ we have defined an element F_Λ of the algebra H_l. Let us now assign to Λ another element of H_l, which will be denoted by G_Λ. Let $\rho \in S_l$ be the permutation such that $\Lambda = \rho \cdot \Lambda^\circ$, as it was in Section 2. For any $j = 1, \dots, l$ denote by $\mathcal{B}_j$ the subsequence of the sequence $\rho(1), \dots, \rho(l)$ consisting of all $i < j$ such that $\rho^{-1}(i) < \rho^{-1}(j)$. Note that we have a reduced decomposition in the symmetric group S_l,

$$\rho\sigma_0 = \prod_{j=1,\dots,l}^{\rightarrow} \left(\prod_{k=1,\dots,|\mathcal{B}_j|}^{\rightarrow} \sigma_{j-k} \right)$$

where $|\mathcal{B}_j|$ is the length of sequence $\mathcal{B}_j$; cf. the reduced decomposition (2.12). Consider the rational function taking values in H_l, of the variables $z_1, \dots, z_l$

$$\prod_{j=1,\dots,l}^{\rightarrow} \left(\prod_{k=1,\dots,|\mathcal{B}_j|}^{\rightarrow} F_{j-k}(q^{2c_i(\Lambda)} z_i, q^{2c_j(\Lambda)} z_j) \right) \qquad \text{where } i = \mathcal{B}_j(k).$$

Denote this rational function by $G_\Lambda(z_1, \dots, z_l)$. Using induction on the length of the element $\rho \in S_l$ as in the proof of Proposition 2.6, one can prove that

$$F_\Lambda(z_1, \dots, z_l) = G_\Lambda(z_1, \dots, z_l)$$
$$\times \prod_{j=1,\dots,l}^{\leftarrow} \left(\prod_{k=1,\dots,|\mathcal{A}_j|}^{\leftarrow} F_{l-j+k}(q^{2c_i(\Lambda)} z_i, q^{2c_j(\Lambda)} z_j) \right) \qquad \text{where } i = \mathcal{A}_j(k).$$

Therefore, restriction of $G_\Lambda(z_1, \dots, z_l)$ to the subspace $\mathcal{Z}_\Lambda \subset \mathbb{C}(q)^{\times l}$ is regular at the point $(1, \dots, 1)$ due to Theorem 2.2. The value of that restriction at $(1, \dots, 1)$ is our

element $G_\Lambda \in H_l$ by definition. Moreover, then F_Λ equals

$$G_\Lambda \cdot \prod_{j=1,\dots,l}^{\leftarrow} \left(\prod_{k=1,\dots,|\mathcal{A}_j|}^{\leftarrow} F_{l-j+k}(q^{2c_i(\Lambda)}, q^{2c_j(\Lambda)}) \right) \qquad \text{where } i = \mathcal{A}_j(k).$$

Using the relation (2.4), this factorization of F_Λ implies that the left hand side of the equality in Proposition 2.6 also equals G_Λ times

$$\prod_{j=1,\dots,l} \left(\prod_{k=1,\dots,|\mathcal{A}_j|} \left(1 - \frac{(q-q^{-1})^2 q^{2c_i(\Lambda)+2c_j(\Lambda)}}{(q^{2c_i(\Lambda)} - q^{2c_j(\Lambda)})^2} \right) \right) \qquad \text{where } i = \mathcal{A}_j(k).$$

Rewriting the factors of the last displayed product, Proposition 2.6 yields

Corollary 3.1. *We have the equality in the algebra H_l*

$$\prod_{j=1,\dots,l} \left(\prod_{k=1,\dots,|\mathcal{A}_j|} \left(1 - \frac{(q-q^{-1})^2}{(q^{c_i(\Lambda)-c_j(\Lambda)} - q^{c_j(\Lambda)-c_i(\Lambda)})^2} \right) \right) \cdot G_\Lambda$$
$$= \prod_{j=1,\dots,l}^{\rightarrow} \left(\prod_{k=1,\dots,|\mathcal{A}_j|}^{\rightarrow} F_{j-k}(q^{2c_i(\Lambda)}, q^{2c_j(\Lambda)}) \right) \cdot F_{\Lambda^\circ} \qquad \textit{where } i = \mathcal{A}_j(k).$$

Yet arguing like in the proof of Proposition 2.3, the definition of G_Λ implies

Proposition 3.2. *The element G_Λ equals $T_{\rho\sigma_0}$ plus a sum of the elements T_σ with certain non-zero coefficients from $\mathbb{C}(q)$, where the length of each $\sigma \in S_l$ is less than that of $\rho\sigma_0$.*

Note that $G_{\Lambda^\circ} = F_{\Lambda^\circ}$ by definition. Denote by V_λ the left ideal in the algebra H_l generated by the element F_{Λ°. Due to Corollary 3.1 we have $G_\Lambda \in V_\lambda$ for any standard tableau Λ of shape λ. Proposition 3.2 shows that the elements $G_\Lambda \in H_l$ for all pairwise distinct standard tableaux Λ of shape λ are linearly independent. The next theorem implies, in particular, that these elements also span the vector space V_λ.

For any $k = 1, \dots, l-1$ use the notation $d_k(\Lambda) = c_k(\Lambda) - c_{k+1}(\Lambda)$. If the tableau $\sigma_k\Lambda$ is not standard, then the numbers k and $k+1$ stand *next to each other* in the same row or in the same column of Λ; that is $k+1 = \Lambda(a, b+1)$ or $k+1 = \Lambda(a+1, b)$ for $k = \Lambda(a, b)$. Then we have $d_k(\Lambda) = -1$ or $d_k(\Lambda) = 1$ respectively. But if the tableau $\sigma_k\Lambda$ is standard, then we have $|d_k(\Lambda)| \geqslant 2$.

Theorem 3.3. *For any standard tableau Λ and any $k = 1, \dots, l-1$ we have:*

(a)
$$T_k G_\Lambda = \begin{cases} qG_\Lambda & \text{if } d_k(\Lambda) = -1, \\ -q^{-1}G_\Lambda & \text{if } d_k(\Lambda) = 1; \end{cases}$$

(b)
$$T_k G_\Lambda = \frac{q-q^{-1}}{1-q^{2d_k(\Lambda)}} G_\Lambda + G_{\sigma_k\Lambda} \times \begin{cases} 1 - \dfrac{(q-q^{-1})^2}{(q^{d_k(\Lambda)} - q^{-d_k(\Lambda)})^2} & \text{if } d_k(\Lambda) \leqslant -2, \\ 1 & \text{if } d_k(\Lambda) \geqslant 2. \end{cases}$$

Proof. The element $G_\Lambda \in H_l$ is obtained by multiplying F_Λ on the right by a certain element of H_l. Hence Part (a) of Theorem 3.3 immediately follows from Propositions 2.5 and 2.7. Now suppose that the tableau $\sigma_k\Lambda$ is standard. Moreover, suppose that $d_k(\Lambda) \geqslant 2$; in this case we have $k \in \mathcal{A}_{k+1}$. Using Corollary 3.1 along with the relations (2.2) and (2.3), one can get the equality

$$\left(1 - \frac{(q-q^{-1})^2}{(q^{d_k(\Lambda)} - q^{-d_k(\Lambda)})^2}\right) G_\Lambda = F_k(q^{2c_k(\Lambda)}, q^{2c_{k+1}(\Lambda)}) G_{\sigma_k\Lambda}. \tag{3.1}$$

Using the relation (2.4), we obtain from (3.1) the equality

$$F_k(q^{2c_{k+1}(\Lambda)}, q^{2c_k(\Lambda)}) G_\Lambda = G_{\sigma_k\Lambda}.$$

The last equality implies Part (b) of Theorem 3.3 in the case when $d_k(\Lambda) \geqslant 2$; see the definition (2.1). Exchanging the tableaux Λ and $\sigma_k\Lambda$ in (3.1), so that the resulting equality applies in the case where $d_k(\Lambda) \leqslant -2$, we prove Part (b) of Theorem 3.3 in this remaining case. □

Thus the elements $G_\Lambda \in H_l$ for all pairwise distinct standard tableaux Λ of shape λ form a basis in the vector space V_λ. This basis is distinguished due to

Proposition 3.4. *We have* $X_i G_\Lambda = q^{2c_i(\Lambda)} G_\Lambda$ *for each* $i = 1, \ldots, l$.

Proof. We will proceed by induction on $i = 1, \ldots, l$. By definition, $X_1 = 1$. On the other hand, $c_1(\Lambda) = 0$ for any standard tableau Λ. Thus Proposition 3.4 is true for $i = 1$. Now suppose that Proposition 3.4 is true for $i = k$ where $k < l$. To show that it is also true for $i = k+1$, we will use Theorem 3.3. Note that $X_{k+1} = T_k X_k T_k$. If $d_k(\Lambda) = \pm 1$, then $T_k X_k T_k G_\Lambda$ equals

$$\mp q^{\mp 1} T_k X_k G_\Lambda = \mp q^{2c_k(\Lambda)\mp 1} T_k G_\Lambda = q^{2c_k(\Lambda)\mp 2} G_\Lambda = q^{2c_{k+1}(\Lambda)} G_\Lambda$$

respectively. If $d_k(\Lambda) \geqslant 2$, then the product $T_k X_k T_k G_\Lambda$ equals

$$\begin{aligned}
&T_k X_k \left(\frac{q-q^{-1}}{1-q^{2d_k(\Lambda)}} G_\Lambda + G_{\sigma_k\Lambda}\right) \\
&\quad = q^{2c_{k+1}(\Lambda)} T_k \left(\frac{q-q^{-1}}{q^{-2d_k(\Lambda)}-1} G_\Lambda + G_{\sigma_k\Lambda}\right) \\
&\quad = q^{2c_{k+1}(\Lambda)} \left(\frac{q-q^{-1}}{q^{-2d_k(\Lambda)}-1}\left(\frac{q-q^{-1}}{1-q^{2d_k(\Lambda)}} G_\Lambda + G_{\sigma_k\Lambda}\right)\right. \\
&\qquad \left. + \frac{q-q^{-1}}{1-q^{-2d_k(\Lambda)}} G_{\sigma_k\Lambda} + \left(1 - \frac{(q-q^{-1})^2}{(q^{d_k(\Lambda)} - q^{-d_k(\Lambda)})^2}\right) G_\Lambda\right) = q^{2c_{k+1}(\Lambda)} G_\Lambda.
\end{aligned}$$

In the case where $d_k(\Lambda) \leqslant -2$, the proof of the equality $X_{k+1} G_\Lambda = q^{2c_{k+1}(\Lambda)} G_\Lambda$ is similar and is omitted here. □

Let us now consider the left ideal $V_\lambda \subset H_l$ as the H_l-module. Here the algebra H_l acts via left multiplication.

Corollary 3.5. *The H_l-module V_λ is irreducible.*

Proof. The vectors $G_\Lambda \in V_\lambda$ where Λ is ranging over the set of all standard tableaux of the given shape λ, form an eigenbasis for the action on V_λ of the Murphy elements $X_1, \dots, X_l \in H_l$. Moreover, the ordered collections of the corresponding eigenvalues $q^{2c_1(\Lambda)}, \dots, q^{2c_l(\Lambda)}$ are pairwise distinct for all different tableaux Λ. On the other hand, by Corollary 3.1 any basis vector $G_\Lambda \in V_\lambda$ can be obtained by acting on the element $G_{\Lambda^\circ} \in V_\lambda$ with a certain invertible element of H_l. □

Corollary 3.6. *The H_l-modules V_λ for different partitions λ of l are pairwise non-equivalent.*

Proof. Take any symmetric polynomial f in l variables over the field $\mathbb{C}(q)$. For all standard tableaux Λ of the same shape λ, the values of this polynomial

$$f(q^{2c_1(\Lambda)}, \dots, q^{2c_l(\Lambda)}) \in \mathbb{C}(q) \tag{3.2}$$

are the same. Hence by Proposition 3.4, the element $f(X_1, \dots, X_l) \in H_l$ acts on V_λ via multiplication by the scalar (3.2). On the other hand, the partition λ can be uniquely restored from the values (3.2) where the polynomial f varies. Thus the H_l-modules V_λ with different partitions λ cannot be equivalent. □

Remark. The centre of the algebra $\widehat{H}_l$ consists of all the Laurent polynomials in the generators $Y_1, \dots, Y_l$ which are invariant under permutations of these generators; see for instance [12, Proposition 3.11]. In particular, the element $f(X_1, \dots, X_l) \in H_l$ is central, as the image of a central element of $\widehat{H}_l$ under the homomorphism π. Moreover, the centre of the algebra H_l coincides with the collection of all elements $f(X_1, \dots, X_l)$ where the symmetric polynomial f varies; cf. [8]. However, we do not use any of these facts in this section. □

For any $k = 1, \dots, l-1$ consider the restriction of the H_l-module V_λ to the subalgebra $H_k \subset H_l$. We use the standard embedding $H_k \to H_l$, where $T_i \mapsto T_i$ for each index $i = 1, \dots, k-1$.

Corollary 3.7. *The vector $G_\Lambda \in V_\lambda$ belongs to the H_k-invariant subspace in V_λ, equivalent to the H_k-module V_κ where the partition κ is the shape of the tableau obtained by removing from Λ the entries $k+1, \dots, l$.*

Proof. It suffices to consider the case $k = l-1$ only. For each index a such that $\lambda_a > \lambda_{a+1}$, denote by V_a the vector subspace in V_λ spanned by the all those vectors G_Λ where $\Lambda(a, \lambda_a) = l$. By Theorem 3.3, the subspace V_a is preserved by the action of the subalgebra $H_{l-1} \subset H_l$ on V_λ. Moreover, Theorem 3.3 shows that the H_{l-1}-module V_a is equivalent to V_κ where the partition κ of $l-1$ is obtained by decreasing the ath part of λ by 1. □

The properties of the vector G_Λ given by Corollary 3.7 for $k = 1, \dots, l-1$, determine this vector in V_λ uniquely up to a non-zero factor from $\mathbb{C}(q)$. These properties can be restated for any irreducible H_l-module V equivalent to V_λ. Explicit formulas for the action of the generators $T_1, \dots, T_{l-1}$ of H_l on the vectors in V determined by these properties are known; cf. [14, Theorem 6.4].

Setting $q = 1$, the algebra H_l specializes to the symmetric group ring $\mathbb{C}S_l$. The element $T_\sigma \in H_l$ then specializes to the permutation $\sigma \in S_l$ itself. The proof of Theorem 2.2 demonstrates that the coefficients in the expansion of the element $F_\Lambda \in H_l$ relative to the basis of the elements T_σ are regular at $q = 1$ as rational functions of the parameter q. Thus the specialization of the element $F_\Lambda \in H_l$ at $q = 1$ is well defined. The same is true for the element $G_\Lambda \in H_l$; see Corollary 3.1. The specializations at $q = 1$ of the basis vectors $G_\Lambda \in V_\lambda$ form the *Young seminormal basis* in the corresponding irreducible representation of the group S_l. The action of the generators $\sigma_1, \ldots, \sigma_{l-1}$ of S_l on the vectors of the latter basis was first given by [19, Theorem IV]. For the interpretation of the elements F_Λ and G_Λ using representation theory of the affine Hecke algebra $\widehat{H}_l$, see [1, Section 3] and references therein.

Let φ_λ be the character of the irreducible H_l-module V_λ. Determine a linear function $\delta : H_l \to \mathbb{C}(q)$ by setting

$$\delta(T_\sigma^{-1}) = \begin{cases} 1 & \text{if } \sigma = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is known that the function δ is central; see for instance [4, Lemma 5.1]. At $q = 1$, this function specializes to the character of the regular representation of the algebra $\mathbb{C}S_l$, normalized so that the value of the character at $1 \in S_l$ is 1. This observation implies that each of the coefficients in the expansion of the function δ relative to the basis of the characters φ_λ in the vector space of central functions on H_l is non-zero. Thus for some scalars $h_\lambda(q) \in \mathbb{C}(q)$,

$$\delta = \sum_\lambda h_\lambda^{-1}(q)\, \varphi_\lambda. \tag{3.3}$$

For any standard tableau Λ of shape λ, denote by E_Λ the element $F_\Lambda T_0^{-1} \in H_l$. Recall that the element $F_\Lambda \in H_l$ can be obtained by multiplying G_Λ on the right by some element of H_l. It follows from Proposition 2.4 and Theorem 3.3, that the element E_Λ belongs to the simple two-sided ideal of the algebra H_l corresponding to the equivalence class of irreducible H_l-module V_λ. Further, Propositions 2.4 and 3.4 imply the equalities

$$X_i E_\Lambda = E_\Lambda X_i = q^{2c_i(\Lambda)} E_\Lambda \qquad \text{for} \qquad i = 1, \ldots, l. \tag{3.4}$$

Proposition 3.8. *Here* $E_\Lambda^2 = h_\lambda(q) E_\Lambda$ *for any standard tableau* Λ *of shape* λ.

Proof. The proofs of Corollaries 3.5 and 3.6 show that the equalities (3.4) determine the element $E_\Lambda \in H_l$ uniquely, up to a multiplier from $\mathbb{C}(q)$. Hence $E_\Lambda^2 = h_\Lambda(q) E_\Lambda$ for some $h_\Lambda(q) \in \mathbb{C}(q)$. Note that by Proposition 2.3, the coefficient of 1 in the expansion of the element $E_\Lambda \in H_l$ relative to the basis of the elements T_σ^{-1} is 1.

To prove that $h_\Lambda(q) = h_\lambda(q)$, we will employ an argument from [5, Section 3]. At $q = 1$, the element E_Λ specializes to the diagonal matrix element of the irreducible representation of S_l parametrized by the partition λ, corresponding to the vector of the Young seminormal basis parametrized by the tableau Λ. As a linear combination the elements of the group S_l, this matrix element is normalized so that its coefficient at $1 \in S_l$ is 1. Therefore $h_\Lambda(q) \neq 0$.

The element $h_\Lambda^{-1}(q)E_\Lambda \in H_l$ is an idempotent, so for any partition ω of l the value $\varphi_\omega(h_\Lambda^{-1}(q)E_\Lambda)$ is an integer. In particular, this value does not depend on the parameter q, and can be determined by specializing to $q = 1$. Thus we get

$$\varphi_\omega(h_\Lambda^{-1}(q)E_\Lambda) = \begin{cases} 1 & \text{if} \quad \omega = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Applying the functions at each side of the equality (3.3) to the element $h_\Lambda^{-1}(q)E_\Lambda \in H_l$, we obtain the equality $h_\Lambda^{-1}(q) = h_\lambda^{-1}(q)$. □

Several formulas are known for the scalars $h_\lambda(q)$. Two different formulas for each $h_\lambda(q)$ were given in [17]; see also [5, Section 3]. Another formula reads as

$$h_\lambda(q) = \prod_{(a,b)} \frac{1 - q^{2(\lambda_a + \lambda_b^* - a - b + 1)}}{1 - q^2} \cdot q^{\lambda_1(1-\lambda_1) + \lambda_2(1-\lambda_2) + \cdots} \tag{3.5}$$

where the product is taken over all nodes (a, b) of the Young diagram (2.5). At $q = 1$, the rational function of q on the right hand side of (3.5) specializes to the product of the *hook-lengths* $\lambda_a + \lambda_b^* - a - b + 1$ corresponding to the nodes (a, b) of the Young diagram (2.5). We will give a new proof of (3.5) by using Theorem 2.2 and Proposition 3.8; see the end of Section 4 for the proof.

From now on until the end of this section, we will assume that Λ is the *row tableau* of shape λ. By definition, here we have $\Lambda(a, b + 1) = \Lambda(a, b) + 1$ for all possible nodes (a, b) of the Young diagram (2.5). According to the notation of Section 2, let $\rho \in S_l$ be the permutation such that the row tableau $\Lambda = \rho \cdot \Lambda^\circ$. Let S_λ be the subgroup in S_l preserving the collections of numbers appearing in every row of the tableau Λ; it is called the *Young subgroup*. Following [5], consider the element $A_\lambda = P_\lambda T_{\rho^{-1}}^{-1} Q_\lambda T_{\rho^{-1}}$ of the algebra H_l, where

$$P_\lambda = \sum_{\sigma \in S_\lambda} q^{-\ell(\sigma)} T_\sigma^{-1} \qquad \text{and} \qquad Q_\lambda = \sum_{\sigma \in S_{\lambda^*}} (-q)^{\ell(\sigma)} T_\sigma^{-1}. \tag{3.6}$$

Here $\ell(\sigma)$ is the length of a permutation σ. At $q = 1$, the element $A_\lambda \in H_l$ specializes [18] to the *Young symmetrizer* in $\mathbb{C}S_l$ corresponding to Λ.

Proposition 3.9. *If Λ is the row tableau of shape λ, then $G_\Lambda T_{\rho\sigma_0}^{-1} = A_\lambda$.*

Proof. Let U be the vector subspace in H_l formed by all elements B such that

$$T_k B = qB \qquad \text{if} \quad \sigma_k \in S_\lambda, \tag{3.7}$$

$$BT_k = -q^{-1}B \qquad \text{if} \quad \sigma_k \in S_{\lambda^*}. \tag{3.8}$$

Then $\dim U = 1$; see for instance [5, Section 1]. Using the definition of A_λ, we can verify that $A_\lambda T_{\rho^{-1}}^{-1} \in U$. On the other hand, consider the element

$$B = G_\Lambda T_{\rho\sigma_0}^{-1} T_{\rho^{-1}}^{-1} = G_\Lambda T_0^{-1}. \tag{3.9}$$

It satisfies the condition (3.7) thanks to Part (a) of Theorem 3.3, because here Λ is the row tableau of shape λ. By Proposition 2.7, we also have

$$T_k F_{\Lambda^\circ} = -q^{-1} F_{\Lambda^\circ} \qquad \text{if} \quad \sigma_k \in S_{\lambda^*}.$$

Due to Corollary 3.1, the element (3.9) can be obtained by multiplying $F_{\Lambda^\circ} T_0^{-1}$ on the left by a certain element of H_l. But the element $F_{\Lambda^\circ} T_0^{-1}$ is α_l-invariant. Hence the element (3.9) also satisfies the condition (3.8). Thus $G_\Lambda T_0^{-1} \in U$.

To complete the proof of Proposition 3.9, it suffices to compare the coefficients at $T_{\rho\sigma_0}$ in the expansions of the elements G_Λ and $A_\lambda T_{\rho\sigma_0}$ of H_l relative to the basis of the elements T_σ. For G_Λ this coefficient is 1 by Proposition 3.2. Let S'_λ be the subgroup $\sigma_0 S_{\lambda^*} \sigma_0 \subset S_l$. Observe that if $\sigma \in S_\lambda$ and $\sigma' \in S'_\lambda$, then

$$\ell(\sigma\rho\sigma_0\sigma') = \ell(\rho\sigma_0) - \ell(\sigma) - \ell(\sigma').$$

In particular, then we have $\sigma\rho\sigma_0\sigma' = \rho\sigma_0$ only for $\sigma = \sigma' = 1$. Therefore

$$\begin{aligned} A_\lambda T_{\rho\sigma_0} &= \left(\sum_{\sigma \in S_\lambda} q^{-\ell(\sigma)} T_\sigma^{-1} \right) T_{\rho^{-1}}^{-1} \left(\sum_{\sigma \in S_{\lambda^*}} (-q)^{\ell(\sigma)} T_\sigma^{-1} \right) T_0 \\ &= \left(\sum_{\sigma \in S_\lambda} q^{-\ell(\sigma)} T_\sigma^{-1} \right) T_{\rho\sigma_0} \left(\sum_{\sigma' \in S'_\lambda} (-q)^{\ell(\sigma')} T_{\sigma'}^{-1} \right) \\ &= \sum_{\sigma \in S_\lambda} \sum_{\sigma' \in S'_\lambda} q^{-\ell(\sigma)} (-q)^{\ell(\sigma')} T_{\sigma^{-1}\rho\sigma_0\sigma'^{-1}}. \end{aligned}$$

The coefficient of $T_{\rho\sigma_0}$ in the sum displayed in the last line above is 1. □

Remark. One can give another expression for the element $A_\lambda \in H_l$ defined via (3.6), by using the identities

$$\sum_{\sigma \in S_l} q^{-\ell(\sigma)} T_\sigma^{-1} = q^{l(1-l)} \sum_{\sigma \in S_l} q^{\ell(\sigma)} T_\sigma,$$

$$\sum_{\sigma \in S_l} (-q)^{\ell(\sigma)} T_\sigma^{-1} = (-q)^{l(l-1)} \sum_{\sigma \in S_l} (-q)^{-\ell(\sigma)} T_\sigma. \quad \square$$

4. Eigenvalues of the operator J

Take any partition λ of l. For any standard tableau Λ of shape λ denote by V_Λ the left ideal in the algebra H_l, generated by the element F_Λ defined in Section 2. If $\Lambda = \Lambda^\circ$ then $V_\Lambda = V_\lambda$ in the notation of Section 3. Recall that the element $F_\Lambda \in H_l$ can be obtained by multiplying F_{Λ° on the left and on the right by certain invertible elements of H_l; see Proposition 2.6. Hence V_Λ is equivalent to V_λ as the H_l-module. The algebra H_l acts on any left ideal $V_\Lambda \subset H_l$ via left multiplication. Also recall that the element F_Λ can be obtained by multiplying G_Λ by certain element of H_l on the right. Thus by Proposition 3.4

$$X_i F_\Lambda = q^{2c_i(\Lambda)} F_\Lambda \qquad \text{for each} \quad i = 1, \ldots, l. \tag{4.1}$$

For any non-zero $z \in \mathbb{C}(q)$, consider the evaluation $\widehat{H}_l$-module $V_\Lambda(z)$. This is the pullback of the H_l-module V_Λ back through the homomorphism π_z; see Section 1. As a vector space $V_\Lambda(z)$ is the left ideal $V_\Lambda \subset H_l$, and the subalgebra $H_l \subset \widehat{H}_l$ acts on this vector space via left multiplication. By (4.1), in the $\widehat{H}_l$-module $V_\Lambda(z)$ we have

$$Y_i \cdot F_\Lambda = zq^{2c_i(\Lambda)} F_\Lambda \qquad \text{for each} \quad i = 1, \dots, l. \tag{4.2}$$

Note that any element of $\widehat{H}_l$ can be written as a sum of certain Laurent monomials in $Y_1, \dots, Y_l$ multiplied by some elements of H_l on the left. Therefore the action of the generators $Y_1, \dots, Y_l$ on the $\widehat{H}_l$-module $V_\Lambda(z)$ is determined by (4.2).

Take a partition μ of m, and any standard tableau M of shape μ. Also take any non-zero element $w \in \mathbb{C}(q)$. Let us realize the $\widehat{H}_{l+m}$-module W induced from the $\widehat{H}_l \otimes \widehat{H}_m$-module $V_\Lambda(z) \otimes V_M(w)$ as the left ideal in H_{l+m} generated by the product $F_\Lambda \bar{F}_M$. Here $\bar{F}_M$ denotes the image of the element $F_M \in H_m$ under the embedding $H_m \to H_{l+m} : T_j \mapsto T_{l+j}$. The action of the generators $Y_1, \dots, Y_{l+m} \in \widehat{H}_{l+m}$ on this left ideal is then determined by setting

$$\begin{aligned} Y_i \cdot F_\Lambda \bar{F}_M &= zq^{2c_i(\Lambda)} F_\Lambda \bar{F}_M \qquad \text{for each} \quad i = 1, \dots, l; \\ Y_{l+j} \cdot F_\Lambda \bar{F}_M &= wq^{2c_j(M)} F_\Lambda \bar{F}_M \qquad \text{for each} \quad j = 1, \dots, m. \end{aligned} \tag{4.3}$$

Further, consider $\widehat{H}_{l+m}$-module W' induced from the $\widehat{H}_m \otimes \widehat{H}_l$-module $V_M(w) \otimes V_\Lambda(z)$. Let us realize W' as the left ideal in H_{l+m} generated by the product $F_M \bar{F}_\Lambda$, where $\bar{F}_\Lambda$ denotes the image of $F_\Lambda \in H_l$ under the embedding $H_l \to H_{l+m} : T_i \mapsto T_{i+m}$. The generators $Y_1, \dots, Y_{l+m}$ act on W' so that

$$\begin{aligned} Y_{i+m} \cdot F_M \bar{F}_\Lambda &= zq^{2c_i(\Lambda)} F_M \bar{F}_\Lambda \qquad \text{for each} \quad i = 1, \dots, l; \\ Y_j \cdot F_M \bar{F}_\Lambda &= wq^{2c_j(M)} F_M \bar{F}_\Lambda \qquad \text{for each} \quad j = 1, \dots, m. \end{aligned} \tag{4.4}$$

Consider the element τ of the symmetric group S_{l+m}, which was defined as the permutation (1.8). We will use one reduced decomposition of this element,

$$\tau = \prod_{i=1,\dots,l}^{\leftarrow} \left(\prod_{j=1,\dots,m}^{\rightarrow} \sigma_{i+j-1} \right).$$

The corresponding element T_τ of the algebra H_{l+m} satisfies the relations

$$T_i T_\tau = T_\tau T_{i+m} \qquad \text{for each} \quad i = 1, \dots, l-1; \tag{4.5}$$

$$T_{l+j} T_\tau = T_\tau T_j \qquad \text{for each} \quad j = 1, \dots, m-1. \tag{4.6}$$

In particular, these relations imply the equality in H_{l+m}

$$F_\Lambda \bar{F}_M T_\tau = T_\tau F_M \bar{F}_\Lambda. \tag{4.7}$$

Now introduce two elements of the algebra $\widehat{H}_{l+m}$,

$$S_{\Lambda M}(z, w) = \prod_{i=1,\dots,l}^{\rightarrow} \left(\prod_{j=1,\dots,m}^{\leftarrow} F_{l+m-i-j+1}(q^{2c_i(\Lambda)} z, q^{2c_j(M)} w) \right), \tag{4.8}$$

$$S'_{\Lambda M}(z,w)=\overleftarrow{\prod_{i=1,\dots,l}}\left(\overrightarrow{\prod_{j=1,\dots,m}}F_{i+j-1}(q^{2c_i(\Lambda)}z,q^{2c_j(M)}w)\right).$$

We have assumed that $z^{-1}w\notin q^{2\mathbb{Z}}$ so that these two elements are well defined; see (2.1). Using the relations (2.2) and (2.3) together with the definitions of the elements $F_\Lambda\in H_l$ and $F_M\in H_m$, we obtain the relation in the algebra H_{l+m}

$$F_\Lambda\bar{F}_M S_{\Lambda M}(z,w)=S'_{\Lambda M}(z,w)F_M\bar{F}_\Lambda. \tag{4.9}$$

We will use one more expression for the element of H_{l+m}, appearing on either side of the equality (4.9). For each $i=1,\dots,l$ denote by $\bar{X}_i$ the image of the Murphy element $X_i\in H_l$ under the embedding $H_l\to H_{l+m}:T_i\mapsto T_{m+i}$.

Proposition 4.1. *The element of the algebra H_{l+m} in (4.9) equals T_τ times*

$$\overleftarrow{\prod_{i=1,\dots,l}}\frac{z^{-1}w-q^{2c_i(\Lambda)}\bar{X}_iX_{i+m}^{-1}}{z^{-1}w-q^{2c_i(\Lambda)}}\cdot F_M\bar{F}_\Lambda.$$

Proof. Using Propositions 2.8 and 2.9 repeatedly, for the standard tableau M instead of Λ and for the element $q^{2c_i(\Lambda)}zw^{-1}\in\mathbb{C}(q)$ instead of z where $i=1,\dots,l$, one shows that the element of the algebra H_{l+m} on the right hand side of (4.9) equals the product

$$\overleftarrow{\prod_{i=1,\dots,l}}\frac{z^{-1}wT_i\cdots T_{i+m-1}-q^{2c_i(\Lambda)}T_i^{-1}\cdots T_{i+m-1}^{-1}}{z^{-1}w-q^{2c_i(\Lambda)}}\cdot F_M\bar{F}_\Lambda. \tag{4.10}$$

The ordered product of the factors in (4.10) corresponding to $i=l,\dots,1$ can be rewritten as T_τ, multiplied on the right by the product over $i=l,\dots,1$ of

$$\begin{aligned}
&\overrightarrow{\prod_{k=1,\dots,i-1}}(T_k\cdots T_{k+m-1})^{-1}\\
&\quad\times\frac{z^{-1}w-q^{2c_i(\Lambda)}T_{i+m-1}^{-1}\cdots T_i^{-1}T_i^{-1}\cdots T_{i+m-1}^{-1}}{z^{-1}w-q^{2c_i(\Lambda)}}\\
&\quad\times\overleftarrow{\prod_{k=1,\dots,i-1}}(T_k\cdots T_{k+m-1})\\
&=T_{i+m-1}\cdots T_{m+1}\cdot\overrightarrow{\prod_{k=1,\dots,i-1}}(T_k\cdots T_{k+m})^{-1}\\
&\quad\times\frac{z^{-1}w-q^{2c_i(\Lambda)}T_{i+m-1}^{-1}\cdots T_i^{-1}T_i^{-1}\cdots T_{i+m-1}^{-1}}{z^{-1}w-q^{2c_i(\Lambda)}}\\
&\quad\times\overleftarrow{\prod_{k=1,\dots,i-1}}(T_k\cdots T_{k+m-1})\cdot T_{m+1}^{-1}\cdots T_{i+m-1}^{-1}.
\end{aligned}$$

We can now complete the proof of Proposition 4.1 by using the definitions of $X_i \in H_l$ and $X_{m+i} \in H_{l+m}$, along with the relations for all $j = 1, \dots, m$

$$\prod_{k=1,\dots,i-1}^{\rightarrow} (T_k \cdots T_{k+m})^{-1} \cdot T_{i+j-1}^{-1} \cdot \prod_{k=1,\dots,i-1}^{\leftarrow} (T_k \cdots T_{k+m}) = T_j^{-1}. \quad \square$$

It follows from the relation (4.9) that the right multiplication in H_{l+m} by the element $S_{\Lambda M}(z, w)$ determines a linear operator $I : W \to W'$.

Proposition 4.2. *The operator $I : W \to W'$ is an $\widehat{H}_{l+m}$-intertwiner.*

Proof. The subalgebra $H_{l+m} \subset \widehat{H}_{l+m}$ acts on W, W' via left multiplication; so the operator I commutes with this action by definition. The left ideal W in H_{l+m} is generated by the element $F_\Lambda \bar{F}_M$; therefore it suffices to check that

$$Y_i \cdot I(F_\Lambda \bar{F}_M) = I(Y_i \cdot F_\Lambda \bar{F}_M) \qquad \text{for each} \quad i = 1, \dots, l+m.$$

Firstly, consider the case where $i \leqslant l$. In this case by using (4.3), (4.4) and (4.9)

$$Y_i \cdot I(F_\Lambda \bar{F}_M) = Y_i \cdot (S'_{\Lambda M}(z, w) F_M \bar{F}_\Lambda) = S'_{\Lambda M}(z, w) \times (Y_{m+i} \cdot F_M \bar{F}_\Lambda) = z q^{2c_i(\Lambda)} S'_{\Lambda M}(z, w) F_M \bar{F}_\Lambda = I(Y_i \cdot F_\Lambda \bar{F}_M).$$

Here we also used the defining relations (1.4) and (1.5) of the algebra $\widehat{H}_{l+m}$; for more details of this argument see [16, Section 2]. The case $i > l$ can be considered similarly. $\square$

Consider the operator of the right multiplication in H_{l+m} by the element

$$R_{\Lambda M}(z, w) = S_{\Lambda M}(z, w) T_\tau^{-1}.$$

Because of the relations (4.7) and (4.9), this operator preserves the subspace $W \subset H_{l+m}$. Restriction of this operator to the subspace W will be denoted by J. The subalgebra $H_{l+m} \subset \widehat{H}_{l+m}$ acts on the $\widehat{H}_{l+m}$-module W via left multiplication, so the operator $J : W \to W$ commutes with this action. Now regard W as a H_{l+m}-module only. Let ν be any partition of $l+m$ such that the H_{l+m}-module W has exactly one irreducible component equivalent to V_ν. The operator J preserves this component, and acts thereon as multiplication by a certain element of $\mathbb{C}(q)$. Denote this element by $r_\nu(z, w)$; it depends on the parameters z and w as a rational function of $z^{-1}w$, and does not depend on the choice of the tableaux Λ and M of the given shapes λ and μ. In this section, we compute the eigenvalues $r_\nu(z, w)$ of J for certain partitions ν.

Choose any sequence $i_1, \dots, i_{\lambda_1^*} \in \{1, 2, \dots\}$ of pairwise distinct indices; this sequence needs not to be increasing. Recall that λ_1^* is the number of non-zero parts in the partition λ. Consider the partition μ as an infinite sequence with finitely many non-zero terms. Define an infinite sequence $\xi = (\xi_1, \xi_2, \dots)$ by

$$\xi_{i_a} = \mu_{i_a} + \lambda_a, \qquad a = 1, \dots, \lambda_1^*;$$
$$\xi_i = \mu_i, \qquad i \neq i_1, \dots, i_{\lambda_1^*}.$$

Suppose we get the inequalities $\xi_1 \geqslant \xi_2 \geqslant \cdots$ so that ξ is a partition of $l+m$. Then the H_{l+m}-module W has exactly one irreducible component equivalent to V_ξ. This follows from the Littlewood–Richardson rule [13, Section I.9]. We will compute the eigenvalue

$r_\xi(z,w)$ by applying the operator J to a certain vector in that irreducible component. For the purposes of this computation, assume that Λ is the column tableau Λ°; the tableau M will remain arbitrary.

The image of the action of the element $F_M\bar{F}_{\Lambda^\circ}$ in the irreducible H_{l+m}-module V_ξ is a one-dimensional subspace. Let us describe this subspace explicitly. Let Ξ be the tableau of shape ξ, defined as follows. Firstly, put $\Xi(c,d)=M(c,d)$ for all nodes (c,d) of the Young diagram of μ. Further, for any positive integer j consider all those parts of λ which are equal to j. These are the parts λ_a where the index a belongs to the sequence

$$\lambda^*_{j+1}+1,\lambda^*_{j+1}+2,\dots,\lambda^*_j. \tag{4.11}$$

The length $\lambda^*_j-\lambda^*_{j+1}$ of this sequence is the multiplicity of the part j in the partition λ; let us denote this multiplicity by n for short. Rearrange the sequence (4.11) to the sequence $a_1,\dots,a_n$ such that the inequalities $i_{a_1}<\dots<i_{a_n}$ hold. Then for every term $a=a_k$ of the rearranged sequence put

$$\Xi(i_a,\mu_{i_a}+b)=m+\Lambda^\circ(\lambda^*_{j+1}+k,b)\qquad\text{where } b=1,\dots,\lambda_a.$$

Proposition 4.3. *The tableau Ξ is standard.*

Proof. For any possible integers c and d, the condition $\Xi(c,d)<\Xi(c,d+1)$ is satisfied by definition, because the tableaux Λ° and M are standard. For any node (c,d) of the Young diagram of μ, the condition $\Xi(c,d)<\Xi(c+1,d)$ is also satisfied by definition. Now suppose there are two different numbers k and k' greater than m that appear in the same column of the tableaux Ξ. Let i and i' be the corresponding rows of Ξ; assume that $i<i'$. Here $i=i_a$ and $i'=i_{a'}$ for certain indices $a,a'\in\{1,\dots,\lambda^*_1\}$. If $\lambda_a\geqslant\lambda_{a'}$ then $k<k'$ because the tableau Λ° is standard. Here we also use the definition of Ξ. Now suppose that $\lambda_a<\lambda_{a'}$. Then $\mu_i>\mu_{i'}$, because the assumption $i<i'$ implies

$$\mu_i+\lambda_a\geqslant\mu_{i'}+\lambda_{a'}.$$

Let b and b' be the columns of the tableau Λ° corresponding to its entries $k-m$ and $k'-m$. Since k and k' appear in the same column of the tableau Ξ while $\mu_i>\mu_{i'}$, we have $b<b'$. Then $k<k'$ by the definition of Λ°. □

Using Proposition 4.3, consider the vector $G_\Xi\in V_\xi$ as defined in Section 3. Take the element $Q_\lambda\in H_l$ as defined in (3.6). Denote by $\bar{Q}_\lambda$ the image of this element under the embedding $H_l\to H_{l+m}:T_i\mapsto T_{i+m}$.

Proposition 4.4. *The image of the action of the element $F_M\bar{F}_{\Lambda^\circ}\in H_{l+m}$ on the H_{l+m}-module V_ξ is spanned by the vector $\bar{Q}_\lambda G_\Xi$.*

Proof. Put $V=F_MV_\xi$. The subspace $V\subset V_\xi$ is spanned by all those vectors $G_{\tilde{\Xi}}$ where, for every node (c,d) of the Young diagram of μ, the standard tableau $\tilde{\Xi}$ of shape ξ satisfies the condition $\tilde{\Xi}(c,d)=M(c,d)$. The action of the element $\bar{F}_{\Lambda^\circ}\in H_{l+m}$ on V_ξ preserves the subspace $V\subset V_\xi$, and the image $\bar{F}_{\Lambda^\circ}V$ is one-dimensional. Moreover, we have $\bar{F}_{\Lambda^\circ}V=\bar{Q}_\lambda V$; see [5, Section 1].

It now remains to check that $\bar{Q}_\lambda G_\Xi\neq 0$. Due to our choice of the tableau Ξ, it suffices to consider the case where each non-zero part of λ equals 1. In this case, the element

$Q_\lambda \in H_l$ is central. On the other hand, any vector of V has the form $\bar{C}G_\Xi$ where $\bar{C}$ is the image of some element $C \in H_l$ under the embedding $H_l \to H_{l+m} : T_i \mapsto T_{i+m}$. So $\bar{Q}_\lambda V \neq \{0\}$ implies $\bar{Q}_\lambda G_\Xi \neq 0$. □

Theorem 4.5. *We have the equality*

$$r_\xi(z,w) = \prod_{(a,b)} \frac{z^{-1}w - q^{-2(\mu_{i_a}+\lambda^*_b - i_a - b+1)}}{z^{-1}w - q^{2b-2a}}$$

where the product is taken over all nodes (a,b) *of the Young diagram* (2.5).

Proof. First consider the case where each non-zero part of λ equals 1. In this case, Λ° is the only one standard tableau of shape λ and we have $c_i(\Lambda^\circ) = 1-i$ for any $i = 1,\dots,l$. The product displayed in Proposition 4.1 then equals

$$\prod_{i=1,\dots,l}^{\longleftarrow} \frac{z^{-1}w - q^{2-2i}\bar{X}_i X_{i+m}^{-1}}{z^{-1}w - q^{2-2i}} \cdot F_M \bar{F}_{\Lambda^\circ}. \tag{4.12}$$

We will prove by induction on $l = 1, 2, \dots$ that the product (4.12) equals

$$\prod_{i=1,\dots,l} \frac{z^{-1}w - q^{2-2l}X_{i+m}^{-1}}{z^{-1}w - q^{2-2i}} \cdot F_M \bar{F}_{\Lambda^\circ}. \tag{4.13}$$

The elements $X_{m+1},\dots,X_{l+m} \in H_{l+m}$ pairwise commute; hence the ordering of the factors corresponding to $i = 1,\dots,l$ in the product (4.13) is irrelevant. Theorem 4.5 will then follow in our special case. Indeed, let Z_ξ be the minimal central idempotent in the algebra H_{l+m} corresponding to the partition ξ. Using Proposition 4.1 together with the equality between (4.12) and (4.13), we get

$$\begin{aligned}
J(Z_\xi F_{\Lambda^\circ}\bar{F}_M) &= Z_\xi S'_{\Lambda^\circ M}(z,w) F_M \bar{F}_{\Lambda^\circ} T_\tau^{-1} \\
&= T_\tau Z_\xi \cdot \prod_{i=1,\dots,l} \frac{z^{-1}w - q^{2-2l}X_{i+m}^{-1}}{z^{-1}w - q^{2-2i}} \cdot F_M \bar{F}_{\Lambda^\circ} T_\tau^{-1} \\
&= T_\tau Z_\xi \cdot \prod_{i=1,\dots,l} \frac{z^{-1}w - q^{2-2l}X_{i+m}^{-1}}{z^{-1}w - q^{2-2i}} \\
&\quad \times \prod_{j=1,\dots,m} \frac{z^{-1}w - q^{2-2l}X_j^{-1}}{z^{-1}w - q^{2-2l-2c_j(M)}} \cdot F_M \bar{F}_{\Lambda^\circ} T_\tau^{-1} \\
&= T_\tau Z_\xi \cdot \prod_{i=1,\dots,l} \frac{z^{-1}w - q^{2-2l-2c_{i+m}(\Xi)}}{z^{-1}w - q^{2-2i}} \\
&\quad \times \prod_{j=1,\dots,m} \frac{z^{-1}w - q^{2-2l-2c_j(\Xi)}}{z^{-1}w - q^{2-2l-2c_j(M)}} \cdot F_M \bar{F}_{\Lambda^\circ} T_\tau^{-1} \\
&= \prod_{a=1,\dots,l} \frac{z^{-1}w - q^{2i_a-2l-2\mu_{i_a}}}{z^{-1}w - q^{2-2a}} \cdot Z_\xi F_{\Lambda^\circ}\bar{F}_M = r_\xi(z,w) Z_\xi F_{\Lambda^\circ}\bar{F}_M,
\end{aligned} \tag{4.14}$$

as Theorem 4.5 claims. Here we used the counterparts of the relations (4.1) for the standard tableau M and Ξ instead of Λ; cf. our proof of Corollary 3.6.

Now let us prove the equality between (4.12) and (4.13). We have $X_1 = 1$ by definition; hence that equality is obvious when $l = 1$. Suppose that $l > 1$. The numerator of the fraction in (4.12) corresponding to the index $i = 1$ equals

$$z^{-1}w - X_{m+1}^{-1} = z^{-1}w - T_m^{-1} \cdots T_1^{-1} T_1^{-1} \cdots T_m^{-1}. \tag{4.15}$$

In our special case, we have the relations in the algebra H_{l+m}

$$T_{m+i} \bar{F}_{\Lambda^\circ} = -q^{-1} \bar{F}_{\Lambda^\circ} \qquad \text{for} \qquad i = 1, \ldots, l-1.$$

Using these relations along with the equality (4.15), we obtain

$$\begin{aligned}(z^{-1}w - X_{m+1}^{-1})\bar{F}_{\Lambda^\circ} \\ = (-q)^{l-1} T_{m+1} \cdots T_{l+m-1} (z^{-1}w - q^{2-2l} X_{l+m}^{-1}) \bar{F}_{\Lambda^\circ}.\end{aligned} \tag{4.16}$$

Further, for any $i = 2, \ldots, l$ the elements $T_1, \ldots, T_{i-2}$ commute with the Murphy element $X_i \in H_l$. So the elements $T_{m+1}, \ldots, T_{i+m-2}$ commute with $\bar{X}_i \in H_{l+m}$; they also commute with X_{i+m}. Therefore for $i = 2, \ldots, l$ we have

$$\begin{aligned}&\bar{X}_i X_{i+m}^{-1} T_{m+1} \cdots T_{l+m-1} \\ &\quad = T_{m+1} \cdots T_{i+m-2} \bar{X}_i X_{i+m}^{-1} T_{i+m-1} \cdots T_{l+m-1} = T_{m+1} \cdots T_{i+m-2} \\ &\qquad \times T_{i+m-1} \cdots T_{m+1} T_m^{-1} \cdots T_1^{-1} T_1^{-1} \cdots T_{i+m-1}^{-1} T_{i+m-1} T_{i+m} \cdots T_{l+m-1} \\ &\quad = T_{m+1} \cdots T_{l+m-1} T_{i+m-2} \cdots T_{m+1} T_m^{-1} \cdots T_1^{-1} T_1^{-1} \cdots T_{i+m-2}^{-1} \\ &\quad = T_{m+1} \cdots T_{l+m-1} \bar{X}_{i-1} X_{i+m-1}^{-1}.\end{aligned}$$

Therefore by using the equality (4.16), the product (4.12) equals

$$\begin{aligned}&(-q)^{l-1} T_{m+1} \cdots T_{l+m-1} \cdot \overleftarrow{\prod_{i=2,\ldots,l}} \frac{z^{-1}w - q^{2-2i} \bar{X}_{i-1} X_{i+m-1}^{-1}}{z^{-1}w - q^{2-2i}} \\ &\qquad \times \frac{z^{-1}w - q^{2-2l} X_{m+l}^{-1}}{z^{-1}w - 1} \cdot F_M \bar{F}_{\Lambda^\circ} = (-q)^{l-1} T_{m+1} \cdots T_{l+m-1} \\ &\qquad \times \overleftarrow{\prod_{i=2,\ldots,l}} \frac{z^{-1}w - q^{2-2l} X_{i+m-1}^{-1}}{z^{-1}w - q^{2-2i}} \cdot \frac{z^{-1}w - q^{2-2l} X_{m+l}^{-1}}{z^{-1}w - 1} \cdot F_M \bar{F}_{\Lambda^\circ} \\ &\quad = (-q)^{l-1} T_{m+1} \cdots T_{l+m-1} \cdot \overleftarrow{\prod_{i=1,\ldots,l}} \frac{z^{-1}w - q^{2-2l} X_{i+m}^{-1}}{z^{-1}w - q^{2-2i}} \cdot F_M \bar{F}_{\Lambda^\circ}.\end{aligned} \tag{4.17}$$

Here we used the equality between the counterparts of the products (4.12) and (4.13) for $l - 1$ instead of l and for $q^2 z^{-1} w$ instead of $z^{-1}w$, which we have by the inductive assumption. We also used commutativity of the Murphy element X_{l+m} with $T_{m+1}, \ldots, T_{l+m-2}$. To establish the equality between the products (4.12) and (4.13) themselves, it now remains to observe that the product over $i = 1, \ldots, l$ in the line (4.17) is symmetric in $X_{m+1}, \ldots, X_{l+m}$ and therefore commutes with $T_{m+1}, \ldots, T_{l+m-1}$; cf. the remark after our proof of Corollary 3.6.

Thus we have proved Theorem 4.5 when each non-zero part of λ is 1. Now let λ be an arbitrary partition of l. Consider the element $\bar{Q}_\lambda G_\Xi \in H_{l+m}$. Due to Proposition 4.4, this element is divisible on the left by $F_M \bar{F}_{\Lambda^\circ}$. The element

$$\alpha_{l+m}(\bar{Q}_\lambda G_\Xi) F_M \bar{F}_{\Lambda^\circ} T_\tau^{-1} = \alpha_{l+m}(G_\Xi) \bar{Q}_\lambda T_\tau^{-1} F_{\Lambda^\circ} \bar{F}_M \tag{4.18}$$

is non-zero, and belongs to the left ideal $W \subset H_{l+m}$. Further, the element (4.18) belongs to the irreducible component of the H_{l+m}-module W equivalent to V_ξ. Thus (4.18) is an eigenvector of the operator $J : W \to W$ with the eigenvalue $r_\xi(z, w)$. On other hand, due to Proposition 4.1 the image of (4.18) under the operator J equals

$$\begin{aligned}
&\alpha_{l+m}(G_\Xi)\bar{Q}_\lambda \cdot \overleftarrow{\prod_{i=1,\dots,l}} \frac{z^{-1}w - q^{2c_i(\Lambda)}\bar{X}_i X_{i+m}^{-1}}{z^{-1}w - q^{2c_i(\Lambda)}} \cdot F_M \bar{F}_{\Lambda^\circ} \\
&\quad = \alpha_{l+m}(G_\Xi)\bar{Q}_\lambda \cdot \prod_{i=1,\dots,l} \frac{z^{-1}w - q^{4c_i(\Lambda)} X_{i+m}^{-1}}{z^{-1}w - q^{2c_i(\Lambda)}} \cdot F_M \bar{F}_{\Lambda^\circ}.
\end{aligned} \tag{4.19}$$

To obtain the latter equality we used the relations (3.4), the divisibility of the element $\alpha_{l+m}(G_\Xi)\bar{Q}_\lambda$ on the right by $\bar{E}_{\Lambda^\circ}$, and the commutativity of the element $\bar{X}_i$ with the Murphy elements $X_{i+m+1}, \dots, X_{l+m}$ for any $i = 1, \dots, l$. Here $\bar{E}_{\Lambda^\circ}$ denotes the image of the element $E_{\Lambda^\circ} \in H_l$ under the embedding $H_l \to H_{l+m} : T_i \mapsto T_{i+m}$. The factors in the product (4.19) corresponding to the indices $i = 1, \dots, l$ pairwise commute; hence their ordering is irrelevant.

Due to Theorem 3.3, the vector $\bar{Q}_\lambda G_\Xi \in V_\xi$ is a linear combination of the vectors $Q_{\tilde{\Xi}}$ where $\tilde{\Xi}$ is any standard tableau of shape ξ, obtained from Ξ by a permutation $\tau^{-1}\sigma\tau \in S_{l+m}$ such that $\sigma \in S_{\lambda^*} \subset S_l \subset S_{l+m}$. Now the expression (4.19) for the J-image of (4.18) shows that the eigenvalue $r_\xi(z, w)$ is multiplicative relative to the columns of the tableau Λ°. That is, by using Theorem 4.5 consecutively for the partitions of $\lambda_1^*, \lambda_2^*, \dots$ with each non-zero part being equal to 1, we get

$$r_\xi(z, w) = \prod_{b=1}^{\lambda_1} \prod_{a=1}^{\lambda_b^*} \frac{z^{-1}w - q^{-2(\mu_{i_a} + \lambda_b^* - i_a - b + 1)}}{z^{-1}w - q^{2b-2a}} \tag{4.20}$$

as required. According to (4.19), the numerator in (4.20) is obtained from the numerator in (4.14) by changing l, μ_{i_a} to $\lambda_b^*, \mu_{i_a} + b - 1$ respectively, and by increasing the exponential by $4(b-1) = 4c_k(\Lambda^\circ)$ where $k = \Lambda^\circ(1, b)$. □

Our next theorem is essentially a reformulation of Theorem 4.5. Choose any sequence $j_1, \dots, j_{\lambda_1} \in \{1, 2, \dots\}$ of pairwise distinct indices; this sequence needs not to be increasing. Consider the partition μ^* conjugate to μ. Define a sequence $\eta^* = (\eta_1^*, \eta_2^*, \dots)$ by

$$\begin{aligned}
&\eta_{j_b}^* = \mu_{j_b}^* + \lambda_b^*, \qquad b = 1, \dots, \lambda_1; \\
&\eta_j^* = \mu_j, \qquad j \neq j_1, \dots, i_{\lambda_1}.
\end{aligned}$$

Suppose we have the inequalities $\eta_1^* \geqslant \eta_2^* \geqslant \dots$, so that η^* is a partition of $l + m$. Then define η as the partition conjugate to η^*. The H_{l+m}-module W has exactly one irreducible component equivalent to V_η; this follows from the Littlewood–Richardson

rule [13, Section I.9]. Consider the corresponding eigenvalue $r_\eta(z,w)$ of the operator $J : W \to W$.

Theorem 4.6. *We have the equality*

$$r_\eta(z,w) = \prod_{(a,b)} \frac{z^{-1}w - q^{2(\lambda_a+\mu^*_{j_b}-a-j_b+1)}}{z^{-1}w - q^{2b-2a}}$$

where the product is taken over all nodes (a,b) *of the Young diagram* (2.5).

Proof. For any positive integer l, the $\mathbb{C}(q)$-algebra H_l may be also regarded as an algebra over the field $\mathbb{C} \subset \mathbb{C}(q)$. The assignments $q \mapsto q^{-1}$ and $T_i \mapsto -T_i$ for $i = 1,\dots,l-1$ determine an involutive automorphism of H_l as a $\mathbb{C}$-algebra. Denote by β_l this automorphism. For the minimal central idempotent Z_λ of the semisimple $\mathbb{C}(q)$-algebra H_l we have $\beta_l(Z_\lambda) = Z_{\lambda^*}$; this can be proved by specializing H_l at $q = 1$ to the symmetric group ring $\mathbb{C}S_l$. Further, for any standard tableau Λ of shape λ, define the standard tableau Λ^* of shape λ^* by setting $\Lambda^*(b,a) = \Lambda(a,b)$ for all nodes (a,b) of the Young diagram (2.5). Then

$$\beta_l(F_\Lambda) = (-1)^{l(l-1)/2} F_{\Lambda^*}.$$

Indeed, the counterparts of the equalities (3.4) for F_{Λ^*} instead of F_Λ determine the element $F_{\Lambda^*} \in H_l$ uniquely up to a factor from $\mathbb{C}(q)$, while $c_i(\Lambda^*) = -c_i(\Lambda)$ and $\beta_l(X_i) = X_i$ for $i = 1,\dots,l$. We also use Proposition 2.3 and the equality

$$\beta_l(T_0) = (-1)^{l(l-1)/2} T_0.$$

Now consider the automorphism β_{l+m} of the $\mathbb{C}$-algebra H_{l+m}. Both sides of the equality to be proved in Theorem 4.6 depend on z, w as rational functions of $z^{-1}w$. Hence it suffices to prove that equality only when $\beta_{l+m}(z^{-1}w) = z^{-1}w$. Our argument will be somewhat simpler then. By using (4.8), we then get

$$\beta_{l+m}(S_{\Lambda M}(z,w)) = (-1)^{lm} S_{\Lambda^* M^*}(z,w).$$

Note that we also have $\beta_{l+m}(T_\tau^{-1}) = (-1)^{lm} T_\tau^{-1}$. For any standard tableaux Λ and M of shapes λ and μ respectively, by definition we have the equality

$$Z_\eta F_\Lambda \bar{F}_M S_{\Lambda M}(z,w) T_\tau^{-1} = r_\eta(z,w) Z_\eta F_\Lambda \bar{F}_M.$$

By applying the automorphism β_{l+m} to both sides of this equality, we get

$$Z_{\eta^*} F_{\Lambda^*} \bar{F}_{M^*} S_{\Lambda^* M^*}(z,w) T_\tau^{-1} = \beta_{l+m}(r_\eta(z,w)) Z_{\eta^*} F_{\Lambda^*} \bar{F}_{M^*}.$$

Hence by using Theorem 4.5 for the partitions λ^*, μ^* and η^* instead of λ, μ and ξ respectively, we get

$$\beta_{l+m}(r_\eta(z,w)) = \prod_{(a,b)} \frac{z^{-1}w - q^{-2(\mu^*_{j_a}+\lambda_b-j_a-b+1)}}{z^{-1}w - q^{2b-2a}}$$

where the product is taken over all nodes (a,b) of the Young diagram of λ^*. Equivalently, this product may be also taken over all nodes (b,a) of the Young diagram of λ. Exchanging

the indices a and b in the last displayed equality, we then obtain Theorem 4.6 due to the involutivity of the mapping β_{l+m}. $\square$

Let us now derive Corollary 1.1 as stated in the beginning of this article. We will use Theorems 4.5 and 4.6 in the simplest situation when $i_a = a$ for every $a = 1, \dots, \lambda'_1$ and $j_b = j$ for every $b = 1, \dots, \lambda_1$. Then we have

$$\xi = \lambda + \mu \qquad \text{and} \qquad \eta = (\lambda^* + \mu^*)^*.$$

By Theorems 4.5 and 4.6, the ratio $r_\xi(z,w)/r_\eta(z,w) = h_{\lambda\mu}(z,w)$ equals the product of the fractions

$$\frac{z^{-1}w - q^{-2(\mu_a+\lambda^*_b-a-b+1)}}{z^{-1}w - q^{2(\lambda_a+\mu^*_b-a-b+1)}} \tag{4.21}$$

taken over all nodes (a,b) of the Young diagram (2.5) of λ. Consider those nodes of (2.5) which do not belong to the Young diagram of μ. Those nodes form the *skew Young diagram*

$$\{(a,b) \in \mathbb{Z}^2 \mid 1 \leqslant a,\ \mu_a < b \leqslant \lambda_a\}. \tag{4.22}$$

To obtain Corollary 1.1, it now suffices to prove the following:

Proposition 4.7. *The product of the fractions* (4.21) *over all the nodes* (a,b) *of the skew Young diagram* (4.22) *equals* 1.

Proof. For any integer c, let us write $\langle c \rangle$ instead of $z^{-1}w - q^{2c}$ for short. We will proceed by induction on the number of nodes in the skew Young diagram (4.22). When the set (4.22) is empty, there is nothing to prove. Let (i,j) be any node of (4.22) such that by removing it from (2.5) we again obtain a Young diagram. Then $\lambda_i = j$ and $\lambda^*_j = i$. By applying the inductive assumption to this Young diagram instead of (2.5), we have to show that the product

$$\frac{\langle j-\mu_i-1\rangle}{\langle \mu^*_j - i + 1\rangle} \prod_{a=\mu^*_j+1}^{i-1} \frac{\langle a+j-\mu_a-i-1\rangle}{\langle a+j-\mu_a-i\rangle} \prod_{b=\mu_i+1}^{j-1} \frac{\langle j+\mu^*_b-i-b\rangle}{\langle j+\mu^*_b-i-b+1\rangle}$$

equals 1. Denote this product by p. Note that here $\mu_i < \lambda_i$ and $\mu^*_j < \lambda^*_j$.

Suppose there is a node (c,d) in (4.22) with $\mu_i < d < \lambda_i$ and $\mu^*_j < c < \lambda^*_j$ such that by adding this node to the Young diagram of μ we again obtain a Young diagram. Then we have $\mu_c = d-1$ and $\mu^*_d = c-1$. The counterpart of the product p for the last Young diagram, instead of that of μ, equals 1 by the inductive assumption. The equality $p = 1$ then follows, by using the identity

$$\frac{\langle j+c-i-d-1\rangle}{\langle j+c-i-d\rangle} \frac{\langle j+c-i-d+1\rangle}{\langle j+c-i-d\rangle} \times \frac{\langle j+c-i-d\rangle}{\langle j+c-i-d+1\rangle} \frac{\langle j+c-i-d\rangle}{\langle j+c-i-d-1\rangle} = 1.$$

It remains to consider the case where there is no node (c,d) in (4.22) with the properties listed above. Then we have $\mu^*_b = i-1$ for all $b = \mu_i+1, \dots, j-1$ and $\mu_a = j-1$ for

all $a = \mu_j^* + 1, \ldots, i-1$. The product p then equals

$$\frac{\langle j-\mu_i-1\rangle}{\langle \mu_j^*-i+1\rangle}\frac{\langle 0\rangle}{\langle j-\mu_i-1\rangle}\frac{\langle \mu_j^*-i+1\rangle}{\langle 0\rangle}=1. \quad \square$$

Finally, let us show how the formula (3.5) can be derived from Theorem 4.5. The element $h_\lambda(q) \in \mathbb{C}(q)$ on the left hand side of (3.5) will be determined by the relation $E_\Lambda^2 = h_\lambda(q)E_\Lambda$ in H_l, where Λ is any standard tableau of shape λ. Below we actually prove another formula for $h_\lambda(q)$ which is equivalent to (3.5).

Corollary 4.8. *We have the equality*

$$h_\lambda(q) = \prod_{(a,b)} \frac{1-q^{-2(\lambda_a+\lambda_b^*-a-b+1)}}{1-q^{-2}} \cdot q^{\lambda_1^*(\lambda_1^*-1)+\lambda_2^*(\lambda_2^*-1)+\cdots} \tag{4.23}$$

where the product is taken over all nodes (a, b) *of the Young diagram* (2.5)*.*

Proof. We will use induction on λ_1, the longest part of the partition λ. First, suppose that $\lambda_1 = 1$. Then each non-zero part of λ equals 1, and there is only one standard tableau Λ of shape λ. In this case, let us write $h_l(q)$ and E_l instead of $h_\lambda(q)$ and E_Λ respectively. Using (2.1) and Theorem 2.2,

$$E_l = \overrightarrow{\prod_{(i,j)}} \left(T_{j-i} + \frac{q-q^{-1}}{q^{2i-2j}-1} \right) \cdot T_0^{-1}$$

where the pairs (i, j) with $1 \leqslant i < j \leqslant l$ are ordered lexicographically. By Proposition 2.5, we have $T_k E_l = -q^{-1}E_l$ for each index $k = 1, \ldots, l-1$. So

$$h_l(q) = \prod_{(i,j)} \left(\frac{q-q^{-1}}{q^{2i-2j}-1} - q^{-1} \right) \cdot (-q)^{l(l-1)/2} = q^{l(l-1)} \prod_{k=1}^{l} \frac{1-q^{-2k}}{1-q^{-2}}.$$

Thus we now have the induction base. To make the induction step, suppose that (4.23) is true for some partition λ of l. Take any positive integer m such that $m \leqslant \lambda_b^*$ for every $b = 1, \ldots, \lambda_1$. Let us show that then the counterpart of the equality (4.23) is true for the partition of $l+m$

$$\theta = (\lambda_1 + 1, \ldots, \lambda_m + 1, \lambda_{m+1}, \lambda_{m+2}, \ldots).$$

Choose any standard tableau Λ of shape λ. Put $\mu = (1, \ldots, 1, 0, 0, \ldots)$ so that $\theta = \lambda + \mu$. In this case, there is only one standard tableau M of shape μ. Let Θ be the unique standard tableau of shape θ agreeing with Λ in the entries $1, \ldots, l$; the numbers $l+1, \ldots, l+m$ then appear in the column λ_1+1 of the tableau Θ. Consider the eigenvalue $r_\theta(z, w)$ of the operator J. We have

$$E_\Theta F_\Lambda \bar{F}_M S_{\Lambda M}(z, w) = r_\theta(z, w) E_\Theta F_\Lambda \bar{F}_M T_\tau. \tag{4.24}$$

By using Theorem 2.2 and its counterpart for the tableau Θ instead of Λ, the element $F_\Theta \in H_{l+m}$ is divisible on the left by the element F_Λ. Therefore the element $E_\Theta \in H_{l+m}$ is divisible on the right by the element E_Λ. Similarly, E_Θ is divisible on the right by the

image $\bar{E}_M$ of the element $E_M \in H_m$ under the embedding $H_l \to H_{l+m} : T_i \mapsto T_{i+m}$. So the right hand side of (4.24) equals

$$h_\lambda(q)h_m(q)r_\theta(z,w)F_\Theta.$$

But, by again using Theorem 2.2 and its counterpart for the tableau Θ instead of Λ, the left hand side of (4.24) takes at $z=1$ and $w=q^{2\lambda_1}$ the value

$$E_\Theta F_\Theta = h_\theta(q)F_\Theta.$$

Hence the equality (4.24) of rational functions in z and w implies that

$$h_\theta(q) = h_\lambda(q)h_m(q)r_\theta(1,q^{2\lambda_1}). \tag{4.25}$$

The factor $r_\theta(1,q^{2\lambda_1})$ in (4.25) can be computed by using Theorem 4.5 when $i_a = a$ for each $a = 1,\dots,\lambda_1^*$. The rational function $r_\theta(z,w)$ of z and w can then be written as the product over $b=1,\dots,\lambda_1$ of the functions

$$\prod_{a=1}^{m}\frac{z^{-1}w-q^{2(a+b-\lambda_b^*-2)}}{z^{-1}w-q^{2(b-a)}}\prod_{a=m+1}^{\lambda_b^*}\frac{z^{-1}w-q^{2(a+b-\lambda_b^*-1)}}{z^{-1}w-q^{2(b-a)}}. \tag{4.26}$$

After changing the running index a to λ_b^*-a+1 in both denominators in (4.26), the product over $a=m+1,\dots,\lambda_b^*$ in (4.26) cancels. Therefore

$$r_\theta(z,w)=\prod_{a=1}^{m}\prod_{b=1}^{\lambda_1}\frac{z^{-1}w-q^{2(a+b-\lambda_b^*-2)}}{z^{-1}w-q^{2(a+b-\lambda_b^*-1)}}.$$

Using (4.25) together with the last expression for the function $r_\theta(z,w)$ we get

$$\begin{aligned}h_\theta(q) &= h_\lambda(q)h_m(q)\prod_{a=1}^{m}\prod_{b=1}^{\lambda_1}\frac{1-q^{-2(\lambda_1+\lambda_b^*-a-b+2)}}{1-q^{-2(\lambda_1+\lambda_b^*-a-b+1)}}\\ &= \prod_{(c,d)}\frac{1-q^{-2(\theta_c+\theta_d^*-c-d+1)}}{1-q^{-2}}\cdot q^{\theta_1^*(\theta_1^*-1)+\theta_2^*(\theta_2^*-1)+\cdots}\end{aligned}$$

where (c,d) is ranging over all nodes of the Young diagram of the partition θ. Here we used the expression for $h_m(q)$ provided by the induction base, and the formula (4.23) for $h_\lambda(q)$ which is true by the inductive assumption. Thus we have made the induction step. □

Remark. Corollary 1.1 shows that the H_{l+m}-module W is reducible, if

$$z^{-1}w = q^{-2(\mu_a+\lambda_b^*-a-b+1)} \qquad \text{or} \qquad z^{-1}w = q^{2(\lambda_a+\mu_b^*-a-b+1)}$$

for some node (a,b) in the intersection of the Young diagrams of λ and μ. The irreducibility criterion for the $\widehat{H}_{l+m}$-module W was given in [10]. That is, the $\widehat{H}_{l+m}$-module W is reducible if and only if $z^{-1}w \in q^{2\mathcal{S}}$ for some finite subset $\mathcal{S}\subset\mathbb{Z}$ explicitly described in [11]. It would be interesting to point out for each $z^{-1}w\in q^{2\mathcal{S}}$ a partition ν of $l+m$, such that W as the H_{l+m}-module has exactly one irreducible component equivalent to V_ν, and that the rational function value $r_{\lambda+\mu}(z,w)/r_\nu(z,w)$ is either 0 or ∞. □

Acknowledgements

I finished this article while visiting Kyoto University. I am grateful to Tetsuji Miwa for hospitality, and to the Leverhulme Trust for supporting this visit. This article is a continuation of our joint work [10] with Bernard Leclerc and Jean-Yves Thibon. I am grateful to them for fruitful collaboration, and for discussion of the present work. I am particularly indebted to Alain Lascoux. It is his chivalrous mind that has rendered the collaboration on [10] possible.

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