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# On Two Problems on Oscillations of Linear Differential Equations of the Third Order

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1. The purpose of this paper is to demonstrate the geometrical (and consequently topological) approach to global problems in linear differential equations by solving two problems proposed by J. M. Dolan in [2]. It was used in [4] for linear differential equations of the 2nd order, and in [5 and 6] for linear differential equations of the $n$th order. The approach makes possible to see globally the whole situation in behavior of solutions.

Consider

$$L(y) \equiv y''' + p_2 y'' + p_1 y' + p_0 y = 0 \tag{1}$$

and its formal adjoint

$$L^*(z) \equiv ((z' - p_2 z)' + p_1 z)' - p_0 z = 0, \tag{1*}$$

both on $[a, b)$, $b \leqslant \infty$, $p_i \in C^0[a, b)$ for $i = 1, 2, 3$, $a$ being a real number. It follows from the general theory [6] that for oscillation problems there is no essential, whether $b < \infty$ or $b = \infty$.

Following the definitions in [2], a nontrivial solution of (1) (or (1*)) is said to be oscillatory if it has infinitely many zeros on $[a, b)$; otherwise it is nonoscillatory. Let $\mathscr{S}(\mathscr{S}^*)$ denote the three-dimensional real vector space of solutions of (1) ((1*)), $\mathscr{S}_1(\mathscr{S}_1^*)$ subspace of $\mathscr{S}(\mathscr{S}^*)$.

$\mathscr{S}_1(\mathscr{S}_1^*)$ is called (i) nonoscillatory, if every nontrivial solution of $\mathscr{S}_1(\mathscr{S}_1^*)$ is nonoscillatory; (ii) weakly oscillatory, if $\mathscr{S}_1(\mathscr{S}_1^*)$ has both oscillatory and nonoscillatory nontrivial solutions; (iii) strongly oscillatory, if every nontrivial solution of $(\mathscr{S}_1\mathscr{S}_1^*)$ is oscillatory; (iv) oscillatory, if (ii) or (iii) holds.

Equation (1)((1*)) is called nonoscillatory, weakly oscillatory, strongly oscillatory, or oscillatory, according to $\mathscr{S}(\mathscr{S}^*)$.

The negative answer to the first proposed problem is given by proving:

THEOREM 1. *There does not exist a linear differential equation* (1) *with the property that every two-dimensional subspace of its solution space is weakly oscillatory.*

The affirmative answer to the second problem is given by constructing *an example of a strongly oscillatory equation* (1) *whose adjoint* (1*) *is also strongly oscillatory.*

2. Here we introduce only some necessary notions and relations from [6]. Let $E_n$ denote an Euclidean real $n$-dimensional vector space with the norm $|\mathbf{x}| = (\sum_{i=1}^{n} x_i^2)^{1/2}$ of its vectors $\mathbf{x} = (x_1, ..., x_n)$. For $\mathbf{c} = (c_1, ..., c_n) \in E_n$, $\mathbf{c} \neq \mathbf{0}$, $H(\mathbf{c}) = \{\mathbf{x} \in E_n ; \mathbf{c} \cdot \mathbf{x} = \sum_{i=1}^{n} c_i x_i = 0\}$ is a hyperplane passing $\mathbf{0} \in E_n$. Hyperplanes $H(\mathbf{c}_1), ..., H(\mathbf{c}_j)$ are called independent if the rank of the matrix $(\mathbf{c}_1, ..., \mathbf{c}_j)$ is $j$.

Let $C^j[c, d)$ denote the set of all functions having continuous derivatives up to and including the order $j$ on $[c, d)$. A curve $\mathbf{u}(s) = (u_1(s), ..., u_n(s))$, $s \in [c, d)$, is of the class $\mathbf{C}^j[c, d)$, if $u_i \in C^j[c, d)$ for $i = 1, ..., n$. $W[\mathbf{u}](s)$ denotes the Wronski determinant $\det(\mathbf{u}(s), \mathbf{u}'(s), ..., \mathbf{u}^{(n-1)}(s))$.

In [6] Theorem 1 was proved:

To every $L_n(y) = 0$ on $[a, b)$ there correspond a curve $\mathbf{u} \in \mathbf{C}^n[c, d)$, a mapping $s$: $[a, b) \to [c, d)$, and a linear correspondence $h$ between all solutions of $L_n(y) = 0$ and all hyperplanes $H(\mathbf{c}) \in E_n$ such that $|\mathbf{u}(s)| = 1$, $ds/dt > 0$. To linearly independent solutions $y_1$ and $y_2$ there correspond linearly independent hyperplanes $h(y_1)$ and $h(y_2)$. Moreover, $t_0 \in [a, b)$ is a $k$ $(0 \leqslant k \leqslant n - 1)$ multiple zero of a solution $y$ iff the curve $\mathbf{u}$ has a contact of the $(k - 1)$st order with $h(y)$ at $s(t_0) \in [c, d)$.

And property 8 from [6] states:

If a curve $\mathbf{u}(s) \in \mathbf{C}^n[c, d)$ corresponds to $L_n(y) = 0$ on $[a, b)$ in the above sense, then the normalized vector product

$$\mathbf{u}^*(s) = \mathbf{u}(s) \times \mathbf{u}'(s) \times \cdots \times \mathbf{u}^{(n-2)}(s)/|\mathbf{u}(s) \times \mathbf{u}'(s) \times \cdots \times \mathbf{u}^{(n-2)}(s)|$$

on $[c, d)$ corresponds to $L^*(y) = 0$ on $[c, d)$, adjoint to $L(y) = 0$ on $[a, b)$.

From this moment, we restrict ourselves to $n = 3$. However, since no other properties such as the compactness of the unit sphere $S_{n-1} \subset E_n$ and its main circles, the continuity of $\mathbf{u}(s)$ and the disconnectness of any two open opposite half-spaces $E_n^+$ and $E_n^-$ are used, all notions and theorems in the next paragraphs can be generalized for $n \geqslant 2$.

3. With respect to the results mentioned in Par. 2 and applied for $n = 3$, Theorem 1 will be proved iff we show the following.

THEOREM 2. *There is no curve* $\mathbf{u} \in \mathbf{C}^3[c, d)$, $W[\mathbf{u}](s) \neq 0$, $|\mathbf{u}(s)| = 1$ (*i.e.* $\mathbf{u} \subset S_2 \subset E_3$) *with the property that for every point* $\mathbf{p} \in S_2$ *there exist two planes passing the origin* (*or two main circles*) *containing* $\mathbf{p}$, *and one of* *them intersects* $\mathbf{u}$ (*on* $[c, d)$) *only in a finite number of points whereas the second plane intersects* $\mathbf{u}$ *in infinite number of points.*

*Note.* In fact, we shall prove Theorem 2 under weaker assumptions, namely not assuming $W[\mathbf{u}](s) \neq 0$ and supposing $\mathbf{u} \in \mathbf{C}^0[c, d)$ instead of $\mathbf{u} \in \mathbf{C}^3[c, d)$.

It is not essential if $d = \infty$ or $d < \infty$.

**4.** Topology on $S_2$ is the topology of $E_3$ induced by the norm $|x|$. According to Par. 2, $H(\mathbf{c})$ denotes a plane in $E_3$ passing the origin. Let $C(\mathbf{c}) \stackrel{\mathrm{df}}{=} S_2 \cap H(\mathbf{c})$ denote a main circle. The angle $\sphericalangle(\mathbf{c}_1, \mathbf{c}_2)$,

$$|\mathbf{c}_1| \cdot |\mathbf{c}_2| \neq 0,$$

is defined as

$$\arccos(\mathbf{c}_1 \cdot \mathbf{c}_2)/(|\mathbf{c}_1| \cdot |\mathbf{c}_2|) \in [0, \pi].$$

The distance $\delta(H(\mathbf{c}_1), H(\mathbf{c}_2))$ of two planes $H(\mathbf{c}_1)$, $H(\mathbf{c}_2)$ (or $\delta(C(\mathbf{c}_1), C(\mathbf{c}_2))$ of two main circles) is defined as the angle $\sphericalangle(\mathbf{c}_1, \mathbf{c}_2)$.

Evidently, $0 \leqslant \delta(H(\mathbf{c}_1), H(\mathbf{c}_2)) \leqslant \pi$. Define:

$$\begin{aligned}
E_+(\mathbf{c}) &= \{\mathbf{x} \in E_3 ; \mathbf{x} \cdot \mathbf{c} > 0\}, & \bar{E}_+(\mathbf{c}) &= E_+(\mathbf{c}) \cup H(\mathbf{c}),\\
E_-(\mathbf{c}) &= \{\mathbf{x} \in E_3 ; \mathbf{x} \cdot \mathbf{c} < 0\}, & \bar{E}_-(\mathbf{c}) &= E_-(\mathbf{c}) \cup H(\mathbf{c}),\\
S_+(\mathbf{c}) &= E_+(\mathbf{c}) \cap S_2, & \bar{S}_+(\mathbf{c}) &= S_+(\mathbf{c}) \cup C(\mathbf{c}),\\
S_-(\mathbf{c}) &= E_-(\mathbf{c}) \cap S_2, & \bar{S}_-(\mathbf{c}) &= S_-(\mathbf{c}) \cup C(\mathbf{c}).
\end{aligned}$$

For a non-empty set $M \subset S_2$ define its *circular radius* $r(M)$ as

$$r(M) \stackrel{\mathrm{df}}{=} \inf_{\mathbf{c} \in S_2}\{\sup_{\mathbf{x} \in M}\{\sphericalangle(\mathbf{x}, \mathbf{c})\}\}.$$

Evidently $r(M) = 0$ iff $M$ is a point. Define $\gamma(\mathbf{a}, \alpha) \stackrel{\mathrm{df}}{=} \{\mathbf{x} \in S_2 ; \sphericalangle(\mathbf{a}, \mathbf{x}) \leqslant \alpha\}$. Since $\sup_{\mathbf{x} \in M} \sphericalangle(\mathbf{x}, \mathbf{c})$, $\mathbf{c} \in S_2$, is a continuous function of $\mathbf{c}$ defined on the compact set $S_2$, we have for every nonempty $M \subset S_2$ such $\mathbf{c}_0 \in S_2$ that $\sup_{\mathbf{x} \in M} \sphericalangle(\mathbf{x}, \mathbf{c}_0) = r(M)$ (hence $\sphericalangle(\mathbf{x}, \mathbf{c}_0) \leqslant r(M)$ for all $\mathbf{x} \in M$). This $\mathbf{c}_0$ is uniquely determined, since for two different $\mathbf{c}_0^*$, $\mathbf{c}_0^{**}$ (with the same $r(M)$), we get the contradiction with the definition of $r(M)$.

Let $M$ be a nonempty set, $M \subset \bar{S}_+(\mathbf{c}_0)$ for some $\mathbf{c}_0$. Define the *angular diameter* $\rho$ of $M$ as

$$\rho(M) \stackrel{\mathrm{df}}{=} \inf\{\delta(H(\mathbf{c}_1), H(\mathbf{c}_2)); M \subset \bar{S}_+(\mathbf{c}_1) \cap \bar{S}_+(\mathbf{c}_2), \mathbf{c}_1 \in E_3, \mathbf{c}_2 \in E_3\}.$$

Evidently $\rho(M) = 0$ iff $M$ is a subset of a main hemicircle on $S_2$ (i.e., of the length $\leqslant \pi$).

For $\mathbf{u} \in \mathbf{C}^0[c, d)$, $\mathbf{u} \subset S_2$, define its limit set $\Omega(\mathbf{u})$ as

$$\Omega(\mathbf{u}) \overset{\text{df}}{=} \bigcap_{t \in [c,d)} \overline{\{\mathbf{u}(s);\, s \in [t, d)\}},$$

where $\bar{M}$ denotes the closure of $M$. Each $\mathbf{p} \in \Omega(\mathbf{u})$ is called a limit point of $\mathbf{u}$. Compare similar notions in dynamical systems in [3] or [7]. Remember that our curve $\mathbf{u}$ may intersect itself even infinitely many times. However, it holds for the following.

LEMMA 1. *For each* $\mathbf{u} \in \mathbf{C}^0[c, d)$, $c < d \leqslant \infty$, $\mathbf{u} \subset S_2$, *the limit set* $\Omega(\mathbf{u})$ *is nonempty, connected, and closed.*

*Proof.* It is essentially the same as in [3, p. 358–362] due to the compactness of $S_2$. The self-intersections of $\mathbf{u}$ do not play any role here.

LEMMA 2. *Let* $\mathbf{u} \in \mathbf{C}^0[c, d)$, $\mathbf{u} \subset S_2$. *For every* $\mathbf{p} \in S_2$, *let there exist two planes* $H({}_{\mathbf{p}}\mathbf{c})$ *and* $H({}_{\mathbf{p}}\mathbf{c}')$, *both containing* $\mathbf{p}$, *such that* $\{s \in [c, d);\, \mathbf{u}(s) \in H({}_{\mathbf{p}}\mathbf{c})\}$ *is finite and* $\{s \in [c, d);\, \mathbf{u}(s) \in H({}_{\mathbf{p}}\mathbf{c}')\}$ *is infinite. Then* $\rho(\Omega(\mathbf{u})) = 0$.

*Note.* The assumptions of Theorem 2 are stronger than that of Lemma 2.

*Proof.* Let $\mathbf{p} \in S_2$. Since $\{s \in [c, d);\, \mathbf{u}(s) \in H({}_{\mathbf{p}}\mathbf{c})\}$ is finite, there exists ${}_{\mathbf{p}}s$ such that either $\{\mathbf{u}(s);\, s \in [{}_{\mathbf{p}}s, d)\} \subset S_+({}_{\mathbf{p}}\mathbf{c})$, or $\{\mathbf{u}(s);\, s \in [{}_{\mathbf{p}}s, d)\} \subset S_-({}_{\mathbf{p}}\mathbf{c})$. Without loss of generality, let ${}_{\mathbf{p}}\mathbf{c}$ be always oriented such that

$$\{\mathbf{u}(s);\, s \in [{}_{\mathbf{p}}s, d)\} \subset S_+({}_{\mathbf{p}}\mathbf{c}) \qquad \text{for every } \mathbf{p} \in S_2.$$

Hence we have $\Omega(\mathbf{u}) \subset \bar{S}_+({}_{\mathbf{p}}\mathbf{c})$ for every $\mathbf{p} \in S_2$.

(i) If $r(\Omega(\mathbf{u})) = 0$, then $\Omega(\mathbf{u})$ is a point and $\rho(\Omega(\mathbf{u})) = 0$.

(ii) Hence let $r(\Omega(\mathbf{u})) = B > 0$ and $\rho(\Omega(\mathbf{u})) = A$. Suppose $A > 0$. It is possible to choose $\mathbf{e}$ such that $\gamma(\mathbf{e}, B) \supset \Omega(\mathbf{u})$, $\Omega(\mathbf{u}) \subset \bar{S}_-({}_{\mathbf{e}}\mathbf{c})$. Hence $\Omega(\mathbf{u}) \subset (\gamma(\mathbf{e}, B) \cap \bar{S}_+({}_{\mathbf{e}}\mathbf{c}))$. Take arbitrarily $\mathbf{f} \in \text{int}(\gamma(\mathbf{a}, B) \cap S_+({}_{\mathbf{e}}\mathbf{c}))$. If $C({}_{\mathbf{f}}\mathbf{c}) \cap (\text{int}\, \gamma(\mathbf{e}, B) \cap C({}_{\mathbf{e}}\mathbf{c})) \neq \varnothing$ or if $S_-({}_{\mathbf{f}}\mathbf{c}) \supset (\gamma(\mathbf{e}, B) \cap C({}_{\mathbf{e}}\mathbf{c}))$, then there exists a $\gamma(\mathbf{e}', B') \supset \Omega(\mathbf{u})$ with $B' < B$, that contradicts to $r(\Omega(\mathbf{u})) = B$. Hence we get:

$\Omega(\mathbf{u}) \subset (\gamma(\mathbf{e}, B) \cap \bar{S}_+({}_{\mathbf{e}}\mathbf{c}))$, $B \leqslant \pi/2$, and for every $\mathbf{p} \in \text{int}(\gamma(\mathbf{e}, B) \cap \bar{S}_+({}_{\mathbf{e}}\mathbf{c}))$ the hemisphere $\bar{S}_+({}_{\mathbf{p}}\mathbf{c})$ contains $(\gamma(\mathbf{e}, B) \cap C({}_{\mathbf{e}}\mathbf{c}))$. Since $\mathbf{p}$ can be chosen arbitrarily close to $\gamma(\mathbf{e}, B) \cap C({}_{\mathbf{e}}\mathbf{c})$, and $\Omega(\mathbf{u}) \subset (\bar{S}_+({}_{\mathbf{e}}\mathbf{c}) \cap \bar{S}_+({}_{\mathbf{p}}\mathbf{c}) \cap \gamma(\mathbf{e}, B))$ where $B > 0$, the value $\rho((\bar{S}_+({}_{\mathbf{e}}\mathbf{c}) \cap \bar{S}_+({}_{\mathbf{p}}\mathbf{c})) = \sphericalangle(H({}_{\mathbf{e}}\mathbf{c}), H({}_{\mathbf{p}}\mathbf{c}))$ can be arbitrarily small. Hence $\rho(\Omega(\mathbf{u})) = 0$.

LEMMA 3. *Under the assumptions of Lemma* 2, $\Omega(\mathbf{u})$ *is a closed arc of a main circle of* $S_2$ *of the length* $2r(\Omega(\mathbf{u})) \leqslant \pi$, *i.e. either*

1° $\Omega(\mathbf{u})$ *is a point* ($r(\Omega(\mathbf{u})) = 0$); *or*

2° $\Omega(\mathbf{u})$ *is a closed arc of the positive length* $<\pi$; *or*

3° $\Omega(\mathbf{u})$ *is a closed hemicircle* ($r(\Omega(\mathbf{u})) = \pi/2$).

*Proof.* Lemma 3 is a direct consequence of Lemmas 1 and 2.

5. Now we prove Theorem 2 by showing that each of the cases 1°–3° leads to a contradiction.

ad 1°. Let $\mathbf{p} = \Omega(\mathbf{u})$ and $\mathbf{q} \in C({}_{\mathbf{p}}\mathbf{c})$, $\mathbf{q} \neq \mathbf{p}$. Consider a plane $Q$, $\mathbf{0} \in Q$, $\mathbf{q} \in Q$.

If $\mathbf{p} \bar{\in} Q$, then $\{s \in [c, d); \mathbf{u}(s) \in Q\}$ is finite, since $Q \cap \Omega(\mathbf{u}) = \varnothing$.

If $\mathbf{p} \in Q$, then $Q \equiv C({}_{\mathbf{p}}\mathbf{c})$ by the choice of $\mathbf{q}$, and $\{s \in [c, d); \mathbf{u}(s) \in Q\}$ is finite by the definition of ${}_{\mathbf{p}}\mathbf{c}$.

Hence we have a contradiction with the assumption concerning planes through the point $\mathbf{q}$.

ad 2°. Let $\Omega(\mathbf{u}) \subset C(\mathbf{k})$, $\lambda$ be its length $(0 < \lambda < \pi)$, $\mathbf{p}$ and $\mathbf{q}$ its end-points.

a. Let $\{s \in [c, d); \mathbf{u}(s) \in C(\mathbf{k})\}$ be infinite. Choose $\mathbf{w} \in \Omega(\mathbf{u})$, $\mathbf{w} \neq \mathbf{p}$, $\mathbf{w} \neq \mathbf{q}$. Let $Q$ denote a plane, $Q \ni \mathbf{w}$, $Q \ni \mathbf{0}$, $Q \bar{\ni} \mathbf{p}$. Then $\{s \in [c, d); \mathbf{u}(s) \in Q\}$ is infinite, since $\mathbf{p} \in \Omega(\mathbf{u})$, $\mathbf{q} \in \Omega(\mathbf{u})$, and their suitable neighborhoods $N(\mathbf{p})$ and $N(\mathbf{q})$ lie on opposite sides of $Q$, and hence $\mathbf{u}(s)$ goes infinitely many times from $N(\mathbf{p})$ to $N(\mathbf{q})$ and back.

We have seen that every plane containing $\mathbf{w}$ and $\mathbf{0}$ (i.e. containing also $\mathbf{p}$ and hence coinciding with $C(\mathbf{k})$, or not containing $\mathbf{p}$ and being denoted by $Q$), has an infinite number of intersections with $\mathbf{u}(s)$; that is a contradiction.

b. Let $\{s \in [c, d); \mathbf{u}(s) \in C(\mathbf{k})\}$ be finite. Choose $\mathbf{w} \in C(\mathbf{k})$, $\mathbf{w} \bar{\in} \Omega(\mathbf{u})$, $\measuredangle(\mathbf{w}, \mathbf{x}) < \pi$ for each $\mathbf{x} \in \Omega(\mathbf{u})$. Every plane containing the origin and $\mathbf{w}$ has only a finite number of intersections with $\mathbf{u}$ since either it coincides with $C(\mathbf{k})$, or it does not contain any point from $\Omega(\mathbf{u})$. But this is a contradiction, since the assumptions are not satisfied at $\mathbf{w}$.

ad 3°. Let $\Omega(\mathbf{u}) \subset C(\mathbf{k})$ be a hemicircle, $\mathbf{p}$ and $\mathbf{q}$ its endpoints. Consider $C({}_{\mathbf{p}}\mathbf{c}')$. Since $\Omega(\mathbf{u})$ is a hemicircle, we have also $\mathbf{q} \in C({}_{\mathbf{p}}\mathbf{c}')$. Let $\mathbf{w} \in C({}_{\mathbf{p}}\mathbf{c}')$, $\mathbf{w} \neq \mathbf{p}$, $\mathbf{w} \neq \mathbf{q}$. If $Q$ is a plane, $Q \ni \mathbf{0}$, $Q \ni \mathbf{w}$, $Q \bar{\ni} \mathbf{p}$, then $Q$ intersects $\mathbf{u}(s)$ infinitely many times, since suitable neighborhoods of $\mathbf{p}$ and $\mathbf{q}$ lie on opposite hemispheres with respect to $Q \cap S_2$. We use the same argument as in 2a. We see that every plane containing $\mathbf{0}$ and $\mathbf{w}$ either coincides with $C({}_{\mathbf{p}}\mathbf{c}')$ or is denoted by $Q$. However, in both cases, it has infinitely many intersections with $\mathbf{u}(s)$, which contradicts the existence of $C({}_{\mathbf{w}}\mathbf{c})$.

Hence Theorem 2 has been proved.

6. Figures 1 and 2.

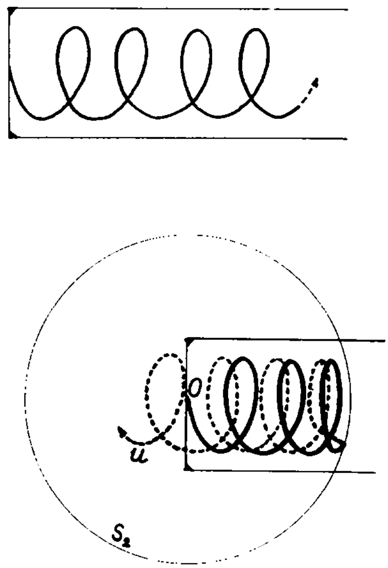


FIGURE 1

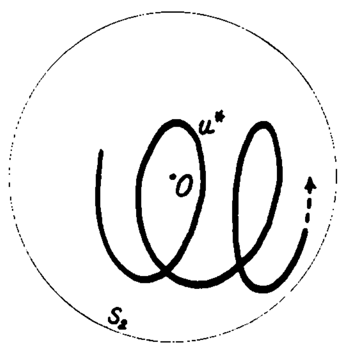


FIGURE 2

If we are not satisfied with the figures showing an approximate behavior of $\mathbf{u}$ and $\mathbf{u}^*$, both intersected by every plane ($\ni \mathbf{0}$) infinitely many times (accordingly to the results mentioned in Par. 2), we may follow the next lines (see [6]).

Let $R > r > 0$, $s \in [0, \infty)$,

$$\varphi(s) = rs + R \sin s, \qquad \omega(s) = r + R \cos s$$

$$v_1(s) = \cos \varphi(s), \qquad v_2(s) = \sin \varphi(s), \qquad v_3(s) = \omega(s).$$

Then $\mathbf{v} = (v_1, v_2, v_3)$ is a curve lying on the cylinder $v_1^2 + v_2^2 = 1$ in $E_3$, $\mathbf{v} \in \mathbf{C}^\infty[0, \infty)$, being intersected infinitely many times by every plane in $E_3$ containing the origin. Moreover,

$$W[\mathbf{v}](s) = -\rho''\omega' + \rho'\omega'' + \varphi'^3\omega = -R(R + r\cos s) + (r + R\cos s)^4$$

that tends to $-R^2 + R^4\cos^4 s = -R^2(1 - R^2\cos^4 s)$ for $r \to 0_+$. For every $R \in (0, 1)$ there exists $r > 0$ such that $W[\mathbf{v}](s) \neq 0$ on $[0, \infty)$. Hence for those $R$ and $r$, the curve $\mathbf{u}(s) =^{\mathrm{df}} \mathbf{v}(s)/|\mathbf{v}(s)| \in S_2$, $\mathbf{u} \in \mathbf{C}^\infty[0, \infty)$, is intersected infinitely many times by every plane passing the origin, and $W[\mathbf{u}](s) = W[\mathbf{v}](s)/|\mathbf{v}(s)|^3 \neq 0$ on $[0, \infty)$.

Now $\mathbf{u}^*(s) = \mathbf{u}(s) \times \mathbf{u}'(s) = (\mathbf{v}(s) \times \mathbf{v}'(s))/|\mathbf{v}(s)|^2$, where

$$\begin{aligned}\mathbf{v}(s) \times \mathbf{v}'(s) &= (\cos\varphi, \sin\varphi, \omega) \times (-\sin\varphi\cdot\varphi', \cos\varphi\cdot\varphi', \omega') \\ &= (\sin\varphi\cdot\omega' - \omega\cdot\cos\varphi\cdot\varphi', -\omega\cdot\sin\varphi\cdot\varphi' - \omega'\cdot\cos\varphi, \varphi')\end{aligned}$$

for $\varphi = rs + R\sin s$, $\omega = r + R\cos s$, $' = d/ds$.

Since $\mathbf{v} \in \mathbf{C}^\infty[0, \infty)$, also $\mathbf{u}^* \in \mathbf{C}^\infty[0, \infty)$. From $W[\mathbf{u}] \neq 0$ we have $W[\mathbf{u}^*] \neq 0$. For $\mathbf{c} = (c_1, c_2, c_3) \neq 0$,

$$\begin{aligned}&\mathbf{c}\cdot(\mathbf{v}(s) \times \mathbf{v}'(s)) \\ &\quad = c_1[-R\sin s\cdot\sin(rs + R\sin s) - (r + R\cos s)^2\cdot\cos(rs + R\sin s)] \\ &\qquad + c_2[-(r + R\cos s)^2\cdot\sin(rs + R\sin s) - R\sin s\cdot\cos(rs + R\sin s)] \\ &\qquad + c_3(r + R\cos s).\end{aligned}$$

Let $\cos s_0 = -r/R$, $s_0 \in [0, \pi]$, $s_k = s_0 + 2k\pi$. Then

$$\begin{aligned}\mathbf{c}\cdot(\mathbf{v}(s) \times \mathbf{v}'(s))_{s=s_k} &= -c_1 R\sin s_k\cdot\sin(rs_k + R\sin s_k) \\ &\quad - c_2 R\sin s_k\cdot\cos(rs_k + R\sin s_k) \\ &= -R\sin s_0\cdot d_1\cdot\sin(rs_k + R\sin s_0 + d_2)\end{aligned}$$

for suitable constants $d_1$ and $d_2$.

Hence for arbitrary $R$, there always exists (a sufficiently small) $r > 0$ such that for every $d_1$ and $d_2$ the last expression is either 0 for every $k$, or it changes its sign in an infinite subsequence $\{s_{k_j}\}_{j=1}^\infty$, $k_j \to \infty$ for $j \to \infty$. (This subsequence of $\{s_k\}$ may depend on $d_1$ and $d_2$.)

In other words, $\{s \in [0, \infty); \mathbf{c}\cdot\mathbf{u}^*(s) = 0\}$ is infinite for every $\mathbf{c} \neq \mathbf{0}$.

Hence the existence of our $\mathbf{u}$ (and $\mathbf{u}^*$) established, according to the results in [6] mentioned here in Par. 2, the existence of a differential equation (1) that together with its adjoint (1*) are strongly oscillatoric. The coordinates of $\mathbf{u}$ may be used as linearly independent solutions of (1) when (1) has to be written explicitly.

*Note.* After finishing the example, Dr. J. Suchomel has informed me about an example of (1) due to Ascoli [1] that also gives the affirmative answer to the second problem of J. M. Dolan.

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